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APPLICATIONS OF TWISTED COUPLING FRAMEWORKS IN C^*-ALGEBRA VALUED G-METRIC SPACES

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Abstract. This paper introduces twisted (α,β)-$(\varphi,\wp)$-contraction type $\mathbb{T}$-coupling and proves a theorem ensuring existence and uniqueness of strong coupled coincidence and common fixed points in C^*-algebra valued G-metric spaces ($\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$). The findings generalize prior work, supported by examples, and demonstrate relevance through applications to functional equations and homotopy theory.

Keywords: (α,β)-$(\varphi,\wp)$-Contraction mappings; SCC-map; $\mathbb{T}$-coupling; altering distance functions; $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$; coupled fixed points.

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1. Introduction

Banach's fixed-point theorem has evolved through weaker contractive conditions and diverse metric spaces. Mustafa and Sims [1] introduced G-metric spaces, expanding its scope. Later, Zhenhua, Jiang, and Sun[2] developed C^*-algebra-valued metric spaces, while Shen et al. [3] combined this with G-metrics to study fixed points in complete C^*-algebra-valued G-metric

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spaces. These frameworks have led to significant results and applications, including differential equations.

The study of coupled fixed points has rapidly advanced within metric fixed point theory. The concept was first introduced by Guo *et al.* in 1987 [5]. In 2003, Kirk *et al.* [6] introduced cyclic contractions, proving that such contractions possess fixed points. Following this, Bhaskar *et al.* [7] established the coupled contraction mapping theorem.

In 2017, S. Binayak Choudhury *et al.* [8] introduced the notion of coupling between two non-empty subsets in a metric space, showing that such couplings yield strong unique fixed points under Banach-type or Chatterjea-type contractive conditions. This idea was extended by G. V. R. Babu *et al.* [9] and S. Mary Anushia *et al.* [10] to complete S-metric spaces. Further generalizations focused on relaxing classical contraction conditions using altering and ultra altering distance functions, as proposed by Khan *et al.* [11] and Ansari *et al.* [12, 13]. Choudhury *et al.* [14] posed open problems regarding couplings satisfying various inequalities. Aydi *et al.* [15] addressed this by proving strong coupled fixed point results for (ϕ,ψ)-contraction type couplings in complete partial metric spaces. Rashid *et al.* [16] and D. Eshi *et al.* [17] advanced this by introducing SCC-Map and ϕ-contraction type T-coupling, establishing coupled coincidence point theorems. Fuad Abdulkerim *et al.* [18] further contributed by proving fixed point results for (ϕ,ψ)-contraction type T-coupling mappings.

The objective of this paper is to establish unique strong common coupled fixed point (USCCFP) theorems in the context of $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$, specifically for twisted (α,β)-$(\varphi,\wp)$-contractive type $\mathbb{T}$-Coupling SCC-maps. Furthermore, we present applications to Functional equations and homotopy theory, along with a discussion on the relevance and impact of the results obtained.

2. Preliminaries

This section provides a brief introduction to some fundamental aspects of C^*-algebra theory [19, 20].

Let $\mathscr{A}$ be a unital C^*-algebra with the unit element $1_{\mathscr{A}}$. Define $\mathscr{A}_h = \{\mathfrak{e} \in \mathscr{A} : \mathfrak{e} = \mathfrak{e}^*\}$. An element $\mathfrak{e} \in \mathscr{A}$ is considered positive, denoted as $\mathfrak{e} \succeq 0_{\mathscr{A}}$, if $\mathfrak{e} = \mathfrak{e}^*$ and its spectrum $\eta(\mathfrak{e}) \subseteq [0,\infty)$. Here, $0_{\mathscr{A}}$ in $\mathscr{A}$ represents the zero element in $\mathscr{A}$, and $\eta(\mathfrak{e})$ denotes the spectrum of $\mathfrak{e}$. On $\mathscr{A}_h$, a natural partial ordering is defined by $\mathfrak{s} \preceq \mathfrak{v}$ if and only if $\mathfrak{v} - \mathfrak{s} \succeq 0_{\mathscr{A}}$. We denote $\mathscr{A}_+ = \{\mathfrak{e} \in \mathscr{A} : \mathfrak{e} \succeq 0_{\mathscr{A}}\}$ and $\mathscr{A}' = \{\mathfrak{e} \in \mathscr{A} : \mathfrak{e}\mathfrak{d} = \mathfrak{d}\mathfrak{e}\ \forall \mathfrak{d} \in \mathscr{A}\}$.

Definition 2.1:([3, 4]) Let $\mathbb{V}$ be a non-empty set and denote the associated C^*-algebra by $\mathscr{A}$. A mapping $\rho_{c^*} : \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathscr{A}$ that satisfies the required conditions is referred to as a C^*-algebra-valued G-metric.

- (i) $\rho_{c^*}(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3) = 0_{\mathscr{A}}$ if $\mathfrak{s}_1 = \mathfrak{s}_2 = \mathfrak{s}_2$,
- (ii) $0_{\mathscr{A}} \prec \rho_{c^*}(\mathfrak{s}_1, \mathfrak{s}_1, \mathfrak{s}_2)$ for all $\mathfrak{s}_1, \mathfrak{s}_2 \in \mathbb{V}$ with $\mathfrak{s}_1 \neq \mathfrak{s}_2$,
- (iii) $\rho_{c^*}(\mathfrak{s}_1, \mathfrak{s}_1, \mathfrak{s}_2) \preceq \rho_{c^*}(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3)$ for all $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3 \in \mathbb{V}$ with $\mathfrak{s}_1 \neq \mathfrak{s}_3$,
- (iv) $\rho_{c^*}(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3) = \rho_{c^*}(P[\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3])$ where P is a permutation of $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3$ (symmetry),
- (v) $\rho_{c^*}(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3) \preceq \rho_{c^*}(\mathfrak{s}_1, \mathfrak{s}_4, \mathfrak{s}_4) + \rho_{c^*}(\mathfrak{s}_4, \mathfrak{s}_2, \mathfrak{s}_3)$ for all $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3, \mathfrak{s}_4 \in \mathbb{V}$ (rectangle inequality)

Then the structure $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ is called a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$.

Example 2.2: ([3, 4]) Let $\mathbb{V} = \mathbb{R}$ and define $\rho_{c^*} : \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathscr{A}$ as
$\rho_{c^*}(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3) = ||\mathfrak{s}_1 - \mathfrak{s}_2|| I_{\mathscr{A}} + ||\mathfrak{s}_2 - \mathfrak{s}_3|| I_{\mathscr{A}} + ||\mathfrak{s}_3 - \mathfrak{s}_1|| I_{\mathscr{A}}$ for all $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3 \in \mathbb{V}$, then $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ is a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$. ρ_{c^*} is a C^*-algebra valued G-metric.

Definition 2.3:([3, 4])A $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$ $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ is said to be symmetric if
$\rho_{c^*}(\mathfrak{s}_1, \mathfrak{s}_1, \mathfrak{s}_2) = \rho_{c^*}(\mathfrak{s}_2, \mathfrak{s}_2, \mathfrak{s}_1)\ \forall\ \mathfrak{s}_1, \mathfrak{s}_2 \in \mathbb{V}$

Definition 2.4: ([3, 4]) Assume that $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ is a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$. According to $\mathscr{A}$ a sequence $\{\mathfrak{s}_k\}$ in $\mathbb{V}$ is defined as:

(1) C^*-algebra valued G-convergent to a point $\mathfrak{s} \in \mathbb{V}$ if, for each $0_{\mathscr{A}} \prec \varepsilon$, there exist $x, y \in \mathbb{N}$ such that $\rho_{c^*}(\mathfrak{s}, \mathfrak{s}_x, \mathfrak{s}_y) \prec \varepsilon$. We can also use different presentations for that as follows:

$$\mathfrak{s}_x \to \mathfrak{s} \text{ or } \lim_{x\to\infty} \rho_{c^*}(\mathfrak{s}, \mathfrak{s}_x, \mathfrak{s}_y) = 0_{\mathscr{A}} \text{ or } \lim_{x\to\infty} \mathfrak{s}_x = \mathfrak{s}.$$

(2) C^*-algebra valued G-Cauchy sequence, if for $0_{\mathscr{A}} \prec \varepsilon$, there exists positive integer $x^* \in \mathbb{N}$ such that $\rho_{c^*}(\mathfrak{s}_x, \mathfrak{s}_y, \mathfrak{s}_z) \prec \varepsilon\ \forall\ x, y, z \geq x^*$ or $\rho_{c^*}(\mathfrak{s}_x, \mathfrak{s}_y, \mathfrak{s}_z) \to 0_{\mathscr{A}}$ as $x, y, z \to \infty$ or $||\rho_{c^*}(\mathfrak{s}_x, \mathfrak{s}_y, \mathfrak{s}_z)|| \to 0$.

(3) It is referred to as being complete when a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$ $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ is present. If each Cauchy sequence in $\mathbb{V}$ converges to a point in $\mathbb{V}$.

Lemma 2.5: ([3, 4]) Let $\mathscr{A}$ be a C^*-algebra with the identity element $I_{\mathscr{A}}$ and $\mathfrak{v}$ be a positive element of $\mathscr{A}$.Then

(i) There is a unique element $\mathfrak{u} \in \mathscr{A}_+$ such that $\mathfrak{u}^2 = \mathfrak{v}$.

(ii) The set $\mathscr{A}_+ = \{\mathfrak{v}^*\mathfrak{v}/\mathfrak{v} \in \mathscr{A}\}$ with a conjugate-linear involution $* : \mathscr{A} \to \mathscr{A}$.

(iii) $\mathfrak{v}, \mathfrak{u} \in \mathscr{A}$, and $0_{\mathscr{A}} \preceq \mathfrak{v} \preceq \mathfrak{u}$ then $||\mathfrak{v}|| \leq ||\mathfrak{u}||$.

(iv) If $\mathfrak{v} \in \mathscr{A}_+$ with $||\mathfrak{v}|| < \frac{1}{2}$ then $(I - \mathfrak{v})$ is invertible and $||\mathfrak{v}(I - \mathfrak{v})^{-1}|| < 1$.

3. Main Results

Definition 3.1: Let $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ is a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$ and a pair $(\mathfrak{s}, \mathfrak{v}) \in \mathbb{V} \times \mathbb{V}$ is called

(a) a CFP of mapping $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ if $\mathbb{Q}(\mathfrak{s}, \mathfrak{v}) = \mathfrak{s}$, $\mathbb{Q}(\mathfrak{v}, \mathfrak{s}) = \mathfrak{v}$;

(a_i) a SCFP of mapping $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ if $(\mathfrak{s}, \mathfrak{v})$ is CFP and $\mathfrak{s} = \mathfrak{v}$ *i.e* $\mathbb{Q}(\mathfrak{s}, \mathfrak{s}) = \mathfrak{s}$;

(b) a CCIP of $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ and $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ if $\mathbb{Q}(\mathfrak{s}, \mathfrak{v}) = \mathbb{T}\mathfrak{s}$, $\mathbb{Q}(\mathfrak{v}, \mathfrak{s}) = \mathbb{T}\mathfrak{v}$;

(b_i) a SCCIP of $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ and $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ if $\mathfrak{s} = \mathfrak{v}$. *i.e* $\mathbb{Q}(\mathfrak{s}, \mathfrak{s}) = \mathbb{T}\mathfrak{s}$;

(c) a CCFP of $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ and $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ if $\mathbb{Q}(\mathfrak{s}, \mathfrak{v}) = \mathbb{T}\mathfrak{s} = \mathfrak{s}$, $\mathbb{Q}(\mathfrak{v}, \mathfrak{s}) = \mathbb{T}\mathfrak{v} = \mathfrak{v}$;

(c_i) a SCCFP of $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ and $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ if $\mathfrak{s} = \mathfrak{v}$. *i.e* $\mathbb{Q}(\mathfrak{s}, \mathfrak{s}) = \mathbb{T}\mathfrak{s} = \mathfrak{s}$;

(d) the pair $(\mathbb{Q}, \mathbb{T})$ is weakly compatible (ω-compt) if $\mathbb{T}(\mathbb{Q}(\mathfrak{s}, \mathfrak{v})) = \mathbb{Q}(\mathbb{T}\mathfrak{s}, \mathbb{T}\mathfrak{v})$ and $\mathbb{T}(\mathbb{Q}(\mathfrak{v}, \mathfrak{s})) = \mathbb{Q}(\mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{s})$ whenever $\mathbb{Q}(\mathfrak{s}, \mathfrak{v}) = \mathbb{T}\mathfrak{s}$, $\mathbb{Q}(\mathfrak{v}, \mathfrak{s}) = \mathbb{T}\mathfrak{v}$.

Definition 3.2: Let $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ is a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$, $\mathscr{F}$ and $\mathscr{G}$ be two nonempty subsets of $\mathbb{V}$. Then a function $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ is said to be a coupling with respect to $\mathscr{F}$ and $\mathscr{G}$ if $\mathbb{Q}(\mathfrak{s}, \mathfrak{v}) \in \mathscr{G}$ and $\mathbb{Q}(\mathfrak{v}, \mathfrak{s}) \in \mathscr{F}$ where $\mathfrak{s} \in \mathscr{F}$ and $\mathfrak{v} \in \mathscr{G}$.

Definition 3.3: Let $\mathscr{F}$ and $\mathscr{G}$ be two nonempty subsets of $\mathbb{V}$. Any function $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ is said to be

(i) a cyclic (with respect to $\mathscr{F}$ and $\mathscr{G}$) if $\mathbb{T}(\mathscr{F}) \subset \mathscr{G}$ and $\mathbb{T}(\mathscr{G}) \subset \mathscr{F}$.

(ii) a self-cyclic (with respect to $\mathscr{F}$ and $\mathscr{G}$) if $\mathbb{T}(\mathscr{F}) \subseteq \mathscr{F}$ and $\mathbb{T}(\mathscr{G}) \subseteq \mathscr{G}$.

Definition 3.4: Let $\mathscr{F}$ and $\mathscr{G}$ be two nonempty subsets of a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$ $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ and a self map $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ is said to be (self-cyclic compatible map) SCC-Map with respect to $\mathscr{F}$ and $\mathscr{G}$, if

(i) $\mathbb{T}$ is self-cyclic with respect to $\mathscr{F}$ and $\mathscr{G}$ *i.e* $\mathbb{T}(\mathscr{F}) \subseteq \mathscr{F}$ *and* $\mathbb{T}(\mathscr{G}) \subseteq \mathscr{G}$

(ii) $\mathbb{T}(\mathscr{F})$ and $\mathbb{T}(\mathscr{G})$ are closed in $\mathbb{V}$.

Definition 3.5: Let $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$, $\mathbb{T}: \mathbb{V} \to \mathbb{V}$ and $\alpha, \beta: \mathbb{V}^3 \to \mathscr{A}_+$ be a functions then $\mathbb{T}$ is called a twisted (α, β)-adm, if for all $\mathfrak{s}, \mathfrak{e} \in \mathbb{V}$

$$\begin{cases} \alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e}) \succeq 1_{\mathscr{A}} \\ \beta(\mathfrak{s},\mathfrak{s},\mathfrak{e}) \succeq 1_{\mathscr{A}} \end{cases} \Rightarrow \begin{cases} \alpha(\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{s}) \succeq 1_{\mathscr{A}} \\ \beta(\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{s}) \succeq 1_{\mathscr{A}} \end{cases}$$

Example 3.6: Let $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$ where $\mathbb{V} = \mathbb{R}^n$, $\mathscr{A} = M_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices,

$\rho_{c^*}: \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathscr{A}$ as $\rho_{c^*}(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3) = ||\mathfrak{s}_1 - \mathfrak{s}_2||^2 I_{\mathscr{A}} + ||\mathfrak{s}_2 - \mathfrak{s}_3||^2 I_{\mathscr{A}} + ||\mathfrak{s}_3 - \mathfrak{s}_1||^2 I_{\mathscr{A}}$ for all $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3 \in \mathbb{V}$ where $I_{\mathscr{A}}$ is the $n \times n$ identity matrix in $\mathscr{A}$. Let $\mathbb{T}: \mathbb{V} \to \mathbb{V}$ be defined by $\mathbb{T}(x) = Ux$ where $U \in M_n(\mathbb{C})$ is a unitary matrix (i.e., $U^*U = UU^* = I_{\mathscr{A}}$). Define functions $\alpha, \beta: \mathbb{V}^3 \to \mathscr{A}_+$ by $\alpha(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3) = A + \rho_{c^*}(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3)$, $\quad \beta(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3) = B + \rho_{c^*}(\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3)$ where $A, B \in \mathscr{A}_+$ are positive definite matrices such that $A \succeq I_{\mathscr{A}}$, $B \succeq I_{\mathscr{A}}$.

Now, verification of twisted admissibility, Suppose for some $\mathfrak{s}, \mathfrak{e} \in \mathbb{V}$, we have $\alpha(\mathfrak{s},\mathfrak{s},\mathfrak{e}) \succeq I_{\mathscr{A}}$, $\quad \beta(\mathfrak{s},\mathfrak{s},\mathfrak{e}) \succeq I_{\mathscr{A}}$. Then, since U is unitary and preserves norms,

$$\alpha(\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{s}) = A + \rho_{c^*}(U\mathfrak{e}, U\mathfrak{e}, U\mathfrak{s}) = A + \rho_{c^*}(\mathfrak{e},\mathfrak{e},\mathfrak{s}) \succeq I_{\mathscr{A}}$$

and similarly, $\beta(\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{s}) = B + \rho_{c^*}(\mathfrak{e},\mathfrak{e},\mathfrak{s}) \succeq I_{\mathscr{A}}$. Hence, $\mathbb{T}$ is a twisted (α, β)-admissible mapping.

Definition 3.7: A function $\varphi: \mathscr{A}_+ \to \mathscr{A}_+$ is called an altering distance function if the following properties are satisfied:

(a) φ is monotonically non-deceasing and continuous;

(b) $\varphi(\mathfrak{s}) = 0_{\mathscr{A}}$ if and only if $\mathfrak{s} = 0_{\mathscr{A}}$.

The family of all altering distance functions is denoted by Ω.

Definition 3.8: Let $\mathscr{F}$ and $\mathscr{G}$ be two nonempty subsets of a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$ $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ and $\mathbb{T}: \mathbb{V} \to \mathbb{V}$ is a SCC-map on $\mathbb{V}$ (with respect to $\mathscr{F}$ and $\mathscr{G}$). Then a coupling $\mathbb{Q}: \mathbb{V}^2 \to \mathbb{V}$ is said to be generalized twisted (α, β)-$(\varphi, \wp)$-contractive mapping of $\mathbb{T}$-coupling type (i) or (ii) or (iii) (with respect to $\mathscr{F}$ and $\mathscr{G}$) if there exist altering distance functions $\varphi, \wp \in \Omega$ and $\alpha, \beta: \mathbb{V}^3 \to \mathscr{A}_+$ such that for all $\mathfrak{s}, \mathfrak{z} \in \mathscr{F}$ and $\mathfrak{e}, \mathfrak{v} \in \mathscr{G}$,

(A) Generalized twisted (α, β)-$(\varphi, \wp)$-contractive mapping of $\mathbb{T}$-coupling type (i), if there exist $a \in \mathscr{A}$ with $||a|| < 1$ such that

$$\alpha(\mathbb{T}\mathfrak{s}, \mathbb{T}\mathfrak{s}, \mathbb{T}\mathfrak{e})\beta(\mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{z})\varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s},\mathfrak{v}), \mathbb{Q}(\mathfrak{s},\mathfrak{v}), \mathbb{Q}(\mathfrak{e},\mathfrak{z}))\right)$$

$$\preceq \varphi\left(a^\star \mathbb{M}(\mathfrak{s},\mathfrak{v},\mathfrak{e},\mathfrak{z})a\right) - \wp(a^\star \mathbb{M}(\mathfrak{s},\mathfrak{v},\mathfrak{e},\mathfrak{z})a)$$

(B) Generalized twisted (α,β)-$(\varphi,\wp)$-contractive mapping of $\mathbb{T}$-coupling type (ii), if there exist $a,b \in \mathscr{A}$ with $||a|| < 1$ and $0 < ||b|| \leq 1$ such that

$$\left(\alpha(\mathbb{T}\mathfrak{s}, \mathbb{T}\mathfrak{s}, \mathbb{T}\mathfrak{e})\beta(\mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{z}) + b\right)^{\varphi(\rho_{c^*}(\mathbb{Q}(\mathfrak{s},\mathfrak{v}), \mathbb{Q}(\mathfrak{s},\mathfrak{v}), \mathbb{Q}(\mathfrak{e},\mathfrak{z})))}$$

$$\preceq (1_{\mathscr{A}} + b)^{\varphi(a^\star \mathbb{M}(\mathfrak{s},\mathfrak{v},\mathfrak{e},\mathfrak{z})a) - \wp(a^\star \mathbb{M}(\mathfrak{s},\mathfrak{v},\mathfrak{e},\mathfrak{z})a)}$$

(C) Generalized twisted (α,β)-$(\varphi,\wp)$-contractive mapping of $\mathbb{T}$-coupling type (iii), if there exist $a,b \in \mathscr{A}$ with $||a|| < 1$ and $||b|| \geq 1$ such that

$$\left(\varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s},\mathfrak{v}), \mathbb{Q}(\mathfrak{s},\mathfrak{v}), \mathbb{Q}(\mathfrak{e},\mathfrak{z}))\right) + b\right)^{\alpha(\mathbb{T}\mathfrak{s}, \mathbb{T}\mathfrak{s}, \mathbb{T}\mathfrak{e})\beta(\mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{z})}$$

$$\preceq \varphi\left(a^\star \mathbb{M}(\mathfrak{s},\mathfrak{v},\mathfrak{e},\mathfrak{z})a\right) - \wp(a^\star \mathbb{M}(\mathfrak{s},\mathfrak{v},\mathfrak{e},\mathfrak{z})a) + b$$

where $\mathbb{M}(\mathfrak{s},\mathfrak{v},\mathfrak{e},\mathfrak{z}) = \max\left\{ \rho_{c^*}(\mathbb{T}\mathfrak{s}, \mathbb{T}\mathfrak{s}, \mathbb{T}\mathfrak{e}), \rho_{c^*}(\mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{z}) \right\}$.

Theorem 3.9: Let $\mathscr{F}$ and $\mathscr{G}$ be a nonempty closed subsets of a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}\mathbb{G}\mathbb{M}\mathbb{S}$ $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$, $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ is a SCC-map on $\mathbb{V}$ (with respect to $\mathscr{F}$ and $\mathscr{G}$), and a coupling $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ is said to be generalized twisted (α,β)-$(\varphi,\wp)$-contractive mapping of $\mathbb{T}$-coupling type (i) or (ii) or (iii) (with respect to $\mathscr{F}$ and $\mathscr{G}$) and assume that

(3.9.1) $\mathbb{T}(\mathscr{F}) \cap \mathbb{T}(\mathscr{G}) \neq \emptyset$ and $\mathbb{Q}(\mathscr{F} \times \mathscr{G}) \subseteq \mathbb{T}(\mathscr{G})$, $\mathbb{Q}(\mathscr{G} \times \mathscr{F}) \subseteq \mathbb{T}(\mathscr{F})$;

(3.9.2) $\mathbb{T}$ is a twisted (α,β)-admissible mapping,

(3.9.3) $\mathbb{Q}$ and $\mathbb{T}$ have a CCIP in $\mathscr{F} \times \mathscr{G}$;

(3.9.4) $\{\mathbb{Q}, \mathbb{T}\}$ is ω-compatible pairs,

(3.9.5) if $\{\mathfrak{s}_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$ and $\{\mathfrak{v}_n\}_{n=1}^{\infty} \subseteq \mathscr{G}$ with $\alpha(\mathbb{T}\mathfrak{s}_n, \mathbb{T}\mathfrak{s}_{n+1}, \mathbb{T}\mathfrak{s}_{n+1}) \succeq I_{\mathscr{A}}$ $\beta(\mathbb{T}\mathfrak{v}_n, \mathbb{T}\mathfrak{v}_{n+1}, \mathbb{T}\mathfrak{v}_{n+1}) \succeq I_{\mathscr{A}}$ for all n and $\lim_{n\to\infty} \mathbb{T}\mathfrak{s}_n = \mathbb{T}\mathfrak{s} \in \mathbb{T}(\mathscr{F})$ and $\lim_{n\to\infty} \mathbb{T}\mathfrak{v}_n = \mathbb{T}\mathfrak{v} \in \mathbb{T}(\mathscr{G})$ then $\alpha(\mathbb{T}\mathfrak{s}_n, \mathbb{T}\mathfrak{s}, \mathbb{T}\mathfrak{s}) \succeq I_{\mathscr{A}}$ $\beta(\mathbb{T}\mathfrak{v}_n, \mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{v}) \succeq I_{\mathscr{A}}$.

(3.9.6) $\alpha(\mathbb{T}\mathfrak{s}, \mathbb{T}\mathfrak{s}^\star, \mathbb{T}\mathfrak{s}^\star) \succeq I_{\mathscr{A}}$ and $\beta(\mathbb{T}\mathfrak{e}, \mathbb{T}\mathfrak{e}^\star, \mathbb{T}\mathfrak{e}^\star) \succeq I_{\mathscr{A}}$ whenever, $\mathbb{T}\mathfrak{s} \neq \mathbb{T}\mathfrak{s}^\star$ and $\mathbb{T}\mathfrak{e} \neq \mathbb{T}\mathfrak{e}^\star$.

Then $\mathbb{Q}$ and $\mathbb{T}$ have a USCCFP in $\mathscr{F}\times\mathscr{G}$.

Proof Since $\mathscr{F}$ and $\mathscr{G}$ are non-empty subsets of $\mathbb{V}$ and $\mathbb{Q}$ is generalized twisted (α,β)-$(\varphi,\wp)$-contractive mapping of $\mathbb{T}$-coupling with respect $\mathscr{F}$ and $\mathscr{G}$, then for $\mathfrak{s}_0\in\mathscr{F}$ and $\mathfrak{e}_0\in\mathscr{G}$ such that $\alpha(\mathbb{Q}(\mathfrak{s}_0,\mathfrak{e}_0),\mathbb{Q}(\mathfrak{s}_0,\mathfrak{e}_0),\mathbb{T}\mathfrak{e}_0)\succeq I_{\mathscr{A}}$ and $\beta(\mathbb{Q}(\mathfrak{e}_0,\mathfrak{s}_0),\mathbb{Q}(\mathfrak{e}_0,\mathfrak{s}_0),\mathbb{T}\mathfrak{s}_0)\succeq I_{\mathscr{A}}$, we define the sequence $\{\mathfrak{s}_\mathfrak{v}\}$ and $\{\mathfrak{e}_\mathfrak{v}\}$ in $\mathscr{F}$ and $\mathscr{G}$ respectively such that

$$\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1}=\mathbb{Q}(\mathfrak{e}_\mathfrak{v},\mathfrak{s}_\mathfrak{v})\quad \mathbb{T}\mathfrak{e}_{\mathfrak{v}+1}=\mathbb{Q}(\mathfrak{s}_\mathfrak{v},\mathfrak{e}_\mathfrak{v})\ \forall\,\mathfrak{v}\in\mathbb{N}\cup\{0\}.$$

If for some $\mathfrak{v}$, $\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1}=\mathbb{T}\mathfrak{e}_\mathfrak{v}$ and $\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1}=\mathbb{T}\mathfrak{s}_\mathfrak{v}$ then, we have $\mathbb{T}\mathfrak{s}_\mathfrak{v}=\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1}=\mathbb{Q}(\mathfrak{s}_\mathfrak{v},\mathfrak{e}_\mathfrak{v})$ and $\mathbb{T}\mathfrak{e}_\mathfrak{v}=\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1}=\mathbb{Q}(\mathfrak{e}_\mathfrak{v},\mathfrak{s}_\mathfrak{v})$. This show that $(\mathfrak{s}_\mathfrak{v},\mathfrak{e}_\mathfrak{v})$ is a coupled coincidence point of $\mathbb{Q}$ and $\mathbb{T}$. So, we are done in this case. Thus we assume that $\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1}\neq\mathbb{T}\mathfrak{e}_\mathfrak{v}$ and $\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1}\neq\mathbb{T}\mathfrak{s}_\mathfrak{v}$ for all $\mathfrak{v}\geq 0$.
Since $\mathbb{T}$ is a twisted (α,β)-adm, then $\alpha(\mathfrak{e}_0,\mathfrak{s}_1,\mathfrak{s}_1)=\alpha(\mathfrak{e}_0,\mathbb{T}\mathfrak{s}_0,\mathbb{T}\mathfrak{s}_0)\succeq 1_{\mathscr{A}}$ then
$\alpha(\mathfrak{s}_2,\mathfrak{s}_2,\mathfrak{e}_1)=\alpha(\mathbb{T}\mathfrak{s}_1,\mathbb{T}\mathfrak{s}_1,\mathbb{T}\mathfrak{e}_0)\succeq 1_{\mathscr{A}}\implies\alpha(\mathfrak{e}_2,\mathfrak{s}_3,\mathfrak{s}_3)=\alpha(\mathbb{T}\mathfrak{e}_1,\mathbb{T}\mathfrak{s}_2,\mathbb{T}\mathfrak{s}_2)\succeq 1_{\mathscr{A}}$.
By repeating similar process, we obtain
$\alpha(\mathbb{T}\mathfrak{s}_\mathfrak{v},\mathbb{T}\mathfrak{s}_\mathfrak{v},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1})\succeq 1_{\mathscr{A}}$, $\alpha(\mathbb{T}\mathfrak{e}_\mathfrak{v},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1})\succeq 1_{\mathscr{A}}\ \forall\,\mathfrak{v}\in\mathbb{N}$.
Similarly, we have $\beta(\mathbb{T}\mathfrak{e}_\mathfrak{v},\mathbb{T}\mathfrak{e}_\mathfrak{v},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1})\succeq 1_{\mathscr{A}}$, $\beta(\mathbb{T}\mathfrak{s}_\mathfrak{v},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1})\succeq 1_{\mathscr{A}}\ \forall\ \mathfrak{v}\in\mathbb{N}$. and again $\alpha(\mathfrak{e}_1,\mathfrak{e}_1,\mathfrak{s}_0)=\alpha(\mathbb{T}\mathfrak{e}_0,\mathbb{T}\mathfrak{e}_0,\mathfrak{s}_0)\succeq 1_{\mathscr{A}}$ then $\alpha(\mathfrak{s}_1,\mathfrak{e}_2,\mathfrak{e}_2)=\alpha(\mathbb{T}\mathfrak{s}_0,\mathbb{T}\mathfrak{e}_1,\mathbb{T}\mathfrak{e}_1)\succeq 1_{\mathscr{A}}\implies$ $\alpha(\mathfrak{e}_3,\mathfrak{e}_3,\mathfrak{s}_2)=\alpha(\mathbb{T}\mathfrak{e}_2,\mathbb{T}\mathfrak{e}_2,\mathbb{T}\mathfrak{s}_1)\succeq 1_{\mathscr{A}}$. By repeating similar process, we obtain
$\alpha(\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_\mathfrak{v},\mathbb{T}\mathfrak{e}_\mathfrak{v})\succeq 1_{\mathscr{A}}$, $\alpha(\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{s}_\mathfrak{v})\succeq 1_{\mathscr{A}}\ \forall\,\mathfrak{v}\in\mathbb{N}$.
Similarly, we have $\beta(\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_\mathfrak{v},\mathbb{T}\mathfrak{s}_\mathfrak{v})\succeq 1_{\mathscr{A}}$, $\beta(\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{e}_\mathfrak{v},\mathbb{T}\mathfrak{e}_\mathfrak{v})\succeq 1_{\mathscr{A}}\ \forall\,\mathfrak{v}\in\mathbb{N}$.
Also, $\alpha(\mathfrak{e}_0,\mathfrak{s}_0,\mathfrak{s}_0)\succeq 1_{\mathscr{A}}$ then
$\alpha(\mathfrak{s}_1,\mathfrak{s}_1,\mathfrak{e}_1)=\alpha(\mathbb{T}\mathfrak{s}_0,\mathbb{T}\mathfrak{s}_0,\mathbb{T}\mathfrak{e}_0)\succeq 1_{\mathscr{A}}\implies\alpha(\mathfrak{e}_2,\mathfrak{s}_2,\mathfrak{s}_2)=\alpha(\mathbb{T}\mathfrak{e}_1,\mathbb{T}\mathfrak{s}_1,\mathbb{T}\mathfrak{s}_1)\succeq 1_{\mathscr{A}}$. By repeating similar process, we obtain $\alpha(\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1})\succeq 1_{\mathscr{A}}$, $\alpha(\mathbb{T}\mathfrak{e}_\mathfrak{v},\mathbb{T}\mathfrak{s}_\mathfrak{v},\mathbb{T}\mathfrak{s}_\mathfrak{v})\succeq 1_{\mathscr{A}}\ \forall$ $\mathfrak{v}\in\mathbb{N}$.
Similarly, we have $\beta(\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1})\succeq 1_{\mathscr{A}}$, $\beta(\mathbb{T}\mathfrak{s}_\mathfrak{v},\mathbb{T}\mathfrak{e}_\mathfrak{v},\mathbb{T}\mathfrak{e}_\mathfrak{v})\succeq 1_{\mathscr{A}}\ \forall\,\mathfrak{v}\in\mathbb{N}$.

Now, we distinguish the following cases:
Case(i): Let $\mathbb{Q}$ be a generalized twisted (α,β)-$(\varphi,\wp)$-contractive mapping of $\mathbb{T}$-coupling type (i) with respect $\mathscr{F}$ and $\mathscr{G}$. Then by condition (A) in definition (3.8), and the fact that $\mathfrak{s}_\mathfrak{v}\in\mathscr{F}$ and $\mathfrak{e}_\mathfrak{v}\in\mathscr{G}$ for all $\mathfrak{v}$, we have

$$\varphi\left(\rho_{c^*}(\mathbb{T}\mathfrak{s}_\mathfrak{v},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1})\right)\ =\ \varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{s}_\mathfrak{v},\mathfrak{e}_\mathfrak{v}),\mathbb{Q}(\mathfrak{s}_\mathfrak{v},\mathfrak{e}_\mathfrak{v}))\right)$$

$$
\begin{aligned}
&= \varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}})),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})\right)\\
&\preceq \alpha(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1})\beta(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1})\\
&\qquad \varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}})),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})\right)\\
&\preceq \varphi\left(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})a\right)-\wp(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})a)
\end{aligned}
$$

where

$$
\mathbb{M}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1}) = \max\left\{ \rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1}\right),\rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1}\right) \right\}.
$$

Using the properties of $\wp$, we have

$$
\varphi\left(\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1})\right) \preceq \varphi\left(a^{\star}\max\left\{\begin{array}{c}\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1}\right),\\ \rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1}\right)\end{array}\right\}a\right).
$$

Again using the properties of φ, we get

$$
\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1}) \preceq a^{\star}\max\left\{\begin{array}{c}\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1}\right),\\ \rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1}\right)\end{array}\right\}a. \tag{1}
$$

Now, using by condition (A) in definition (3.8), and the fact that $\mathfrak{s}_{\mathfrak{v}} \in \mathscr{F}$ and $\mathfrak{e}_{\mathfrak{v}} \in \mathscr{G}$ for all $\mathfrak{v}$, we have

$$
\begin{aligned}
\varphi\left(\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1})\right) &= \varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}},\mathfrak{s}_{\mathfrak{v}}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}},\mathfrak{s}_{\mathfrak{v}}))\right)\\
&\preceq \alpha(\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}})\beta(\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}})\\
&\qquad \varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}},\mathfrak{s}_{\mathfrak{v}}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}},\mathfrak{s}_{\mathfrak{v}}))\right)\\
&\preceq \varphi\left(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}},\mathfrak{s}_{\mathfrak{v}})a\right)-\wp(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}},\mathfrak{s}_{\mathfrak{v}})a)
\end{aligned}
$$

where

$$
\mathbb{M}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}},\mathfrak{s}_{\mathfrak{v}}) = \max\left\{ \rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}}\right),\rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}}\right) \right\}.
$$

Now, using the properties of φ and $\wp$, we get

$$
\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1}) \preceq a^{\star}\max\left\{\begin{array}{c}\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}}\right),\\ \rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}}\right)\end{array}\right\}a. \tag{2}
$$

By using (1) and (2), we get

$$\max\left\{\begin{array}{c}\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1})\\ \rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1})\end{array}\right\} \preceq a^\star\max\left\{\begin{array}{c}\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}}),\\ \rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}})\end{array}\right\}a.$$

Put $\Delta_{\mathfrak{v}}=\max\left\{\begin{array}{c}\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1})\\ \rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1})\end{array}\right\}$, we obtain

$$\begin{aligned}\Delta_{\mathfrak{v}} &\preceq a^\star\Delta_{\mathfrak{v}-1}a\\ &\preceq (a^\star)^2\Delta_{\mathfrak{v}-2}(a)^2\\ &\vdots\\ &\preceq (a^\star)^{\mathfrak{v}}\Delta_0(a)^{\mathfrak{v}}.\end{aligned}$$

For each $\mathfrak{v}\in\mathbb{N}$, we may see the following by keeping in mind the property where if $a,b\in\mathscr{A}_h$ then $a\preceq b$ gives $u^*au\preceq u^*bu$.

$$\Delta_{\mathfrak{v}} \preceq (a^\star)^{\mathfrak{v}}\Delta_0(a)^{\mathfrak{v}} \tag{3}$$

Case(ii): Let $\mathbb{Q}$ be a generalized twisted (α,β)-$(\varphi,\wp)$-contractive mapping of $\mathbb{T}$-coupling type (ii) with respect $\mathscr{F}$ and $\mathscr{G}$. Then by condition (B) in definition (3.8), and the fact that $\mathfrak{s}_{\mathfrak{v}}\in\mathscr{F}$ and $\mathfrak{e}_{\mathfrak{v}}\in\mathscr{G}$ for all $\mathfrak{v}$, we have

$$\begin{aligned}&(1_{\mathscr{A}}+b)^{\varphi(\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1}))}=(1_{\mathscr{A}}+b)^{\varphi(\rho_{c^*}(\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}}),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}})))}\\ =\ &(1_{\mathscr{A}}+b)^{\varphi(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}})),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1}))}\\ \preceq\ &(\alpha(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1})\beta(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1})+b)^{\varphi(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}})),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1}))}\\ \preceq\ &(1_{\mathscr{A}}+b)^{\varphi(a^\star\mathbb{M}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})a)-\wp(a^\star\mathbb{M}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})a)}.\end{aligned}$$

Then we have

$$\varphi(\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1}))\preceq\varphi(a^\star\mathbb{M}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})a)-\wp(a^\star\mathbb{M}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})a).$$

Similarly,

$$\varphi(\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1})) \preceq \varphi(a^\star\mathbb{M}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}},\mathfrak{s}_{\mathfrak{v}})a)-\wp(a^\star\mathbb{M}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}},\mathfrak{s}_{\mathfrak{v}})a).$$

For all $\mathfrak{v} \in \mathbb{N}$, we can deduce by induction from Eq. (1) and Eq. (2), we get

$$\Delta_{\mathfrak{v}} \preceq (a^{\star})^{\mathfrak{v}} \Delta_0 (a)^{\mathfrak{v}} \tag{4}$$

Case(iii): Let $\mathbb{Q}$ be a generalized twisted (α,β)-$(\varphi,\wp)$-contractive mapping of $\mathbb{T}$-coupling type (iii) with respect $\mathscr{F}$ and $\mathscr{G}$. Then by condition (C) in definition (3.8), and the fact that $\mathfrak{s}_{\mathfrak{v}} \in \mathscr{F}$ and $\mathfrak{e}_{\mathfrak{v}} \in \mathscr{G}$ for all $\mathfrak{v}$, we have

$$\begin{aligned}
&\varphi\left(\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}+1}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}+1})\right) + b = \varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1}, \mathfrak{s}_{\mathfrak{v}-1}), \mathbb{Q}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}), \mathbb{Q}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}))\right) + b \\
= \quad & \varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}})), \mathbb{Q}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}), \mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1}, \mathfrak{s}_{\mathfrak{v}-1})\right) + b \\
\preceq \quad & \left(\varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}})), \mathbb{Q}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}), \mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1}, \mathfrak{s}_{\mathfrak{v}-1})\right) + b\right)^{\alpha(\mathbb{T}\mathfrak{s}_{\mathfrak{v}}, \mathbb{T}\mathfrak{s}_{\mathfrak{v}}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}-1})\beta(\mathbb{T}\mathfrak{e}_{\mathfrak{v}}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}}, \mathbb{T}\mathfrak{s}_{\mathfrak{v}-1})} \\
\preceq \quad & \varphi\left(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}-1}, \mathfrak{s}_{\mathfrak{v}-1})a\right) - \wp(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}-1}, \mathfrak{s}_{\mathfrak{v}-1})a) + b.
\end{aligned}$$

Then we have

$$\varphi\left(\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}+1}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}+1})\right) \preceq \varphi\left(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}-1}, \mathfrak{s}_{\mathfrak{v}-1})a\right) - \wp(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}-1}, \mathfrak{s}_{\mathfrak{v}-1})a).$$

Similarly,

$$\varphi\left(\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}}, \mathbb{T}\mathfrak{s}_{\mathfrak{v}+1}, \mathbb{T}\mathfrak{s}_{\mathfrak{v}+1})\right) \preceq \varphi\left(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}-1}, \mathfrak{e}_{\mathfrak{v}-1}, \mathfrak{e}_{\mathfrak{v}}, \mathfrak{s}_{\mathfrak{v}})a\right) - \wp(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}-1}, \mathfrak{e}_{\mathfrak{v}-1}, \mathfrak{e}_{\mathfrak{v}}, \mathfrak{s}_{\mathfrak{v}})a).$$

For all $\mathfrak{v} \in \mathbb{N}$, we can deduce by induction from Eq. (1) and Eq. (2), we get

$$\Delta_{\mathfrak{v}} \preceq (a^{\star})^{\mathfrak{v}} \Delta_0 (a)^{\mathfrak{v}}. \tag{5}$$

From Eq.(3), Eq.(4), Eq.(5) and using Lemma 2.5, we have

$$||\Delta_{\mathfrak{v}}|| \leq ||a||^{2\mathfrak{v}} ||\Delta_0|| \to 0 \text{ as } \mathfrak{v} \to \infty.$$

Thus, $\lim_{\mathfrak{v}\to\infty} \rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}+1}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}+1}) = 0_{\mathscr{A}}$ and $\lim_{\mathfrak{v}\to\infty} \rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}}, \mathbb{T}\mathfrak{s}_{\mathfrak{v}+1}, \mathbb{T}\mathfrak{s}_{\mathfrak{v}+1}) = 0_{\mathscr{A}}$

Now, we define a sequence $\{\Gamma_{\mathfrak{v}}\}$ by $\Gamma_{\mathfrak{v}} = \rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}})$ and show that $\Gamma_{\mathfrak{v}} \to 0_{\mathscr{A}}$ as $\mathfrak{v} \to \infty$. By using condition (A) in definition (3.8), and the fact that $\mathfrak{s}_{\mathfrak{v}} \in \mathscr{F}$ and $\mathfrak{e}_{\mathfrak{v}} \in \mathscr{G}$ for all $\mathfrak{v}$, we have

$$\begin{aligned}
\varphi(\Gamma_{\mathfrak{v}}) &= \varphi\left(\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}})\right)=\varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1}))\right)\\
&= \varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1})),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})\right)\\
&\preceq \alpha(\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1})\beta(\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1})\\
&\quad\ \varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1})),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})\right)\\
&\preceq \varphi\left(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})a\right)-\wp(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})a)\\
&\preceq \varphi\left(a^{\star}\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1}\right)a\right)-\wp(a^{\star}\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1}\right)a)\\
&\preceq \varphi\left(a^{\star}\rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1}\right)a\right)-\wp(a^{\star}\rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1}\right)a) \qquad (6)
\end{aligned}$$

and by condition (B) in definition (3.8), we have

$$\begin{aligned}
&(1_{\mathscr{A}}+b)^{\varphi(\Gamma_{\mathfrak{v}})}=(1_{\mathscr{A}}+b)^{\varphi(\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}}))}=(1_{\mathscr{A}}+b)^{\varphi(\rho_{c^*}(\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1})))}\\
=\ &(1_{\mathscr{A}}+b)^{\varphi(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1})),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1}))}\\
\preceq\ &(\alpha(\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1})\beta(\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1})+b)^{\varphi(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1})),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1}))}\\
(7)\preceq\ &(1_{\mathscr{A}}+b)^{\varphi(a^{\star}\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1})a)-\wp(a^{\star}\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1})a)}
\end{aligned}$$

also, by condition (C) in definition (3.8), we have

$$\begin{aligned}
&\varphi(\Gamma_{\mathfrak{v}})+b=\varphi\left(\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}})\right)+b\\
=\ &\varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1}))\right)+b\\
=\ &\varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1})),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})\right)+b\\
\preceq\ &(\varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1})),\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})\right)+b)^{\alpha(\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1})\beta(\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1})}\\
\preceq\ &\varphi\left(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})a\right)-\wp(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{e}_{\mathfrak{v}-1},\mathfrak{s}_{\mathfrak{v}-1})a)+b\\
\preceq\ &\varphi\left(a^{\star}\rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1}\right)a\right)-\wp(a^{\star}\rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}-1}\right)a)+b.
\end{aligned}$$

(8)

Thus, from (6), (7), (8) and using properties of φ, $\wp$ and $||a||<1$, we conclude that $||\Gamma_{\mathfrak{v}}||<||\Gamma_{\mathfrak{v}-1}||$ for all $\mathfrak{v}\geq 1$. Thus, $||\{\Gamma_{\mathfrak{v}}\}||$ is monotone decreasing sequence of non-negative real numbers which implies that there exists $\mathfrak{z}\geq 0$ such that $\lim_{\mathfrak{v}\to\infty}||\Gamma_{\mathfrak{v}}||=\lim_{\mathfrak{v}\to\infty}||\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}})||=\mathfrak{z}$. Taking $\mathfrak{v}\to\infty$ in any one of Eq.(6), Eq.(7), Eq.(8)

and using continuities of φ and $\wp$ and $||a||<1$, we have $\varphi(\mathfrak{z})\leq\varphi(\mathfrak{z})-\wp(\mathfrak{z})<\varphi(\mathfrak{z})$. Since $\wp$ is an altering distance function, it follows that $\wp(\mathfrak{z})=0$ which in turn implies that $\mathfrak{z}=0$. That is, $\lim_{\mathfrak{v}\to\infty}\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}})=0_{\mathscr{A}}$ Now, we have

$$\lim_{\mathfrak{v}\to\infty}\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1})\preceq\lim_{\mathfrak{v}\to\infty}\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}})+\lim_{\mathfrak{v}\to\infty}\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1})=0_{\mathscr{A}}$$

and

$$\lim_{\mathfrak{v}\to\infty}\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1})\preceq\lim_{\mathfrak{v}\to\infty}\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}})+\lim_{\mathfrak{v}\to\infty}\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1})=0_{\mathscr{A}}.$$

(9)

Now, we will prove that the sequences $\{\mathbb{T}\mathfrak{s}_{\mathfrak{v}}\}$ and $\{\mathbb{T}\mathfrak{e}_{\mathfrak{v}}\}$ are Cauchy sequences in $\mathbb{T}(\mathscr{F})$ and $\mathbb{T}(\mathscr{G})$ with regard to $\tilde{\mathscr{A}}$ respectively. If possible, let $\{\mathbb{T}\mathfrak{s}_{\mathfrak{v}}\}$ or $\{\mathbb{T}\mathfrak{e}_{\mathfrak{v}}\}$ is not a Cauchy sequence. Then there exist $\varepsilon\succ 0_{\mathscr{A}}$, and a sequence of positive integer there exists two subsequences $\{\mathfrak{u}(j)\}$ and $\{\mathfrak{v}(j)\}$ such that for all positive integers j with $\mathfrak{v}(j)>\mathfrak{u}(j)>j$, we have

$$\theta_j=\max\left\{\ \rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)}),\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)})\ \right\}\succeq\varepsilon. \tag{10}$$

Furthermore, corresponding to $\mathfrak{u}(j)$, we can choose $\mathfrak{v}(j)$ such that j is the smallest positive integer with $\mathfrak{v}(j)\geq\mathfrak{u}(j)>j$ and satisfying (10), then

$$\max\left\{\ \rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1}),\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1})\ \right\}\prec\varepsilon. \tag{11}$$

Sub-Case(i): Let $\mathbb{Q}$ be a generalized twisted (α,β)-$(\varphi,\wp)$-contractive mapping of $\mathbb{T}$-coupling type (i) with respect $\mathscr{F}$ and $\mathscr{G}$. Then by condition (A) in definition (3.8), we have

$$\begin{aligned}
&\varphi\left(\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1})\right)=\varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{v}(j)-1}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}))\right)\\
\preceq\ &\alpha(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)})\beta(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)})\\
&\quad\varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{v}(j)-1}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}))\right)\\
\preceq\ &\varphi\left(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)})a\right)-\wp\left(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)})a\right).
\end{aligned}$$

Sub-Case(ii): Let $\mathbb{Q}$ be a generalized twisted (α,β)-$(\varphi,\wp)$-contractive mapping of $\mathbb{T}$-coupling type (ii) with respect $\mathscr{F}$ and $\mathscr{G}$. Then by condition (B) in definition (3.8), we have

$$\begin{aligned}
&(1_{\mathscr{A}}+b)^{\varphi\left(\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1})\right)}=(1_{\mathscr{A}}+b)^{\varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{v}(j)-1}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}))\right)}\\
\preceq\ &\begin{pmatrix}\alpha(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)})\\ \beta(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)})\end{pmatrix}^{\varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{v}(j)-1}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}))\right)}
\end{aligned}$$

$$\preceq \varphi\left(a^\star \mathbb{M}(\mathfrak{s}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)})a\right) - \wp\left(a^\star \mathbb{M}(\mathfrak{s}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)})a\right).$$

Sub-Case(iii): Let $\mathbb{Q}$ be a generalized twisted (α,β)-$(\varphi,\wp)$-contractive mapping of $\mathbb{T}$-coupling type (iii) with respect $\mathscr{F}$ and $\mathscr{G}$. Then by condition (C) in definition (3.8), we have

$$\begin{aligned}
& \varphi\left(\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1})\right)+b\\
= & \varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{v}(j)-1}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}))\right)+b\\
\preceq & \left(\varphi\left(\rho_{c^*}(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{v}(j)-1}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}),\mathbb{Q}(\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}))\right)+b\right)^{\begin{pmatrix}\alpha(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)})\\ \beta(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)})\end{pmatrix}}\\
\preceq & \varphi\left(a^\star \mathbb{M}(\mathfrak{s}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)})a\right) - \wp\left(a^\star \mathbb{M}(\mathfrak{s}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)})a\right)+b.
\end{aligned}$$

In all sub-cases, we have

$$\mathbb{M}(\mathfrak{s}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{v}(j)-1},\mathfrak{e}_{\mathfrak{u}(j)},\mathfrak{s}_{\mathfrak{u}(j)}) = \max\left\{\begin{array}{c}\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)}\right),\\ \rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}\right)\end{array}\right\}.$$

Using properties of φ and $\wp$, we conclude that

$$\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1}) \preceq a^\star \max\left\{\begin{array}{c}\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)}\right),\\ \rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}\right)\end{array}\right\}a.$$

Similarly, we can show by the same steps that

$$\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1}) \preceq a^\star \max\left\{\begin{array}{c}\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)}\right),\\ \rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}\right)\end{array}\right\}a.$$

Therefore, we conclude that

$$\max\left\{\begin{array}{c}\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1}),\\ \rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1})\end{array}\right\} \preceq a^\star \max\left\{\begin{array}{c}\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)}\right),\\ \rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}\right)\end{array}\right\}a$$

implies that

$$\max\left\{\begin{array}{c}||\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1})||,\\ ||\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1})||\end{array}\right\} < \max\left\{\begin{array}{c}||\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)}\right)||,\\ ||\rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}\right)||\end{array}\right\}.$$

Now consider,

$$\max\left\{\begin{array}{c}||\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}\right)||\\ ||\rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1}\right)||\end{array}\right\}$$

$$
\begin{aligned}
&\leq \max\left\{\begin{array}{c}\left(\begin{array}{c}||\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1}\right)|| \\ +||\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}\right)||\end{array}\right) \\ \\ \left(\begin{array}{c}||\rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}\right)|| \\ +||\rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1}\right)||\end{array}\right)\end{array}\right\} \\
&\leq ||\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}\right)|| \\
&\quad +\max\left\{\begin{array}{c}||\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1}\right)||, \\ ||\rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}\right)||\end{array}\right\} \\
&< ||\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}\right)||+\varepsilon.
\end{aligned}
$$

Thus,

$$
\max\left\{\begin{array}{c}||\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1})||, \\ ||\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1})||\end{array}\right\} < ||\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}\right)||+\varepsilon.
$$

Now from (10) and using the triangle inequality, we have

$$
\begin{aligned}
\varepsilon \preceq \theta_j &= \max\left\{\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}), \rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)})\right\} \\
&\preceq \max\left\{\begin{array}{c}\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}) + \rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}), \\ \rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)}) + \rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)})\end{array}\right\} \\
&\preceq \rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}) + \max\left\{\begin{array}{c}\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}), \\ \rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)})\end{array}\right\} \\
&\preceq \rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}) \\
&\quad +\max\left\{\begin{array}{c}\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1}) + \rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}), \\ \rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1}) + \rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)})\end{array}\right\} \\
&\preceq \rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}) \\
&\quad +\max\left\{\begin{array}{c}\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1}), \\ \rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1})\end{array}\right\} + \max\left\{\begin{array}{c}\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1}), \\ \rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1})\end{array}\right\}
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\varepsilon \leq \theta_j &= \max\left\{||\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)})||, ||\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)})||\right\} \\
&< ||\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)})|| + ||\rho_{c^*}\left(\mathbb{T}\mathfrak{s}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}, \mathbb{T}\mathfrak{e}_{\mathfrak{v}(j)-1}\right)||+\varepsilon
\end{aligned}
$$

$$+\max\left\{\begin{array}{c} ||\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1},\mathbb{T}\mathfrak{s}_{\mathfrak{u}(j)+1})||, \\ ||\rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1},\mathbb{T}\mathfrak{e}_{\mathfrak{u}(j)+1})|| \end{array}\right\}.$$

Taking $j\to\infty$ and using (9), we have $\varepsilon<\varepsilon$, which is a contradiction. Hence $\{\mathbb{T}\mathfrak{s}_{\mathfrak{v}}\}$ and $\{\mathbb{T}\mathfrak{e}_{\mathfrak{v}}\}$ are Cauchy sequences in $\mathbb{T}(\mathscr{F})$ and $\mathbb{T}(\mathscr{G})$ with regard to $\tilde{\mathscr{A}}$ respectively. Since $\mathbb{T}(\mathscr{F})$ and $\mathbb{T}(\mathscr{G})$ are closed subset of a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$ $(\mathbb{V},\mathscr{A},\rho_{c^*})$, $\{\mathbb{T}\mathfrak{s}_{\mathfrak{v}}\}$ and $\{\mathbb{T}\mathfrak{e}_{\mathfrak{v}}\}$ are convergent in $\mathbb{T}(\mathscr{F})$ and $\mathbb{T}(\mathscr{G})$ respectively. Thus, there exist $\mathfrak{p}\in\mathbb{T}(\mathscr{F})$ and $\mathfrak{q}\in\mathbb{T}(\mathscr{G})$ such that

$$\lim_{\mathfrak{v}\to\infty}\mathbb{T}\mathfrak{s}_{\mathfrak{v}}=\mathfrak{p} \text{ and } \lim_{\mathfrak{v}\to\infty}\mathbb{T}\mathfrak{e}_{\mathfrak{v}}=\mathfrak{q}. \tag{12}$$

Since $\lim_{\mathfrak{v}\to\infty}\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}})=0_{\mathscr{A}} \implies \rho_{c^*}(\mathfrak{p},\mathfrak{q},\mathfrak{q})=0_{\mathscr{A}}$ then, we have $\mathfrak{p}=\mathfrak{q}$. As $\mathfrak{p}\in\mathbb{T}(\mathscr{F})$ and $\mathfrak{q}\in\mathbb{T}(\mathscr{G})$ it follows that $\mathfrak{p}=\mathfrak{q}\in\mathbb{T}(\mathscr{F})\cap\mathbb{T}(\mathscr{G})$ and hence, $\mathbb{T}(\mathscr{F})\cap\mathbb{T}(\mathscr{G})\neq\emptyset$. Now, since $\mathfrak{p}\in\mathbb{T}(\mathscr{F})$ and $\mathfrak{q}\in\mathbb{T}(\mathscr{G})$, there exist $\mathfrak{s}\in\mathscr{F}$ and $\mathfrak{e}\in\mathscr{G}$ such that $\mathfrak{p}=\mathbb{T}(\mathfrak{s})$ and $\mathfrak{q}=\mathbb{T}(\mathfrak{e})$. From Eq.(12), we ahve $\lim_{\mathfrak{v}\to\infty}\mathbb{T}\mathfrak{s}_{\mathfrak{v}}=\mathfrak{p}=\mathbb{T}(\mathfrak{s})$ and $\lim_{\mathfrak{v}\to\infty}\mathbb{T}\mathfrak{e}_{\mathfrak{v}}=\mathfrak{q}=\mathbb{T}(\mathfrak{e})$ and hence, $\mathbb{T}(\mathfrak{s})=\mathbb{T}(\mathfrak{e})$. From condition (3.9.5), we have $\{\mathfrak{s}_{\mathfrak{v}}\}_{\mathfrak{v}=1}^{\infty}\subseteq\mathscr{F}$ and $\{\mathfrak{e}_{\mathfrak{v}}\}_{\mathfrak{v}=1}^{\infty}\subseteq\mathscr{G}$ with $\alpha(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{s}_{\mathfrak{v}+1})\succeq I_{\mathscr{A}}$ $\beta(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1})\succeq I_{\mathscr{A}}$ for all $\mathfrak{v}$ and $\lim_{\mathfrak{v}\to\infty}\mathbb{T}\mathfrak{s}_{\mathfrak{v}}=\mathfrak{p}=\mathbb{T}\mathfrak{s}\in\mathbb{T}(\mathscr{F})$ and $\lim_{\mathfrak{v}\to\infty}\mathbb{T}\mathfrak{e}_{\mathfrak{v}}=\mathfrak{q}=\mathbb{T}\mathfrak{e}\in\mathbb{T}(\mathscr{G})$ then $\alpha(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s},\mathbb{T}\mathfrak{s})\succeq I_{\mathscr{A}}$ $\beta(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e})\succeq I_{\mathscr{A}}$. Then from condition (A) in definition (3.8), we have

$\rho_{c^*}\left(\mathfrak{p},\mathbb{Q}(\mathfrak{s},\mathfrak{e}),\mathbb{Q}(\mathfrak{s},\mathfrak{e})\right)\preceq\rho_{c^*}\left(\mathfrak{p},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1}\right)+\rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{Q}(\mathfrak{s},\mathfrak{e}),\mathbb{Q}(\mathfrak{s},\mathfrak{e})\right).$

Letting $\mathfrak{v}\to\infty$, we get

$\rho_{c^*}\left(\mathfrak{p},\mathbb{Q}(\mathfrak{s},\mathfrak{e}),\mathbb{Q}(\mathfrak{s},\mathfrak{e})\right)\preceq\lim_{\mathfrak{v}\to\infty}\rho_{c^*}\left(\mathbb{T}\mathfrak{e}_{\mathfrak{v}+1},\mathbb{Q}(\mathfrak{s},\mathfrak{e}),\mathbb{Q}(\mathfrak{s},\mathfrak{e})\right).$

It follows that $\mathbb{Q}$ is a generalized twisted (α,β)-$(\varphi,\wp)$-contractive mapping of $\mathbb{T}$-coupling type (i), then, we have

$$\begin{aligned}
&\varphi\left(\rho_{c^*}\left(\mathfrak{p},\mathbb{Q}(\mathfrak{s},\mathfrak{e}),\mathbb{Q}(\mathfrak{s},\mathfrak{e})\right)\right)\\
\preceq\;&\lim_{\mathfrak{v}\to\infty}\varphi\left(\rho_{c^*}\left(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}}),\mathbb{Q}(\mathfrak{s},\mathfrak{e}),\mathbb{Q}(\mathfrak{s},\mathfrak{e})\right)\right)\\
\preceq\;&\lim_{\mathfrak{v}\to\infty}\alpha(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s},\mathbb{T}\mathfrak{s})\beta(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e})\varphi\left(\rho_{c^*}\left(\mathbb{Q}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}}),\mathbb{Q}(\mathfrak{s},\mathfrak{e}),\mathbb{Q}(\mathfrak{s},\mathfrak{e})\right)\right)\\
\preceq\;&\lim_{\mathfrak{v}\to\infty}\varphi\left(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{s},\mathfrak{e})a\right)-\wp(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{s},\mathfrak{e})a)\\
\preceq\;&\lim_{\mathfrak{v}\to\infty}\varphi\left(a^{\star}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{s},\mathfrak{e})a\right)
\end{aligned}$$

$$\begin{aligned}
&\preceq \lim_{\mathfrak{v}\to\infty} \varphi\left(a^{\star}\max\left\{\begin{array}{c}\rho_{c^*}(\mathbb{T}\mathfrak{s}_{\mathfrak{v}},\mathbb{T}\mathfrak{s},\mathbb{T}\mathfrak{s}),\\ \rho_{c^*}(\mathbb{T}\mathfrak{e}_{\mathfrak{v}},\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e})\end{array}\right\}a\right)\\
&\preceq \varphi\left(a^{\star}\max\left\{\begin{array}{c}\rho_{c^*}(\mathbb{T}\mathfrak{s},\mathbb{T}\mathfrak{s},\mathbb{T}\mathfrak{s}),\\ \rho_{c^*}(\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e})\end{array}\right\}a\right)=0_{\mathscr{A}}.
\end{aligned}$$

Likewise, if we assume that $\mathbb{Q}$ be a generalized twisted (α,β)-$(\varphi,\wp)$-contractive mapping of $\mathbb{T}$-coupling type (ii) and (iii) with respect $\mathscr{F}$ and $\mathscr{G}$, we obtain the same result. Hence, $\rho_{c^*}(\mathfrak{p},\mathbb{Q}(\mathfrak{s},\mathfrak{e}),\mathbb{Q}(\mathfrak{s},\mathfrak{e}))=0_{\mathscr{A}}$ implies $\mathbb{Q}(\mathfrak{s},\mathfrak{e})=\mathfrak{p}$. Similarly, we can prove $\mathbb{Q}(\mathfrak{e},\mathfrak{s})=\mathfrak{q}$. Thus, $\mathbb{Q}(\mathfrak{s},\mathfrak{e})=\mathfrak{p}=\mathbb{T}\mathfrak{s}$ and $\mathbb{Q}(\mathfrak{e},\mathfrak{s})=\mathfrak{q}=\mathbb{T}\mathfrak{e}$.Therefore, $(\mathfrak{s},\mathfrak{e})\in\mathscr{F}\times\mathscr{G}$ is the coupled coincidence point, and $(\mathbb{T}(\mathfrak{s}),\mathbb{T}(\mathfrak{e}))$ is the coupled point of coincidence of $\mathbb{Q}$ and $\mathbb{T}$. Now, we will show that the coupled point of coincidence of $\mathbb{Q}$ and $\mathbb{T}$ is unique. Let $(\mathfrak{s}^{\star},\mathfrak{e}^{\star})$ be another coupled coincidence point of $\mathbb{Q}$ and $\mathbb{T}$. So, we will prove that $\mathbb{T}(\mathfrak{s})=\mathbb{T}(\mathfrak{s}^{\star})$ and $\mathbb{T}(\mathfrak{e})=\mathbb{T}(\mathfrak{e}^{\star})$. Suppose $\mathbb{T}(\mathfrak{s})\neq\mathbb{T}(\mathfrak{s}^{\star})$ or $\mathbb{T}(\mathfrak{e})\neq\mathbb{T}(\mathfrak{e}^{\star})$, from condition (3.9.6), we have $\alpha(\mathbb{T}\mathfrak{s},\mathbb{T}\mathfrak{s}^{\star},\mathbb{T}\mathfrak{s}^{\star})\succeq I_{\mathscr{A}}$ and $\beta(\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e}^{\star},\mathbb{T}\mathfrak{e}^{\star})\succeq I_{\mathscr{A}}$. Then from contraction type (i) of condition (A)

$$\begin{aligned}
&\varphi(\rho_{c^*}(\mathbb{T}(\mathfrak{s}),\mathbb{T}(\mathfrak{s}^{\star}),\mathbb{T}(\mathfrak{s}^{\star})))\\
=\;&\varphi(\rho_{c^*}(\mathbb{Q}(\mathfrak{s},\mathfrak{e}),\mathbb{Q}(\mathfrak{s}^{\star},\mathfrak{e}^{\star}),\mathbb{Q}(\mathfrak{s}^{\star},\mathfrak{e}^{\star})))\\
\preceq\;&\alpha(\mathbb{T}\mathfrak{s},\mathbb{T}\mathfrak{s}^{\star},\mathbb{T}\mathfrak{s}^{\star})\beta(\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e}^{\star},\mathbb{T}\mathfrak{e}^{\star})\varphi(\rho_{c^*}(\mathbb{Q}(\mathfrak{s},\mathfrak{e}),\mathbb{Q}(\mathfrak{s}^{\star},\mathfrak{e}^{\star}),\mathbb{Q}(\mathfrak{s}^{\star},\mathfrak{e}^{\star})))\\
\preceq\;&\varphi(a^{\star}\mathbb{M}(\mathfrak{s},\mathfrak{e},\mathfrak{s}^{\star},\mathfrak{e}^{\star})a)-\wp(a^{\star}\mathbb{M}(\mathfrak{s},\mathfrak{e},\mathfrak{s}^{\star},\mathfrak{e}^{\star})a)\\
\preceq\;&\varphi(a^{\star}\mathbb{M}(\mathfrak{s},\mathfrak{e},\mathfrak{s}^{\star},\mathfrak{e}^{\star})a)\\
\preceq\;&\varphi\left(a^{\star}\max\left\{\begin{array}{c}\rho_{c^*}(\mathbb{T}\mathfrak{s},\mathbb{T}\mathfrak{s}^{\star},\mathbb{T}\mathfrak{s}^{\star}),\\ \rho_{c^*}(\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e}^{\star},\mathbb{T}\mathfrak{e}^{\star})\end{array}\right\}a\right).
\end{aligned}$$

Likewise, if we assume that contraction type (ii) of (B) and (iii) of (C) , we obtain the same result. Hence, we conclude that

$$\begin{aligned}
\max\left\{\begin{array}{c}||\rho_{c^*}(\mathbb{T}\mathfrak{s},\mathbb{T}\mathfrak{s}^{\star},\mathbb{T}\mathfrak{s}^{\star})||,\\ ||\rho_{c^*}(\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e}^{\star},\mathbb{T}\mathfrak{e}^{\star})||\end{array}\right\} &\leq ||a||^2\max\left\{\begin{array}{c}||\rho_{c^*}(\mathbb{T}\mathfrak{s},\mathbb{T}\mathfrak{s}^{\star},\mathbb{T}\mathfrak{s}^{\star})||,\\ ||\rho_{c^*}(\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e}^{\star},\mathbb{T}\mathfrak{e}^{\star})||\end{array}\right\}\\
&< \max\left\{\begin{array}{c}||\rho_{c^*}(\mathbb{T}\mathfrak{s},\mathbb{T}\mathfrak{s}^{\star},\mathbb{T}\mathfrak{s}^{\star})||,\\ ||\rho_{c^*}(\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e}^{\star},\mathbb{T}\mathfrak{e}^{\star})||\end{array}\right\}
\end{aligned}$$

which is a contradiction, unless $\mathbb{T}(\mathfrak{s}) = \mathbb{T}(\mathfrak{s}^\star)$ and $\mathbb{T}(\mathfrak{e}) = \mathbb{T}(\mathfrak{e}^\star)$. Using contraction type (i) of (A), (ii) of (B) and (iii) of (C) we get that $\mathbb{T}(\mathfrak{s}) = \mathbb{T}(\mathfrak{e})$. Thus, $(\mathbb{T}(\mathfrak{s}), \mathbb{T}(\mathfrak{s}))$ is the unique coupled point of coincidence of the mapping $\mathbb{Q}$ and $\mathbb{T}$ with respect to $\mathscr{F}$ and $\mathscr{G}$. Now, we show that $\mathbb{Q}$ and $\mathbb{T}$ have unique coupled common fixed point. For this let $\mathbb{T}(\mathfrak{s}) = \mathfrak{z}$, then, we have $\mathfrak{z} = \mathbb{T}(\mathfrak{s}) = \mathbb{Q}(\mathfrak{s},\mathfrak{s})$, by the weakly compatibility of $\mathbb{Q}$ and $\mathbb{T}$, we have $\mathbb{T}\mathfrak{z} = \mathbb{T}(\mathbb{T}(\mathfrak{s})) = \mathbb{T}\mathbb{Q}(\mathfrak{s},\mathfrak{s}) = \mathbb{Q}(\mathbb{T}\mathfrak{s},\mathbb{T}\mathfrak{s}) = \mathbb{Q}(\mathfrak{z},\mathfrak{z})$. Thus, $(\mathbb{T}(\mathfrak{z}), \mathbb{T}(\mathfrak{z}))$ is coupled point of coincidence of $\mathbb{Q}$ and $\mathbb{T}$. By the uniqueness of coupled point of coincidence of $\mathbb{Q}$ and $\mathbb{T}$, we have $\mathbb{T}(\mathfrak{z}) = \mathbb{T}(\mathfrak{s})$. Thus, we obtain $\mathfrak{z} = \mathbb{T}(\mathfrak{z}) = \mathbb{Q}(\mathfrak{z},\mathfrak{z})$. Therefore, $(\mathfrak{z},\mathfrak{z})$ is the unique strong coupled common fixed point of $\mathbb{Q}$ and $\mathbb{T}$.

Corollary 3.10: Let $\mathscr{F}$ and $\mathscr{G}$ be a nonempty closed subsets of a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$ $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$, $\mathbb{T} : \mathbb{V} \to \mathbb{V}$ is a SCC-map on $\mathbb{V}$ (with respect to $\mathscr{F}$ and $\mathscr{G}$), and a coupling $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ is said to be a $(\varphi, \wp)$-contractive mapping of $\mathbb{T}$-coupling type (i) or (ii) or (iii) (with respect to $\mathscr{F}$ and $\mathscr{G}$) and assume that

(3.10.1) $\mathbb{T}(\mathscr{F}) \cap \mathbb{T}(\mathscr{G}) \neq \emptyset$ and $\mathbb{Q}(\mathscr{F} \times \mathscr{G}) \subseteq \mathbb{T}(\mathscr{G})$, $\mathbb{Q}(\mathscr{G} \times \mathscr{F}) \subseteq \mathbb{T}(\mathscr{F})$;

(3.10.2) $\mathbb{Q}$ and $\mathbb{T}$ have a CCIP in $\mathscr{F} \times \mathscr{G}$;

(3.10.3) $\{\mathbb{Q}, \mathbb{T}\}$ is ω-compatible pairs,

Then $\mathbb{Q}$ and $\mathbb{T}$ have a unique SCCFP in $\mathscr{F} \times \mathscr{G}$.

proof The proof follows from Theorems (3.9) by taking $\alpha(\mathbb{T}\mathfrak{s}, \mathbb{T}\mathfrak{s}, \mathbb{T}\mathfrak{e}) = 1_{\mathscr{A}}$, $\beta(\mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{z}) = 1_{\mathscr{A}}$ in contraction type (i), (ii) and (iii) of definition (3.8).

Corollary 3.11: Let $\mathscr{F}$ and $\mathscr{G}$ be a nonempty closed subsets of a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-G-$\mathbb{MS}$ $(\mathbb{V}, \mathscr{A}, \rho_{c^*})$ with $\mathscr{F} \cap \mathscr{G} \neq \emptyset$, let a coupling $\mathbb{Q} : \mathbb{V}^2 \to \mathbb{V}$ satisfying $(\varphi, \wp)$-contractive mapping of type (i) or (ii) or (iii) (with respect to $\mathscr{F}$ and $\mathscr{G}$). Then $\mathbb{Q}$ has a USCFP in $\mathscr{F} \times \mathscr{G}$.

proof Using the identity map on $\mathbb{V}$ w.r.t $\mathscr{A}$ and $\mathbb{T} = I_{\mathscr{A}}$, we can determine from Corollary (3.10) that $\mathbb{Q}$ has a USCFP.

Example 3.12: Let $\mathbb{V} = M_2(\mathbb{C})$ and $\mathscr{A} = M_2(\mathbb{C})$ with identity $I_{\mathscr{A}}$. Define the $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-$\mathbb{G}$-$\mathbb{MS}$, $\rho_{c^*} : \mathbb{V}^3 \to \mathscr{A}_+$ by $\rho_{c^*}(X,Y,Z) = \|X - Z\| I_{\mathscr{A}} + \|Y - Z\| I_{\mathscr{A}}, \quad$ for all $X,Y,Z \in \mathbb{V}$ where $\|\cdot\|$ is the operator norm. Let $\mathscr{F} = \{X \in \mathbb{V} : \|X - A_0\| \leq r\}$ and $\mathscr{G} = \{Y \in \mathbb{V} : \|Y + B_0\| \leq r\}$ be closed nonempty subsets of $\mathbb{V}$ for some fixed matrices $A_0, B_0 \in \mathbb{V}$ and radius $r > 0$, so that $0 \in \mathscr{F} \cap \mathscr{G}$

when $A_0 = B_0 = 0$. Define the mappings $\mathbb{T}$ and $\mathbb{Q}$ as $\mathbb{T}(X) = \frac{1}{2}X, \quad \mathbb{Q}(X,Y) = \frac{1}{2}(X+Y)$. Let the altering distance functions φ and $\wp$ be as $\varphi(t) = t, \quad \wp(t) = \frac{t}{2}$.

Define variable functions α and β as

$$\alpha(X,Y,Z) = \begin{cases} I_{\mathscr{A}} + \frac{1}{4}\|X-Z\| I_{\mathscr{A}} & \text{if } X,Y \in \mathscr{F}, Z \in \mathscr{G} \\ 0_{\mathscr{A}} & \text{otherwise} \end{cases},$$

$$\beta(X,Y,Z) = \begin{cases} I_{\mathscr{A}} + \frac{1}{4}\|Y-Z\| I_{\mathscr{A}} & \text{if } X,Y \in \mathscr{G}, Z \in \mathscr{F} \\ 0_{\mathscr{A}} & \text{otherwise .} \end{cases}$$

Choose:

- For contraction type (i): $a = \frac{1}{2}I_{\mathscr{A}}$, with $\|a\| < 1$.
- For contraction type (ii): $a = \frac{1}{2}I_{\mathscr{A}},\ b = \frac{1}{2}I_{\mathscr{A}}$, with $0 < \|b\| \leq 1$.
- For contraction type (iii): $a = \frac{1}{2}I_{\mathscr{A}},\ b = 2I_{\mathscr{A}}$, with $\|b\| \geq 1$.

Then all conditions of Theorem 3.9 are satisfied and hence, by Theorem 3.9, $\mathbb{Q}$ is a generalized twisted (α,β)-$(\varphi,\wp)$-contractive mapping of $\mathbb{T}$-coupling type (i), (ii) and (iii) with respect to $\mathscr{F}$ and $\mathscr{G}$, and the USCCFP is the zero matrix: $X^\star = Y^\star = 0_{\mathscr{A}}$.

Example 3.13: Let $\mathbb{V} = \mathbb{R}$ and let $\mathscr{A} = \mathbb{R}$, which is trivially a commutative unital C*-algebra. Define the G-metric: $\rho_{c^*}(x,y,z) = |x-z| + |y-z|$, for all $x,y,z \in \mathbb{R}$. Let $\mathscr{F} = [0,2]$ and $\mathscr{G} = [1,3]$ be the nonempty closed subsets of $\mathbb{V}$.Define the mappings $\mathbb{T}(x) = \frac{1}{2}x$ and $\mathbb{Q}(x,y) = \frac{1}{2}(x+y)$. Let the altering distance functions $\varphi(t) = t, \quad \wp(t) = \frac{t}{2}$. Then

- $\mathbb{T}(\mathscr{F}) = [0,1]$, $\mathbb{T}(\mathscr{G}) = [0.5,1.5] \implies \mathbb{T}(\mathscr{F}) \cap \mathbb{T}(\mathscr{G}) = [0.5,1] \neq \emptyset$.
- $\mathbb{Q}(\mathscr{F} \times \mathscr{G}) \subseteq [0.5,2.5] \subseteq \mathbb{T}(\mathscr{G})$ and $\mathbb{Q}(\mathscr{G} \times \mathscr{F}) \subseteq \mathbb{T}(\mathscr{F})$ hold for selected values.
- $\mathbb{Q}$ and $\mathbb{T}$ are continuous and commute at the fixed point $\implies$ they have a CCIP.
- $\{\mathbb{Q},\mathbb{T}\}$ is ω-compatible due to continuity and symmetry.

Now, we verify the contraction condition of type (i), (ii), and (iii) with $\alpha = \beta = 1_{\mathscr{A}} = 1$. Let $x,y \in \mathscr{F}$, $u,v \in \mathscr{G}$. Then

$$\rho_{c^*}(\mathbb{Q}(x,v),\mathbb{Q}(x,v),\mathbb{Q}(u,y)) = 2\left|\frac{x+v}{2} - \frac{u+y}{2}\right| = |x+v-u-y|.$$

Let $a = \frac{1}{2} \in \mathscr{A}$, then $\mathbb{M}(x,v,u,y) = \max\{|\mathbb{T}(x)-\mathbb{T}(u)|, |\mathbb{T}(v)-\mathbb{T}(y)|\} = \frac{1}{2}\max\{|x-u|,|v-y|\}$. So the contraction condition becomes

$$|x+v-u-y| \leq \frac{1}{2}\max\{|x-u|,|v-y|\},$$

which holds for all $x,y\in\mathscr{F}$, $u,v\in\mathscr{G}$ due to the averaging nature of $\mathbb{Q}$ and scaling of $\mathbb{T}$. Hence, all conditions of Corollary 3.10 are satisfied, and the unique SCC fixed point is: $x^\star = y^\star = 0$.

4. Applications

4.1. Application to Functional Equations.

In this section we denote by $\mathbb{V} = L^\infty(\mathscr{E})$ the space of essentially bounded measurable functions on $\mathscr{E}$ and $\mathscr{H} = L^2(\mathscr{E})$ where $\mathscr{E}$ is Lebesgue space. The set of bounded linear operators on Hilbert space $\mathscr{H}$ denoted by $L(\mathscr{H})$ is C^* algebra with operator norm:
$||\mathbb{A}|| = \sup_{a\in\mathscr{E}}||\langle\mathbb{A}a,\mathbb{A}a\rangle||$. We equip $\mathbb{V}$ with $\rho_{c^*}:\mathbb{V}^3\to L(\mathscr{H})$, which is ascertained by $\rho_{c^*}(\mathfrak{p},\mathfrak{r},\mathfrak{q}) = \mathbb{M}_{|\mathfrak{p}-\mathfrak{q}|+|\mathfrak{r}-\mathfrak{q}|}$ for all $\mathfrak{p},\mathfrak{r},\mathfrak{q}\in\mathbb{V}$, where $\mathbb{M}_\phi: L(\mathscr{H})\to L(\mathscr{H})$ be ascertained by $\mathbb{M}_\phi(\alpha) = \phi\diamond\alpha$ composit of these operators where $\alpha\in\mathscr{H}$ and $\phi\in L(\mathscr{H})$. Therefore, $(\mathbb{V},L(\mathscr{H}),\rho_{c^*})$ is a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}\mathbb{G}\mathbb{M}\mathbb{S}$.

In this setting, we discuss the problem of dynamic programming related to multistage process[21, 22]. Indeed, this problem reduces to the problem of solving the system of functional equations

$$\begin{cases} \mathfrak{s}(\mathfrak{v}) = \sup_{\mathfrak{u}\in\mathscr{D}}\{\mathfrak{f}(\mathfrak{v},\mathfrak{u}) + \mathscr{K}(\mathfrak{v},\mathfrak{u},\mathfrak{s}(\theta(\mathfrak{v},\mathfrak{u})),\mathfrak{e}(\theta(\mathfrak{v},\mathfrak{u}))\},\ \mathfrak{v}\in\mathscr{E} \\ \mathfrak{e}(\mathfrak{v}) = \sup_{\mathfrak{u}\in\mathscr{D}}\{\mathfrak{f}(\mathfrak{v},\mathfrak{u}) + \mathscr{K}(\mathfrak{v},\mathfrak{u},\mathfrak{e}(\theta(\mathfrak{v},\mathfrak{u})),\mathfrak{s}(\theta(\mathfrak{v},\mathfrak{u}))\},\ \mathfrak{v}\in\mathscr{E} \end{cases} \tag{13}$$

where $\theta:\mathscr{E}\times\mathscr{D}\to\mathscr{E}$, $\mathfrak{f}:\mathscr{E}\times\mathscr{D}\to\mathbb{R}$ and $\mathscr{K}:\mathscr{E}\times\mathscr{D}\times\mathbb{R}^2\to\mathbb{R}$.

Specifically, we will prove the following theorem.

Theorem 4.1: Let $\mathscr{K}:\mathscr{E}\times\mathscr{D}\times\mathbb{R}^2\to\mathbb{R}$ and $\mathfrak{f}:\mathscr{E}\times\mathscr{D}\to\mathbb{R}$ be two bounded functions and let $\mathbb{Q}:\mathbb{V}^2\to\mathbb{V}$ be as $\mathbb{Q}(\mathfrak{s},\mathfrak{e})(\mathfrak{v}) = \sup_{\mathfrak{u}\in\mathscr{D}}\{\mathfrak{f}(\mathfrak{v},\mathfrak{u}) + \mathscr{K}(\mathfrak{v},\mathfrak{u},\mathfrak{s}(\theta(\mathfrak{b},\mathfrak{u})),\mathfrak{e}(\theta(\mathfrak{b},\mathfrak{u}))\}$, and $\mathbb{T}:\mathbb{V}\to\mathbb{V}$ be as $\mathbb{T}(\mathfrak{s})(\mathfrak{v}) = \sup_{\mathfrak{u}\in\mathscr{D}}\{\mathfrak{f}(\mathfrak{v},\mathfrak{u}) + \mathscr{K}(\mathfrak{v},\mathfrak{u},\mathfrak{s}(\theta(\mathfrak{v},\mathfrak{u})),\mathfrak{e}(\theta(\mathfrak{v},\mathfrak{u}))\}$ for all $\mathfrak{s},\mathfrak{e}\in\mathbb{V}$ and $\mathfrak{v}\in\mathscr{E}$. Assume that there exists $\eta,\zeta:\mathbb{V}^3\to\mathscr{A}_+$ such that

$$(i)\ \begin{cases} \eta(\mathfrak{s},\mathfrak{s},\mathfrak{e}) \succeq 0_{\mathscr{A}} \\ \zeta(\mathfrak{s},\mathfrak{s},\mathfrak{e}) \succeq 0_{\mathscr{A}} \end{cases} \Rightarrow \begin{cases} \eta(\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{s}) \succeq 0_{\mathscr{A}} \\ \zeta(\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{s}) \succeq 0_{\mathscr{A}} \end{cases} \text{ where } \mathfrak{s},\mathfrak{e}\in\mathbb{V},$$

(ii) let $\mathscr{F}$ and $\mathscr{G}$ be a nonempty closed subsets of $\mathbb{V}$ such that $\mathfrak{s},\mathfrak{z} \in \mathscr{F}$, $\mathfrak{e},\mathfrak{x} \in \mathscr{G}$ and $\iota \in (0,1)$, then

$$|\mathscr{K}(\mathfrak{v},\mathfrak{u},\mathfrak{s}(\mathfrak{b}),\mathfrak{e}(\mathfrak{b})) - \mathscr{K}(\mathfrak{v},\mathfrak{u},\mathfrak{x}(\mathfrak{b}),\mathfrak{z}(\mathfrak{b}))| \leq \frac{\iota}{2}\max\left\{\begin{array}{c} ||\mathbb{T}(\mathfrak{s})(\mathfrak{b}) - \mathbb{T}(\mathfrak{x})(\mathfrak{b})|| \\ ||\mathbb{T}(\mathfrak{e})(\mathfrak{b}) - \mathbb{T}(\mathfrak{z})(\mathfrak{b})|| \end{array}\right\}$$

(iii) if $\{\mathfrak{s}_n\}_{n=1}^{\infty} \subseteq \mathscr{F}$ and $\{\mathfrak{v}_n\}_{n=1}^{\infty} \subseteq \mathscr{G}$ with $\eta(\mathbb{T}\mathfrak{s}_n, \mathbb{T}\mathfrak{s}_{n+1}, \mathbb{T}\mathfrak{s}_{n+1}) \succeq 0_{\mathscr{A}}$ $\zeta(\mathbb{T}\mathfrak{v}_n, \mathbb{T}\mathfrak{v}_{n+1}, \mathbb{T}\mathfrak{v}_{n+1}) \succeq 0_{\mathscr{A}}$ for all n and $\lim_{n\to\infty} \mathbb{T}\mathfrak{s}_n = \mathbb{T}\mathfrak{s} \in \mathbb{T}(\mathscr{F})$ and $\lim_{n\to\infty} \mathbb{T}\mathfrak{v}_n = \mathbb{T}\mathfrak{v} \in \mathbb{T}(\mathscr{G})$ then $\eta(\mathbb{T}\mathfrak{s}_n, \mathbb{T}\mathfrak{s}, \mathbb{T}\mathfrak{s}) \succeq 0_{\mathscr{A}}$ $\zeta(\mathbb{T}\mathfrak{v}_n, \mathbb{T}\mathfrak{v}, \mathbb{T}\mathfrak{v}) \succeq 0_{\mathscr{A}}$.

(iv) $\exists\, \mathfrak{s}_0 \in \mathscr{F}$ and $\mathfrak{e}_0 \in \mathscr{G}$ such that $\eta(\mathbb{T}\mathfrak{s}_0, \mathbb{T}\mathfrak{e}_0, \mathbb{T}\mathfrak{e}_0) \succeq 0_{\mathscr{A}}$ and $\zeta(\mathbb{T}\mathfrak{e}_0, \mathbb{T}\mathfrak{s}_0, \mathbb{T}\mathfrak{s}_0) \succeq 0_{\mathscr{A}}$.

Then the system of functional equations (13) have a bounded solution.

Proof Note that $(\mathbb{V}, L(\mathscr{H}), \rho_{c^*})$ is a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$. Let $\varepsilon \succ 0_{\mathscr{A}}$ be an arbitrary and $\mathfrak{s},\mathfrak{z} \in \mathscr{F}$, $\mathfrak{e},\mathfrak{x} \in \mathscr{G}$ such that $\eta(\mathfrak{s},\mathfrak{x},\mathfrak{x}) \succeq 0_{\mathscr{A}}$ and $\zeta(\mathfrak{e},\mathfrak{z},\mathfrak{z}) \succeq 0_{\mathscr{A}}$, then there exist $\mathfrak{u}_1,\mathfrak{u}_1 \in \mathscr{D}$ such that

$$\mathbb{Q}(\mathfrak{s},\mathfrak{e})(\mathfrak{v}) \prec \mathfrak{f}(\mathfrak{v},\mathfrak{u}_1) + \mathscr{K}(\mathfrak{v},\mathfrak{u}_1,\mathfrak{s}(\theta(\mathfrak{b},\mathfrak{u}_1)),\mathfrak{e}(\theta(\mathfrak{b},\mathfrak{u}_1))) + \varepsilon$$

$$\mathbb{Q}(\mathfrak{x},\mathfrak{z})(\mathfrak{v}) \prec \mathfrak{f}(\mathfrak{v},\mathfrak{u}_2) + \mathscr{K}(\mathfrak{v},\mathfrak{u}_2,\mathfrak{x}(\theta(\mathfrak{b},\mathfrak{u}_2)),\mathfrak{z}(\theta(\mathfrak{b},\mathfrak{u}_2))) + \varepsilon$$

$$\mathbb{Q}(\mathfrak{s},\mathfrak{e})(\mathfrak{v}) \succeq \mathfrak{f}(\mathfrak{v},\mathfrak{u}_2) + \mathscr{K}(\mathfrak{v},\mathfrak{u}_2,\mathfrak{s}(\theta(\mathfrak{b},\mathfrak{u}_2)),\mathfrak{e}(\theta(\mathfrak{b},\mathfrak{u}_2)))$$

$$\mathbb{Q}(\mathfrak{x},\mathfrak{z})(\mathfrak{v}) \succeq \mathfrak{f}(\mathfrak{v},\mathfrak{u}_1) + \mathscr{K}(\mathfrak{v},\mathfrak{u}_1,\mathfrak{x}(\theta(\mathfrak{b},\mathfrak{u}_1)),\mathfrak{z}(\theta(\mathfrak{b},\mathfrak{u}_1))).$$

Then, we conclude that

$$\begin{aligned}
& \mathbb{Q}(\mathfrak{s},\mathfrak{e})(\mathfrak{v}) - \mathbb{Q}(\mathfrak{x},\mathfrak{z})(\mathfrak{v}) \\
\prec\ & \mathscr{K}(\mathfrak{v},\mathfrak{u}_1,\mathfrak{s}(\theta(\mathfrak{b},\mathfrak{u}_1)),\mathfrak{e}(\theta(\mathfrak{b},\mathfrak{u}_1))) - \mathscr{K}(\mathfrak{v},\mathfrak{u}_1,\mathfrak{x}(\theta(\mathfrak{b},\mathfrak{u}_1)),\mathfrak{z}(\theta(\mathfrak{b},\mathfrak{u}_1))) + \varepsilon \\
\leq\ & |\mathscr{K}(\mathfrak{v},\mathfrak{u}_1,\mathfrak{s}(\theta(\mathfrak{b},\mathfrak{u}_1)),\mathfrak{e}(\theta(\mathfrak{b},\mathfrak{u}_1))) - \mathscr{K}(\mathfrak{v},\mathfrak{u}_1,\mathfrak{x}(\theta(\mathfrak{b},\mathfrak{u}_1)),\mathfrak{z}(\theta(\mathfrak{b},\mathfrak{u}_1)))| + \varepsilon \\
\leq\ & \frac{\iota}{2}\max\left\{\begin{array}{c} ||\mathbb{T}(\mathfrak{s})(\mathfrak{b}) - \mathbb{T}(\mathfrak{x})(\mathfrak{b})|| \\ ||\mathbb{T}(\mathfrak{e})(\mathfrak{b}) - \mathbb{T}(\mathfrak{z})(\mathfrak{b})|| \end{array}\right\} + \varepsilon
\end{aligned}$$

and similarly,

$$\mathbb{Q}(\mathfrak{x},\mathfrak{z})(\mathfrak{v}) - \mathbb{Q}(\mathfrak{s},\mathfrak{e})(\mathfrak{v}) \leq \frac{\iota}{2}\max\left\{\begin{array}{c} ||\mathbb{T}(\mathfrak{s})(\mathfrak{b}) - \mathbb{T}(\mathfrak{x})(\mathfrak{b})|| \\ ||\mathbb{T}(\mathfrak{e})(\mathfrak{b}) - \mathbb{T}(\mathfrak{z})(\mathfrak{b})|| \end{array}\right\} + \varepsilon.$$

Since $\varepsilon \succ 0_{\mathscr{A}}$ is arbitrary, then

$$|\mathbb{Q}(\mathfrak{s},\mathfrak{e})(\mathfrak{v})-\mathbb{Q}(\mathfrak{x},\mathfrak{z})(\mathfrak{v})| \leq \frac{\iota}{2}\max\left\{\begin{array}{c}||\mathbb{T}(\mathfrak{s})(\mathfrak{b})-\mathbb{T}(\mathfrak{x})(\mathfrak{b})||\\ ||\mathbb{T}(\mathfrak{e})(\mathfrak{b})-\mathbb{T}(\mathfrak{z})(\mathfrak{b})||\end{array}\right\}.$$

Now, consider

$$\rho_{c^*}(\mathbb{Q}(\mathfrak{s},\mathfrak{e}),\mathbb{Q}(\mathfrak{x},\mathfrak{z}),\mathbb{Q}(\mathfrak{x},\mathfrak{z}))=\mathbb{M}_{(2|\mathbb{Q}(\mathfrak{s},\mathfrak{e})-\mathbb{Q}(\mathfrak{x},\mathfrak{z})|)}$$

We obtain that

$$\begin{aligned}||\rho_{c^*}(\mathbb{Q}(\mathfrak{s},\mathfrak{e}),\mathbb{Q}(\mathfrak{x},\mathfrak{z}),\mathbb{Q}(\mathfrak{x},\mathfrak{z}))|| &= \sup_{||h||=1}\left\langle \mathbb{M}_{((2|\mathbb{Q}(\mathfrak{s},\mathfrak{e})(\mathfrak{v})-\mathbb{Q}(\mathfrak{x},\mathfrak{z})(\mathfrak{v})|))}h,h\right\rangle\\ &= \sup_{||h||=1}\int_{\mathscr{E}} 2|\mathbb{Q}(\mathfrak{s},\mathfrak{e})(\mathfrak{v})-\mathbb{Q}(\mathfrak{x},\mathfrak{z})(\mathfrak{v})|h(\varsigma)\overline{h(\varsigma)}d\varsigma\\ &\leq 2\sup_{||h||=1}\int_{\mathscr{E}}|h(\varsigma)|^2\frac{\iota}{2}\max\left\{\begin{array}{c}||\mathbb{T}(\mathfrak{s})(\mathfrak{b})-\mathbb{T}(\mathfrak{x})(\mathfrak{b})||_\infty\\ ||\mathbb{T}(\mathfrak{e})(\mathfrak{b})-\mathbb{T}(\mathfrak{z})(\mathfrak{b})||_\infty\end{array}\right\}\\ &\leq \frac{\iota}{2}\sup_{||h||=1}\int_{\mathscr{E}}|h(\varsigma)|^2\max\left\{\begin{array}{c}2||\mathbb{T}(\mathfrak{s})(\mathfrak{b})-\mathbb{T}(\mathfrak{x})(\mathfrak{b})||_\infty\\ 2||\mathbb{T}(\mathfrak{e})(\mathfrak{b})-\mathbb{T}(\mathfrak{z})(\mathfrak{b})||_\infty\end{array}\right\}.\end{aligned}$$

By setting $a=\iota 1_{B(L^2(\mathscr{E}))}$, then $a\in B(L^2(\mathscr{E}))$ so that $||a||=\iota<1$, then it follows that

$$||\rho_{c^*}(\mathbb{Q}(\mathfrak{s},\mathfrak{e}),\mathbb{Q}(\mathfrak{x},\mathfrak{z}),\mathbb{Q}(\mathfrak{x},\mathfrak{z}))|| \leq \tfrac{1}{2}||a||^2\max\left\{\begin{array}{c}||\rho_{c^*}(\mathbb{T}(\mathfrak{s}),\mathbb{T}(\mathfrak{x}),\mathbb{T}(\mathfrak{x}))||\\ ||\rho_{c^*}(\mathbb{T}(\mathfrak{e}),\mathbb{T}(\mathfrak{z}),\mathbb{T}(\mathfrak{z}))||\end{array}\right\}.$$

Let $\varphi,\wp:\mathscr{A}_+\to\mathscr{A}$ as $\varphi(x)=x$ and $\wp(x)=\frac{x}{2}\ \forall\, x\in\mathscr{A}_+$. For $\varsigma\in\mathscr{E}$ the following is defined:

$\alpha,\beta:\mathbb{V}\to\mathscr{A}_+$ as $\alpha(\mathfrak{x},\mathfrak{x},\mathfrak{s})=\begin{cases}1_{\mathscr{A}} & \text{if } \eta(\mathfrak{x}(\varsigma),\mathfrak{x}(\varsigma),\mathfrak{s}(\varsigma))\succeq 0_{\mathscr{A}}\\ 0_{\mathscr{A}} & \text{Otherwise}\end{cases}$

and $\beta(\mathfrak{z},\mathfrak{e},\mathfrak{e})=\begin{cases}1_{\mathscr{A}} & \text{if } \zeta(\mathfrak{z}(\varsigma),\mathfrak{e}(\varsigma),\mathfrak{e}(\varsigma))\succeq 0_{\mathscr{A}}\\ 0_{\mathscr{A}} & \text{Otherwise}\end{cases}$ implies that

$\alpha(\mathbb{T}\mathfrak{s},\mathbb{T}\mathfrak{x},\mathbb{T}\mathfrak{e})=\begin{cases}1_{\mathscr{A}} & \text{if } \eta(\mathbb{T}\mathfrak{s}(\varsigma),\mathbb{T}\mathfrak{x}(\varsigma),\mathbb{T}\mathfrak{x}(\varsigma))\succeq 0_{\mathscr{A}}\\ 0_{\mathscr{A}} & \text{Otherwise}\end{cases}$ and

$\beta(\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{e},\mathbb{T}\mathfrak{z})=\begin{cases}1_{\mathscr{A}} & \text{if } \zeta(\mathbb{T}\mathfrak{e}(\varsigma),\mathbb{T}\mathfrak{e}(\varsigma),\mathbb{T}\mathfrak{z}(\varsigma))\succeq 0_{\mathscr{A}}\\ 0_{\mathscr{A}} & \text{Otherwise}\end{cases}$. Consequently, we have

$$\alpha(\mathbb{T}\mathfrak{s},\mathbb{T}\mathfrak{s},\mathbb{T}\mathfrak{e})\beta(\mathbb{T}\mathfrak{v},\mathbb{T}\mathfrak{v},\mathbb{T}\mathfrak{z})\varphi(\rho_{c^*}(\mathbb{Q}(\mathfrak{s},\mathfrak{v}),\mathbb{Q}(\mathfrak{s},\mathfrak{v}),\mathbb{Q}(\mathfrak{e},\mathfrak{z})))$$

$$\preceq \varphi(a^\star\mathbb{M}(\mathfrak{s},\mathfrak{v},\mathfrak{e},\mathfrak{z})a)-\wp(a^\star\mathbb{M}(\mathfrak{s},\mathfrak{v},\mathfrak{e},\mathfrak{z})a)$$

that is, $\mathbb{Q}$ is Generalized twisted (α,β)-$(\varphi,\wp)$-contractive mapping of $\mathbb{T}$-coupling type (i). Thus, by Theorem (3.9), $\mathbb{Q}$ and $\mathbb{T}$ have a common fixed point, that is, the system of functional equations (13) has a bounded solution.

4.2. Application to Homotopy. In this part, we examine the possibility that homotopy theory has a unique solution.

Theorem 4.2: Let $(\mathbb{V},\mathscr{A},\rho_{c^*})$ be complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$, (Δ_1,Δ_2) and $(\overline{\Delta}_1,\overline{\Delta}_2)$ be an open and closed subset of $\mathbb{V}$ such that $(\Delta_1,\Delta_2)\subseteq(\overline{\Delta}_1,\overline{\Delta}_2)$ with $\Delta_1\cap\Delta_2\neq\emptyset$.

Suppose $\mathscr{H}:(\overline{\Delta}_1,\overline{\Delta}_2)\cup(\overline{\Delta}_2,\overline{\Delta}_1)\times[0,1]\to\mathbb{V}$ be an operator with following conditions are satisfying,

$i)$ $\mathfrak{s}\neq\mathscr{H}(\mathfrak{s},\mathfrak{e},\iota)$, $\mathfrak{e}\neq\mathscr{H}(\mathfrak{e},\mathfrak{s},\iota)$, for each $\mathfrak{s}\in\partial\Delta_1$, $\mathfrak{e}\in\partial\Delta_2$ and $\iota\in[0,1]$ (Here $\partial\Delta_1\cup\partial\Delta_2$ is boundary of $\Delta_1\cup\Delta_2$ in $\mathbb{V}$);

$ii)$ for all $\mathfrak{s},\mathfrak{z}\in\overline{\Delta}_1$, $\mathfrak{e},\mathfrak{x}\in\overline{\Delta}_2$, $\iota\in[0,1]$ and $\varphi,\wp\in\Omega$and $a\in\mathscr{A}$ with $||a||<1$ such that

$$\begin{aligned}\varphi\left(\rho_{c^*}\left(\mathscr{H}(\mathfrak{s},\mathfrak{e},\iota),\mathscr{H}(\mathfrak{x},\mathfrak{z},\iota),\mathscr{H}(\mathfrak{x},\mathfrak{z},\iota)\right)\right) \quad \preceq \quad & \varphi\left(a\max\left\{\begin{array}{c}\rho_{c^*}(\mathfrak{s},\mathfrak{x},\mathfrak{x}),\\ \rho_{c^*}(\mathfrak{e},\mathfrak{z},\mathfrak{z})\end{array}\right\}a^*\right)\\ & -\wp\left(a\max\left\{\begin{array}{c}\rho_{c^*}(\mathfrak{s},\mathfrak{x},\mathfrak{x}),\\ \rho_{c^*}(\mathfrak{e},\mathfrak{z},\mathfrak{z})\end{array}\right\}a^*\right)\end{aligned}$$

$iii)$ $\exists\,\mathbb{M}\in\mathscr{A}_+\ni\rho_{c^*}\left(\mathscr{H}(\mathfrak{s},\mathfrak{e},\iota),\mathscr{H}(\mathfrak{s},\mathfrak{e},\ell),\mathscr{H}(\mathfrak{s},\mathfrak{e},\ell)\right)\preceq||\mathbb{M}|||\iota-\ell|$ for every $\mathfrak{s}\in\overline{\Delta}_1$, $\mathfrak{e}\in\overline{\Delta}_2$, $\iota,\ell\in[0,1]$

Then $\mathscr{H}(.,0)$ has a coupled fixed point $\iff$ $\mathscr{H}(.,1)$ has a coupled fixed point.

Proof Let the set $\mathbb{B}=\left\{\ \iota\in[0,1]:\mathscr{H}(\mathfrak{s},\mathfrak{e},\iota)=\mathfrak{s},\mathscr{H}(\mathfrak{e},\mathfrak{s},\iota)=\mathfrak{e}\text{ for some }\mathfrak{s}\in\Delta_1,\mathfrak{e}\in\Delta_2\ \right\}$.

Suppose that $\mathscr{H}(.,0)$ has a coupled fixed point in $\Delta_1\times\Delta_2$, we have that $(0_{\mathscr{A}},0_{\mathscr{A}})\in\mathbb{B}\times\mathbb{B}$. So that $\mathbb{B}\neq\emptyset$. Now we show that $\mathbb{B}$ is both closed and open in $[0,1]$ and hence by the connectedness $\mathbb{B}=[0,1]$. As a result, $\mathscr{H}(.,1)$ has a coupled fixed point in $\Delta_1\times\Delta_2$. First we show that $\mathbb{B}$ closed in $[0,1]$. To see this, Let $\{\mathfrak{i}_{\mathfrak{v}}\}_{\mathfrak{v}=1}^{\infty}\subseteq\mathbb{B}$ with $\mathfrak{i}_{\mathfrak{v}}\to\mathfrak{i}\in[0,1]$ as $\mathfrak{v}\to\infty$. We must show that $\mathfrak{i}\in\mathbb{B}$. Since $\mathfrak{i}_{\mathfrak{v}}\in\mathbb{B}$ for $\mathfrak{v}=0,1,2,3,\cdots$, there exists sequences $\{\mathfrak{s}_{\mathfrak{v}}\}\subseteq\Delta_1,\{\mathfrak{e}_{\mathfrak{v}}\}\subseteq\Delta_2$ with

$\mathfrak{s}_{\mathfrak{v}} = \mathscr{H}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}, \mathfrak{i}_{\mathfrak{v}})$, $\mathfrak{e}_{\mathfrak{v}} = \mathscr{H}(\mathfrak{e}_{\mathfrak{v}}, \mathfrak{s}_{\mathfrak{v}}, \mathfrak{i}_{\mathfrak{v}})$. Consider

$$\begin{aligned}
&\rho_{c^*}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{e}_{\mathfrak{v}+1}) \\
= \;& \rho_{c^*}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}, \mathfrak{i}_{\mathfrak{v}}), \mathscr{H}(\mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{i}_{\mathfrak{v}+1}), \mathscr{H}(\mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{i}_{\mathfrak{v}+1})\right) \\
\preceq \;& \begin{array}{c} \rho_{c^*}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}, \mathfrak{i}_{\mathfrak{v}}), \mathscr{H}(\mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{i}_{\mathfrak{v}}), \mathscr{H}(\mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{i}_{\mathfrak{v}})\right) \\ +\rho_{c^*}\left(\mathscr{H}(\mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{i}_{\mathfrak{v}}), \mathscr{H}(\mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{i}_{\mathfrak{v}+1}), \mathscr{H}(\mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{i}_{\mathfrak{v}+1})\right) \end{array} \\
\preceq \;& \rho_{c^*}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}, \mathfrak{i}_{\mathfrak{v}}), \mathscr{H}(\mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{i}_{\mathfrak{v}}), \mathscr{H}(\mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{i}_{\mathfrak{v}})\right) + ||\mathbb{M}|||\mathfrak{i}_{\mathfrak{v}} - \mathfrak{i}_{\mathfrak{v}+1}|.
\end{aligned}$$

Letting $\mathfrak{v} \to \infty$, and applying φ properties, we get

$$\begin{aligned}
&\lim_{\mathfrak{v}\to\infty} \varphi\left(\rho_{c^*}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{e}_{\mathfrak{v}+1})\right) \\
\preceq \;& \lim_{\mathfrak{v}\to\infty} \varphi\left(\rho_{c^*}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}, \mathfrak{i}_{\mathfrak{v}}), \mathscr{H}(\mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{i}_{\mathfrak{v}}), \mathscr{H}(\mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{i}_{\mathfrak{v}})\right)\right) \\
\preceq \;& \lim_{\mathfrak{v}\to\infty} \left(\begin{array}{c} \varphi\left(a \max\left\{ \begin{array}{c} \rho_{c^*}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{e}_{\mathfrak{v}+1}), \\ \rho_{c^*}(\mathfrak{e}_{\mathfrak{v}}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}) \end{array} \right\} a^*\right) \\ -\wp\left(a \max\left\{ \begin{array}{c} \rho_{c^*}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{e}_{\mathfrak{v}+1}), \\ \rho_{c^*}(\mathfrak{e}_{\mathfrak{v}}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}) \end{array} \right\} a^*\right) \end{array} \right).
\end{aligned}$$

Since $\varphi, \wp$ are non-decreasing continuous functions and $||a|| < 1$, we conclude that

$$\lim_{\mathfrak{v}\to\infty} \max\left\{ \begin{array}{c} ||\rho_{c^*}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{e}_{\mathfrak{v}+1})||, \\ ||\rho_{c^*}(\mathfrak{e}_{\mathfrak{v}}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1})|| \end{array} \right\} \leq \lim_{\mathfrak{v}\to\infty} ||a||^2 \max\left\{ \begin{array}{c} ||\rho_{c^*}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{e}_{\mathfrak{v}+1})||, \\ ||\rho_{c^*}(\mathfrak{e}_{\mathfrak{v}}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1})|| \end{array} \right\}.$$

So that

$$\lim_{p\to\infty} \rho_{c^*}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{e}_{\mathfrak{v}+1}) = \tilde{0}_{\tilde{\mathscr{A}}} \text{ and } \lim_{p\to\infty} \rho_{c^*}(\mathfrak{e}_{\mathfrak{v}}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}) = \tilde{0}_{\tilde{\mathscr{A}}}.$$

By following similar steps, we can establish the result $\lim_{p\to\infty} \rho_{c^*}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}) = \tilde{0}_{\tilde{\mathscr{A}}}$ and $\lim_{p\to\infty} \rho_{c^*}(\mathfrak{e}_{\mathfrak{v}}, \mathfrak{s}_{\mathfrak{v}}, \mathfrak{s}_{\mathfrak{v}}) = \tilde{0}_{\tilde{\mathscr{A}}}$. Now for $\mathfrak{u} > \mathfrak{v}$, by use of rectangle inequality , we have

$$\begin{aligned}
&\rho_{c^*}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{s}_{\mathfrak{u}}, \mathfrak{s}_{\mathfrak{u}}) \\
\preceq \;& \rho_{c^*}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}) + \rho_{c^*}(\mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+2}, \mathfrak{s}_{\mathfrak{v}+2}) + \ldots + \rho_{c^*}(\mathfrak{s}_{\mathfrak{u}-2}, \mathfrak{s}_{\mathfrak{u}-1}, \mathfrak{s}_{\mathfrak{u}-1}) \\
&+\rho_{c^*}(\mathfrak{s}_{\mathfrak{u}-1}, \mathfrak{s}_{\mathfrak{u}}, \mathfrak{s}_{\mathfrak{u}}) \\
\preceq \;& \rho_{c^*}(\mathfrak{s}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}, \mathfrak{e}_{\mathfrak{v}}) + \rho_{c^*}(\mathfrak{e}_{\mathfrak{v}}, \mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+1}) + \rho_{c^*}(\mathfrak{s}_{\mathfrak{v}+1}, \mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{e}_{\mathfrak{v}+1}) \\
&+\rho_{c^*}(\mathfrak{e}_{\mathfrak{v}+1}, \mathfrak{s}_{\mathfrak{v}+2}, \mathfrak{s}_{\mathfrak{v}+2}) \ldots + \rho_{c^*}(\mathfrak{s}_{\mathfrak{u}-2}, \mathfrak{e}_{\mathfrak{u}-2}, \mathfrak{e}_{\mathfrak{u}-2})
\end{aligned}$$

$$+\rho_{c^*}(\mathfrak{e}_{\mathfrak{u}-2},\mathfrak{s}_{\mathfrak{u}-1},\mathfrak{s}_{\mathfrak{u}-1})+\rho_{c^*}(\mathfrak{s}_{\mathfrak{u}-1},\mathfrak{e}_{\mathfrak{u}-1},\mathfrak{e}_{\mathfrak{u}-1})+\rho_{c^*}(\mathfrak{e}_{\mathfrak{u}-1},\mathfrak{s}_{\mathfrak{u}},\mathfrak{s}_{\mathfrak{u}})\to 0_{\mathscr{A}} \text{ as } \mathfrak{v},\mathfrak{u}\to\infty.$$

Hence $\{\mathfrak{s}_{\mathfrak{v}}\}$ is a Cauchy sequence in $\mathscr{C}^*$-$\mathscr{A}\mathbb{V}$-$\mathbb{G}$-$\mathbb{MS}$ $(\mathbb{V},\mathscr{A},\rho_{c^*})$. Similarly we can show that $\{\mathfrak{e}_{\mathfrak{v}_p}\}$ is CS in $(\mathbb{V},\mathscr{A},\rho_{c^*})$ and by the completeness of $(\mathbb{V},\mathscr{A},\rho_{c^*})$, there exist $\mathfrak{p}\in\Delta_1$ and $\mathfrak{q}\in\Delta_2$ such that $\lim_{\mathfrak{v}\to\infty}\mathfrak{s}_{\mathfrak{v}}=\mathfrak{p}$ and $\lim_{\mathfrak{v}\to\infty}\mathfrak{e}_{\mathfrak{v}}=\mathfrak{q}$. Since $\lim\limits_{\mathfrak{v}\to\infty}\rho_{c^*}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}})=0_{\mathscr{A}} \implies \rho_{c^*}(\mathfrak{p},\mathfrak{q},\mathfrak{q})=0_{\mathscr{A}}$ then, we have $\mathfrak{p}=\mathfrak{q}$.

As $\mathfrak{p}\in\Delta_1$ and $\mathfrak{q}\in\Delta_2$ it follows that $\mathfrak{p}=\mathfrak{q}\in\Delta_1\cap\Delta_2$ and hence, $\Delta_1\cap\Delta_2\neq\emptyset$. Now, we have

$$\begin{aligned}\varphi(\rho_{c^*}(\mathfrak{p},\mathscr{H}(\mathfrak{p},\mathfrak{q},\mathfrak{i}),\mathscr{H}(\mathfrak{p},\mathfrak{q},\mathfrak{i}))) &= \lim_{\mathfrak{v}\to\infty}\varphi(\rho_{c^*}(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{e}_{\mathfrak{v}},\mathfrak{i}),\mathscr{H}(\mathfrak{p},\mathfrak{q},\mathfrak{i}),\mathscr{H}(\mathfrak{p},\mathfrak{q},\mathfrak{i})))\\ &\preceq \lim_{\mathfrak{v}\to\infty}\left(\begin{array}{c}\varphi\left(a\max\left\{\begin{array}{c}\rho_{c^*}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{p},\mathfrak{p}),\\ \rho_{c^*}(\mathfrak{e}_{\mathfrak{v}},\mathfrak{q},\mathfrak{q})\end{array}\right\}a^*\right)\\ -\wp\left(a\max\left\{\begin{array}{c}\rho_{c^*}(\mathfrak{s}_{\mathfrak{v}},\mathfrak{p},\mathfrak{p}),\\ \rho_{c^*}(\mathfrak{e}_{\mathfrak{v}},\mathfrak{q},\mathfrak{q})\end{array}\right\}a^*\right)\end{array}\right)=0_{\mathscr{A}}.\end{aligned}$$

It follows that $\mathscr{H}(\mathfrak{p},\mathfrak{q},\mathfrak{i})=\mathfrak{p}$. Similarly, we can prove $\mathscr{H}(\mathfrak{q},\mathfrak{p},\mathfrak{i})=\mathfrak{q}$. Thus $\mathfrak{i}\in\mathbb{B}$. Hence $\mathbb{B}$ is closed in $[0,1]$.

Let $\mathfrak{i}_0\in\mathbb{B}$, then there exist $\mathfrak{s}_0\in\Delta_1,\mathfrak{e}_0\in\Delta_2$ with $\mathfrak{s}_0=\mathscr{H}(\mathfrak{s}_0,\mathfrak{e}_0,\mathfrak{i}_0)$,
$\mathfrak{e}_0=\mathscr{H}(\mathfrak{e}_0,\mathfrak{s}_0,\mathfrak{i}_0)$. Since (Δ_1,Δ_2) is open, then there exist $r>0$ such that $B_{\rho_{c^*}}(\mathfrak{s}_0,r)\subseteq\Delta_1$ and $B_{\rho_{c^*}}(\mathfrak{e}_0,r)\subseteq\Delta_2$. Choose $\mathfrak{i}\in(\mathfrak{i}_0-\varepsilon,\mathfrak{i}_0+\varepsilon)$ such that $|\mathfrak{i}-\mathfrak{i}_0|\leq\frac{1}{||\mathbb{M}^{\mathfrak{v}}||}<\frac{\varepsilon}{2}$, then for
$\mathfrak{s}\in\overline{B_{\rho_{c^*}}(\mathfrak{s}_0,r)}=\{\mathfrak{s}\in\Delta_1/\rho_{c^*}(\mathfrak{s},\mathfrak{s}_0,\mathfrak{s}_0)\preceq r+\rho_{c^*}(\mathfrak{s}_0,\mathfrak{s}_0,\mathfrak{s}_0)\}$
and $\mathfrak{e}\in\overline{B_{\rho_{c^*}}(\mathfrak{e}_0,r)}=\{\mathfrak{e}\in\Delta_2/\rho_{c^*}(\mathfrak{e},\mathfrak{e}_0,\mathfrak{e}_0)\preceq r+\rho_{c^*}(\mathfrak{e}_0,\mathfrak{e}_0,\mathfrak{e}_0)\}$. Now we have

$$\begin{aligned}\rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{e},\mathfrak{i}),\mathfrak{s}_0,\mathfrak{s}_0) &= \rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{e},\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{e}_0,\mathfrak{i}_0),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{e}_0,\mathfrak{i}_0))\\ &\preceq \rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{e},\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{e}_0,\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{e}_0,\mathfrak{i}))\\ &\quad+\rho_{c^*}(\mathscr{H}(\mathfrak{s}_0,\mathfrak{e}_0,\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{e}_0,\mathfrak{i}_0),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{e}_0,\mathfrak{i}_0))\\ &\preceq \rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{e},\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{e}_0,\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{e}_0,\mathfrak{i}))+\frac{1}{||\mathbb{M}^{\mathfrak{v}-1}||}.\end{aligned}$$

Letting $\mathfrak{v}\to\infty$ and applying φ properties, we obtain

$$\begin{aligned}&\varphi(\rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{e},\mathfrak{i}),\mathfrak{s}_0,\mathfrak{s}_0))\\ \preceq\ &\varphi(\rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{e},\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{e}_0,\mathfrak{i}),\mathscr{H}_b(\mathfrak{s}_0,\mathfrak{e}_0,\mathfrak{i})))\end{aligned}$$

$$\preceq \varphi\left(a\max\left\{\begin{array}{c}\rho_{c^*}(\mathfrak{s},\mathfrak{s}_0,\mathfrak{s}_0),\\ \rho_{c^*}(\mathfrak{e},\mathfrak{e}_0,\mathfrak{e}_0)\end{array}\right\}a^*\right)-\wp\left(a\max\left\{\begin{array}{c}\rho_{c^*}(\mathfrak{s},\mathfrak{s}_0,\mathfrak{s}_0),\\ \rho_{c^*}(\mathfrak{e},\mathfrak{e}_0,\mathfrak{e}_0)\end{array}\right\}a^*\right).$$

Since $\varphi,\wp$ are continuous and non-decreasing, we obtain

$$\max\left\{\begin{array}{c}\rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{e},\mathfrak{i}),\mathfrak{s}_0,\mathfrak{s}_0),\\ \rho_{c^*}(\mathscr{H}(\mathfrak{e},\mathfrak{s},\mathfrak{i}),\mathfrak{e}_0,\mathfrak{e}_0)\end{array}\right\}\preceq a\max\left\{\begin{array}{c}\rho_{c^*}(\mathfrak{s},\mathfrak{s}_0,\mathfrak{s}_0),\\ \rho_{c^*}(\mathfrak{e},\mathfrak{e}_0,\mathfrak{e}_0)\end{array}\right\}a^*$$

which implies that

$$\begin{aligned}\max\left\{\begin{array}{c}||\rho_{c^*}(\mathscr{H}(\mathfrak{s},\mathfrak{e},\mathfrak{i}),\mathfrak{s}_0,\mathfrak{s}_0)||,\\ ||\rho_{c^*}(\mathscr{H}(\mathfrak{e},\mathfrak{s},\mathfrak{i}),\mathfrak{e}_0,\mathfrak{e}_0)||\end{array}\right\} &\leq ||a||^2\max\left\{\begin{array}{c}||\rho_{c^*}(\mathfrak{s},\mathfrak{s}_0,\mathfrak{s}_0)||,\\ ||\rho_{c^*}(\mathfrak{e},\mathfrak{e}_0,\mathfrak{e}_0)||\end{array}\right\}\\ &< \max\left\{\begin{array}{c}r+||\rho_{c^*}(\mathfrak{s}_0,\mathfrak{s}_0,\mathfrak{s}_0)||,\\ r+||\rho_{c^*}(\mathfrak{e}_0,\mathfrak{e}_0,\mathfrak{e}_0)||\end{array}\right\}.\end{aligned}$$

Thus for each fixed $\mathfrak{i}\in(\mathfrak{i}_0-\varepsilon,\mathfrak{i}_0+\varepsilon)$, $\mathscr{H}(.,\mathfrak{i}):\overline{B_{\rho_{c^*}}(\mathfrak{s}_0,r)}\to\overline{B_{\rho_{c^*}}(\mathfrak{s}_0,r)}$, $\mathscr{H}(.,\mathfrak{i}):\overline{B_{\rho_{c^*}}(\mathfrak{e}_0,r)}\to\overline{B_{\rho_{c^*}}(\mathfrak{e}_0,r)}$. Since also (ii) holds and $\varphi,\wp$ is continuous and non-decreasing, then all conditions of Theorem 4.2 are satisfied. Thus we conclude that $\mathscr{H}(.,\mathfrak{i})$ has a coupled fixed point in $\overline{\Delta}_1\times\overline{\Delta}_2$. But this must be in $\Delta_1\times\Delta_2$ since (i) holds. Thus, $\mathfrak{i}\in\mathbb{B}$ for any $\mathfrak{i}\in(\mathfrak{i}_0-\varepsilon,\mathfrak{i}_0+\varepsilon)$. Hence $(\mathfrak{i}_0-\varepsilon,\mathfrak{i}_0+\varepsilon)\subseteq\mathbb{B}$. Clearly $\mathbb{B}$ is open in [0, 1]. For the reverse implication, we use the same strategy.

CONCLUSION

This work concludes with the successful establishment of SUCCFP theorems for twisted (α,β)-$(\varphi,\wp)$ -contractive $\mathbb{T}$-Coupling SCC-maps in $\mathscr{C}^*$-$\mathscr{A}\mathbb{VGMS}$. The results enrich fixed point theory and demonstrate practical relevance through applications to functional equations and homotopy theory, paving the way for future mathematical exploration.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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