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NOVEL FIXED POINT OUTCOMES IN COMPACT GENERALIZED METRIC SPACES

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Abstract. This work presents new advancements in fixed point theory within the framework of G-metric spaces, initially introduced by Mustafa and Sims. Contrary to prior findings that often reduced such results to their standard metric counterparts, our approach yields genuinely intrinsic and non-reducible theorems. We establish extended versions of the Banach, Kannan, and Reich fixed point theorems, leveraging the assumption of compactness in the G-metric setting. Our methodology eschews reliance on equivalence to standard metrics, instead furnishing stronger conclusions inherent to the G-metric structure. Furthermore, we explore applications involving mappings that contract the perimeters of triangles—a geometric condition with significant implications for nonlinear analysis. Included examples demonstrate the necessity of our hypotheses and delineate scenarios where existing results fail. These contributions propel the theory of generalized metric structures and their practical use.

Keywords: fixed point; G-metric; triangle perimeter contractions.

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1. Introduction and Preliminary Concepts

The G-metric space, a generalization of the standard metric space, was proposed by Mustafa and Sims [1]. It employs a function $G : X \times X \times X \to \mathbb{R}_+$ adhering to axioms including sym-

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metry, positivity, and a reinforced rectangle inequality, facilitating a broader geometric context for fixed point theory.

Subsequent to [1], numerous fixed point results were established in this framework, encompassing analogues of theorems by Banach [3], Kannan, and Reich [4]. A pivotal observation by Jleli and Samet [5], however, indicated that many such results were recoverable from classical metric space theory via the induced metric $\delta(x,y) = \max\{G(x,x,y), G(y,x,x)\}$, prompting a decline in research interest.

This paper introduces fixed point theorems that are intrinsic to the G-metric structure and not reducible to the standard metric case. We prove new forms of the Banach, Kannan, and Reich theorems for compact G-metric spaces, utilizing techniques that inherently depend on the three-variable nature of the G-metric. Inspired by Jleli et al. [2], we also investigate contractive maps and, motivated by Petrov [6] and Popescu-Pacurar [7], apply our results to mappings contracting triangle perimeters—a condition naturally expressed in G-metric terms but not in ordinary metrics.

Section 2 recalls essential definitions. Section 3 states and proves our main theorems. Section 4 details applications, emphasizing triangle-perimeter contractions.

Notation: Let X be a nonempty set, and denote by $|X|$ its cardinality. For a map $T : X \to X$, $Fix(T)$, the set of fixed points is denoted by

$$Fix(T) = \{x \in X | T(x) = x\}.$$

Definition 1.1 ([1])**.** Let X be nonempty. A function $G : X \times X \times X \to [0,+\infty)$ is a *G-metric* if for all $x,y,z,a \in X$:

(G1) $G(x,y,z) = 0 \iff x = y = z$,

(G2) $0 < G(x,x,y)$ for $x \neq y$,

(G3) $G(x,x,y) \leq G(x,y,z)$ for $z \neq y$,

(G4) G is symmetric in all three variables,

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ (rectangle inequality).

The pair (X,G) is a *G-metric space*.

1.1. Convergence, Continuity, and Compactness.

Definition 1.2 ([1]). A sequence (x_n) in a G-metric space (X,G) *G-converges* to x if it converges to x in the topology $\tau(G)$ induced by G.

Proposition 1.3 ([1]). *For $(x_n) \subseteq X$ and $x \in X$, the following are equivalent:*

(1) $x_n \to x$ *in* $\tau(G)$.
(2) $G(x_n,x_n,x) \to 0$ *as* $n \to \infty$.
(3) $G(x_n,x,x) \to 0$ *as* $n \to \infty$.
(4) $G(x_m,x_n,x) \to 0$ *as* $m,n \to \infty$.

Definition 1.4 ([1]). A function $f:(X,G) \to (X',G')$ is *G-continuous at* x_0 if $f^{-1}(B_{G'}(f(x_0),r)) \in \tau(G)$ for all $r > 0$. It is *G-continuous* if this holds for all $x_0 \in X$.

Proposition 1.5 ([1]). *A function is G-continuous at x iff it is sequentially G-continuous at x.*

Proposition 1.6 ([1]). *The function $G(x,y,z)$ is jointly continuous in its three variables.*

Definition 1.7 ([1]). For $\varepsilon > 0$, a set $A \subseteq X$ is an *ε-net* if $X = \bigcup_{a\in A} B_G(a;\varepsilon)$, where $B_G(a;\varepsilon) = \{y \in X : G(a,y,y) < \varepsilon\}$. A finite ε-net is one where A is finite.

Definition 1.8 ([1]). A G-metric space is *G-totally bounded* if it has a finite ε-net for every $\varepsilon > 0$.

Definition 1.9 ([1]). A G-metric space is *compact* if it is G-complete and G-totally bounded.

Proposition 1.10 ([1]). *For a G-metric space (X,G), these are equivalent:*

(1) (X,G) *is compact.*
(2) $(X,\tau(G))$ *is a compact topological space.*
(3) Every bounded sequence (x_n) *in* X *(i.e.,* $\sup_{n,m,l} G(x_n,x_m,x_l) < \infty$*) has a G-convergent subsequence.*

2. Principal Findings

Assuming $|X| \geq 3$, we present new fixed-point results for compact G-metric spaces, inspired by [2].

Theorem 2.1. *Let (X,G) be compact, and $T : X \to X$ satisfy:*

(i) If $Tx \neq x$, then $T(Tx) \neq x$ for all $x \in X$.

(ii) $G(Tx,Ty,Tz) < G(x,y,z)$ for all distinct $x,y,z \in X$.

Then $Fix(T) \neq \emptyset$ and $|Fix(T)| \leq 2$.

Proof. Condition (ii) implies T is G-continuous. Define $f(x) = G(x,Tx,T(Tx))$, which is continuous. By compactness, f attains its infimum at some $z \in X$.

Assume $z \neq Tz$ and $Tz \neq T(Tz)$. By (i), $z \neq T(Tz)$. Applying (ii) to the distinct triplet $z,Tz,T(Tz)$ gives:

$$f(Tz) = G(Tz,T(Tz),T(T(Tz))) < G(z,Tz,T(Tz)) = f(z),$$

contradicting the minimality of $f(z)$. Thus, $z = Tz$ or $Tz = T(Tz)$, making z or Tz a fixed point.

If three distinct fixed points z_1,z_2,z_3 existed, (ii) would yield:

$$G(z_1,z_2,z_3) = G(Tz_1,Tz_2,Tz_3) < G(z_1,z_2,z_3),$$

a contradiction. Hence, at most two fixed points exist. □

Example 2.2. Let $X = \{1,2,3\}$ with $G(x,y,z) = |x-y|+|x-z|+|y-z|$. Define T by $T1 = T2 = 3$, $T3 = 2$. Here, $T(T2) = 2$, violating (i), while (ii) holds: $G(T1,T2,T3) = 2 < 4 = G(1,2,3)$. X is compact, but $Fix(T) = \emptyset$, showing condition (i) is essential.

Example 2.3. On $X = \{1,2,3\}$ with the same G-metric, define $T1 = 1, T2 = 1, T3 = 2$. Condition (i) holds. Condition (ii) is satisfied: $G(T1,T2,T3) = 2 < 4$. All conditions are met, and $Fix(T) = \{1\}$.

Example 2.4. Let $X = [0,1]$ with the same G-metric. Define:

$$Tx = \begin{cases} \frac{n}{n+2}x + \frac{n+1}{n+2} & \text{if } x \in [\frac{1}{n+1}, \frac{1}{n}],\, n \in \mathbb{N}, \\ 0 & \text{if } x = 0. \end{cases}$$

X is compact, $T0 = 0$, $T1 = 1$, and $T(Tx) \neq x$ for $x \in (0,1)$. For any $\lambda \in (0,1)$, choosing $n > 2\lambda/(1-\lambda)$ gives $G(Tx,Ty,Tz) > \lambda G(x,y,z)$ for distinct x,y,z in some interval, so the contractive condition from [2, Thm. 3.1] is not met. However, $G(Tx,Ty,Tz) < G(x,y,z)$ holds for all distinct x,y,z, fulfilling Theorem 2.1's conditions. Thus, $Fix(T) = \{0,1\}$.

Theorem 2.5. *Let (X,G) be compact, and let $T : X \to X$ be G-continuous satisfying:*

(i) If $Tx \neq x$, then $T(Tx) \neq x$ for all $x \in X$.

(ii) $G(Tx,Ty,Tz) < \frac{1}{3}[G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)]$ for all distinct $x,y,z \in X$.

Then $Fix(T) \neq \emptyset$ and $|Fix(T)| \leq 2$.

Proof. Define $f(x) = G(x,Tx,T(Tx))$. By compactness and continuity, f attains its minimum at some z. Suppose $z \neq Tz$ and $Tz \neq T(Tz)$; then $z \neq T(Tz)$ by (i). Applying (ii) to $z,Tz,T(Tz)$:

$$\begin{aligned} f(Tz) = G(Tz,T(Tz),T(T(Tz))) <& \frac{1}{3}[G(z,Tz,Tz)+G(Tz,T(Tz),T(Tz)) \\ &+G(T(Tz),T(T(Tz)),T(T(Tz)))] \end{aligned}$$

Using G-metric properties (G3 and symmetry), we find:

$$\begin{aligned} G(z,Tz,Tz) &\leq G(z,Tz,T(Tz)) = f(z), \\ G(Tz,T(Tz),T(Tz)) &\leq G(Tz,T(Tz),z) = f(z), \\ G(T(Tz),T(T(Tz)),T(T(Tz))) &\leq G(T(Tz),T(T(Tz)),Tz) \\ &= G(Tz,T(Tz),T(T(Tz))) = f(Tz). \end{aligned}$$

Substituting yields:

$$f(Tz) < \frac{1}{3}[f(z)+f(z)+f(Tz)] = \frac{2}{3}f(z)+\frac{1}{3}f(Tz),$$

which simplifies to $f(Tz) < f(z)$, a contradiction. Therefore, $z = Tz$ or $Tz = T(Tz)$. The proof that at most two fixed points can exist proceeds as in Theorem 2.1, leading to a contradiction from assumption (ii). □

Theorem 2.6. *Let (X,G) be compact, and let $T : X \to X$ be G-continuous satisfying:*

(i) If $Tx \neq x$, then $T(Tx) \neq x$ for all $x \in X$.

(ii) $G(Tx,Ty,Tz) < \max\{G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz),G(x,y,z)\}$ for all distinct $x,y,z \in X$.

Then $Fix(T) \neq \emptyset$ and $|Fix(T)| \leq 2$.

Proof. Define $f(x) = G(x, Tx, T(Tx))$. Let z be a point where f attains its minimum. Assume $z \neq Tz$ and $Tz \neq T(Tz)$, implying $z \neq T(Tz)$. Apply (ii) to $z, Tz, T(Tz)$:

$$f(Tz) < \max\{G(z,Tz,Tz), G(Tz,T(Tz),T(Tz)), G(T(Tz),T(T(Tz)),T(T(Tz))),$$
$$G(z,Tz,T(Tz))\}.$$

Again, using G-metric properties:

$$G(z,Tz,Tz) \leq f(z),\ G(Tz,T(Tz),T(Tz)) \leq f(z),\ G(T(Tz),T(T(Tz)),T(T(Tz))) \leq f(Tz).$$

Thus, $f(Tz) < \max\{f(z), f(Tz)\}$. This forces $f(Tz) < f(z)$, contradicting the minimality of $f(z)$. Hence, z or Tz is a fixed point. The uniqueness argument is similar to previous theorems. □

Corollary 2.7. *Let (X,G) be compact, and let $T : X \to X$ be G-continuous satisfying:*

(i) If $Tx \neq x$, then $T(Tx) \neq x$ for all $x \in X$.

(ii) There exist nonnegative constants a,b,c,d with $a+b+c+d=1$ such that for all distinct $x,y,z \in X$:

$$G(Tx,Ty,Tz) < aG(x,Tx,Tx) + bG(y,Ty,Ty) + cG(z,Tz,Tz) + dG(x,y,z).$$

Then $Fix(T) \neq \emptyset$ and $|Fix(T)| \leq 2$.

3. Application: Stability in Triplet-Based Network Models

Consider a network model where agent interactions occur in triplets, not pairs (e.g., social dynamics, multi-robot systems). Let X (a compact set) represent all possible agent configurations. An update rule is given by $T : X \to X$. Suppose the interaction follows a perimeter-contraction pattern: for distinct configurations x,y,z, the G-metric $G(x,y,z) = \|x-y\| + \|x-z\| + \|y-z\|$ (the triangle perimeter) contracts under T: $G(Tx,Ty,Tz) < G(x,y,z)$.

If T is G-continuous and satisfies $T(Tx) \neq x$ whenever $Tx \neq x$, then by Theorem 2.1thm:main1, T has either one or two fixed points. These represent stable system configurations, with the contraction condition ensuring convergence behavior specific to this generalized geometric setting.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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