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GEOMETRIC RADII FOR SALAGEAN-KOMATU TYPE FUNCTIONS

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Abstract. In this paper, we investigate the geometric properties of the subclass $TS_k^{a,\delta}(\hbar,l)$ of analytic functions with negative coefficients, defined via a composite differential operator combining the Salagean and Komatu operators. Using the coefficient characterization of Murthy and Reddy together with Silverman's exact coefficient criteria for negative coefficients, we establish sharp radii of starlikeness and convexity of order α $(0 \leq \alpha < 1)$ for functions belonging to this class. The classical radii are obtained as immediate corollaries. We also record a sharp radius of close-to-convexity of order α as an application and show that, whenever the corresponding radius is strictly less than 1, the extremal problem is attained by a one-term function. The methodology presented here yields a rigorous radius theory for this recently introduced operator-defined class.

Keywords: analytic functions; starlike functions; convex functions; differential operator; radius of starlikeness; radius of convexity.

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1. INTRODUCTION AND PRELIMINARIES

1.1. Basic Definitions and Notation. Let $\mathscr{A}$ denote the class of functions $u(z)$ that are analytic in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$ and have the normalized Taylor series expansion

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

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The normalization $u(0) = 0$ and $u'(0) = 1$ ensures uniqueness of representation. Let $\mathscr{S}$ denote the subclass of $\mathscr{A}$ consisting of functions that are univalent in E. The study of geometric properties of univalent functions has been a central theme in complex analysis since the early work of Bieberbach [7] and the resolution of his famous conjecture by de Branges [8].

Definition 1.1 (Starlike and Convex Functions)**.** *A function $u \in \mathscr{A}$ is said to be* starlike of order α $(0 \le \alpha < 1)$ *if*

$$\operatorname{Re}\left\{\frac{zu'(z)}{u(z)}\right\} > \alpha \quad \text{for all } z \in E. \tag{2}$$

This class is denoted by $S^*(\alpha)$. Similarly, $u \in \mathscr{A}$ is convex of order α *if*

$$\operatorname{Re}\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > \alpha \quad \text{for all } z \in E, \tag{3}$$

and this class is denoted by $K(\alpha)$.

Geometrically, $u \in S^*(\alpha)$ means that $u(E)$ is a starlike domain with respect to the origin. Similarly, $u \in K(\alpha)$ implies that $u(E)$ is a convex domain. The classical Alexander theorem [1, 9] establishes the relationship

$$u \in K(\alpha) \iff zu'(z) \in S^*(\alpha). \tag{4}$$

1.2. Functions with Negative Coefficients. Following Silverman [23], we denote by T the family of functions in $\mathscr{A}$ of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \ge 0). \tag{5}$$

We write $T^*(\alpha) = T \cap S^*(\alpha)$ and $C(\alpha) = T \cap K(\alpha)$. The importance of the family T lies in the fact that, for many geometric subclasses, the usual coefficient conditions become both necessary and sufficient. This phenomenon allows one to identify extremal functions explicitly and to obtain sharp radius results.

1.3. Differential and Integral Operators. Differential and integral operators have become indispensable tools in geometric function theory. We recall two fundamental operators.

Definition 1.2 (Salagean Operator [22])**.** *For $u \in \mathscr{A}$, the Salagean differential operator D^k : $\mathscr{A} \to \mathscr{A}$ is defined recursively by*

$$D^0 u(z) = u(z), \tag{6}$$

$$D^1 u(z) = zu'(z), \tag{7}$$

$$D^k u(z) = z(D^{k-1}u(z))' \quad (k \in \mathbb{N}). \tag{8}$$

For $u(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we have

$$D^k u(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n \quad (k \in \mathbb{N}_0). \tag{9}$$

The Salagean operator has been extensively studied in connection with various subclasses of univalent functions [22, 19, 15].

Definition 1.3 (Komatu Integral Operator [13])**.** *For $u \in \mathscr{A}$, parameters $a > 0$ and $\delta > 0$, the Komatu integral operator $L_a^\delta : \mathscr{A} \to \mathscr{A}$ is defined by*

$$L_a^\delta u(z) = \frac{a^\delta}{\Gamma(\delta)} \int_0^1 t^{a-2} \left(\log \frac{1}{t}\right)^{\delta-1} u(zt)\, dt. \tag{10}$$

By convention, $L_a^0 u(z) = u(z)$. If $u(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then

$$L_a^\delta u(z) = z + \sum_{n=2}^{\infty} \left(\frac{a}{a+n-1}\right)^\delta a_n z^n \qquad (\delta \geq 0). \tag{11}$$

The Komatu operator generalizes several classical operators: when $a = 2$ and $\delta = 1$, it reduces to the Libera operator [14]; when $a = c+1$ and $\delta = 1$, it becomes the Bernardi operator [6]. Further generalizations and applications can be found in [12, 25, 10].

1.4. The Composite Operator and Function Class.

Following Arkan et al. [4] and the recent work of Murthy and Reddy [17], we consider the composite linear operator $D^k L_a^\delta = D^k \circ L_a^\delta$.

Definition 1.4 (Composite Operator)**.** *For $u \in \mathscr{A}$, the operator $D^k L_a^\delta : \mathscr{A} \to \mathscr{A}$ is defined by*

$$D^k L_a^\delta u(z) = z + \sum_{n=2}^{\infty} \mathfrak{F}_k^{a,\delta}(n) a_n z^n, \tag{12}$$

where the multiplier sequence $\mathfrak{F}_k^{a,\delta}(n)$ *is given by*

$$\mathfrak{F}_k^{a,\delta}(n) = n^k \left(\frac{a}{a+n-1}\right)^{\delta}, \quad (a>0,\ \delta \geq 0,\ k \in \mathbb{N}_0). \tag{13}$$

Remark 1.5. *The operator* $D^k L_a^{\delta}$ *interpolates between classical operators:*

- *when* $\delta = 0$, $D^k L_a^{\delta}$ *reduces to the Salagean operator* D^k*;*
- *when* $k = 0$, $D^k L_a^{\delta}$ *reduces to the Komatu operator* L_a^{δ}*;*
- *when* $k = 0$ *and* $\delta = 0$, $D^k L_a^{\delta}$ *is the identity operator.*

Using this operator, Murthy and Reddy [17] introduced the following function class.

Definition 1.6 (The Class $S_k^{a,\delta}(\hbar, l)$)**.** *For parameters* $\hbar \geq 0$ *and* $0 \leq l < 1$*, a function* $u \in \mathscr{A}$ *belongs to the class* $S_k^{a,\delta}(\hbar, l)$ *if it satisfies*

$$\operatorname{Re}\left(\frac{D^k L_a^{\delta} u(z)}{z}\right) \geq \hbar \left| (D^k L_a^{\delta} u(z))' - \frac{D^k L_a^{\delta} u(z)}{z} \right| + l. \tag{14}$$

We denote by $TS_k^{a,\delta}(\hbar, l) = S_k^{a,\delta}(\hbar, l) \cap T$ *the subclass of functions with negative coefficients.*

The following coefficient theorem is the basic input for the radius problems considered here.

Lemma 1.7 (Coefficient Bound [17, Theorem 2.2])**.** *A function* $u(z) = z - \sum_{n=2}^{\infty} a_n z^n$ *with* $a_n \geq 0$ *belongs to the class* $TS_k^{a,\delta}(\hbar, l)$ *if and only if*

$$\sum_{n=2}^{\infty} [1 + \hbar(n-1)] \mathfrak{F}_k^{a,\delta}(n) a_n \leq 1 - l. \tag{15}$$

1.5. Radii Problems in Geometric Function Theory. The determination of radii of starlikeness and convexity is a classical problem in geometric function theory. Given a class $\mathscr{F}$ of analytic functions, one seeks the largest radius r such that every function in $\mathscr{F}$ is starlike (or convex) in the disk $|z| < r$. Pioneering work in this direction was carried out by Robertson [21], MacGregor [16], and Goodman [11]. For functions with negative coefficients, Silverman [23] established exact coefficient characterizations for starlikeness and convexity of order α, and these criteria are especially well suited to radius questions.

For a different negative-coefficient class involving the same Salagean-Komatu operator, radii of starlikeness, convexity, and close-to-convexity were obtained by Viswanathan et al. [26].

However, for the recently introduced class $TS_k^{a,\delta}(\hbar,l)$, only coefficient estimates and related structural properties were obtained in [17]. The purpose of the present paper is to develop a rigorous radius theory for this class. More precisely, we establish sharp radii of starlikeness and convexity of order α $(0 \leq \alpha < 1)$ for functions in $TS_k^{a,\delta}(\hbar,l)$. The classical order-zero radii then follow immediately, and a close-to-convexity radius of order α is obtained as an application.

2. Radius of Starlikeness

We begin with Silverman's exact coefficient criterion for starlikeness in the negative-coefficient setting (see [23, Theorem 2]).

Lemma 2.1 (Silverman)**.** *Let* $0 \leq \alpha < 1$ *and let*

$$f(z) = z - \sum_{n=2}^{\infty} b_n z^n, \qquad b_n \geq 0.$$

Then $f \in T^*(\alpha)$ *if and only if*

$$\sum_{n=2}^{\infty} (n-\alpha) b_n \leq 1-\alpha. \tag{16}$$

Definition 2.2 (Radius of Starlikeness of Order α)**.** *For* $0 \leq \alpha < 1$*, the radius of starlikeness of order* α *for the class* $TS_k^{a,\delta}(\hbar,l)$*, denoted by* $R^*(\alpha)$*, is the largest number* $r \in (0,1]$ *such that every function in* $TS_k^{a,\delta}(\hbar,l)$ *is starlike of order* α *in the disk* $|z| < r$*.*

Theorem 2.3. *Let* $0 \leq \alpha < 1$ *and let* $u \in TS_k^{a,\delta}(\hbar,l)$*. Then* u *is starlike of order* α *in the disk* $|z| < R^*(\alpha)$*, where*

$$R^*(\alpha) = \inf_{n\geq 2} \left(\frac{(1-\alpha)[1+\hbar(n-1)]\mathfrak{F}_k^{a,\delta}(n)}{(n-\alpha)(1-l)} \right)^{\frac{1}{n-1}}. \tag{17}$$

Moreover,

$$\lim_{n\to\infty} \left(\frac{(1-\alpha)[1+\hbar(n-1)]\mathfrak{F}_k^{a,\delta}(n)}{(n-\alpha)(1-l)} \right)^{\frac{1}{n-1}} = 1. \tag{18}$$

Hence $0 < R^*(\alpha) \leq 1$*. If* $R^*(\alpha) < 1$*, then the infimum in* (17) *is attained at some index* $n_0 \geq 2$*, and the result is sharp for the function*

$$u_{n_0}(z) = z - \frac{1-l}{[1+\hbar(n_0-1)]\mathfrak{F}_k^{a,\delta}(n_0)} z^{n_0}. \tag{19}$$

If $R^*(\alpha) = 1$, then every function in $TS_k^{a,\delta}(\hbar,l)$ is starlike of order α in the whole unit disk E.

Proof. Write

$$W_n = [1+\hbar(n-1)]\mathfrak{F}_k^{a,\delta}(n) = [1+\hbar(n-1)]n^k\left(\frac{a}{a+n-1}\right)^{\delta} \qquad (n \geq 2).$$

Let $0 < r < 1$ and define the dilated function

$$u_r(z) = \frac{u(rz)}{r} = z - \sum_{n=2}^{\infty} a_n r^{n-1} z^n.$$

Since multiplication by a positive constant does not change the quantity $zu'(z)/u(z)$, the function u is starlike of order α in $|z| < r$ if and only if u_r belongs to $T^*(\alpha)$.

Now assume that

$$r \leq \left(\frac{(1-\alpha)W_n}{(n-\alpha)(1-l)}\right)^{\frac{1}{n-1}} \qquad (n \geq 2). \tag{20}$$

Then

$$(n-\alpha)r^{n-1} \leq \frac{(1-\alpha)W_n}{1-l} \qquad (n \geq 2).$$

Multiplying by a_n and summing, and then using Lemma 1.7, we obtain

$$\sum_{n=2}^{\infty}(n-\alpha)a_n r^{n-1} \leq \frac{1-\alpha}{1-l}\sum_{n=2}^{\infty} W_n a_n \leq 1-\alpha.$$

Hence, by Lemma 2.1, $u_r \in T^*(\alpha)$. Therefore u is starlike of order α in $|z| < r$. Since (20) holds for every $r < R^*(\alpha)$, formula (17) follows.

To prove (18), set

$$Q_n = \frac{(1-\alpha)[1+\hbar(n-1)]\mathfrak{F}_k^{a,\delta}(n)}{(n-\alpha)(1-l)}.$$

Then $\log Q_n = O(\log n)$ as $n \to \infty$, because Q_n is a positive constant times a rational combination of powers of n and $(a+n-1)$. Consequently,

$$\lim_{n\to\infty}\frac{\log Q_n}{n-1} = 0,$$

which yields (18). In particular, $R^*(\alpha) \leq 1$.

Suppose now that $R^*(\alpha) < 1$. By (18), the sequence under the infimum tends to 1, so the infimum is attained by some finite index $n_0 \geq 2$. Consider the one-term function (19). By construction,

$$\sum_{n=2}^{\infty} W_n a_n = W_{n_0} \cdot \frac{1-l}{W_{n_0}} = 1-l,$$

so $u_{n_0} \in TS_k^{a,\delta}(\hbar, l)$ by Lemma 1.7. For the dilated function

$$(u_{n_0})_r(z) = z - \frac{1-l}{W_{n_0}} r^{n_0-1} z^{n_0},$$

Lemma 2.1 shows that $(u_{n_0})_r \in T^*(\alpha)$ if and only if

$$(n_0 - \alpha)\frac{1-l}{W_{n_0}} r^{n_0-1} \leq 1-\alpha,$$

that is,

$$r \leq \left(\frac{(1-\alpha)W_{n_0}}{(n_0-\alpha)(1-l)} \right)^{\frac{1}{n_0-1}} = R^*(\alpha).$$

Hence the radius in (17) is sharp.

Finally, if $R^*(\alpha) = 1$, then for every $r < 1$ we have $r < R^*(\alpha)$, and the first part of the proof shows that each $u \in TS_k^{a,\delta}(\hbar, l)$ is starlike of order α in $|z| < r$. Since $r < 1$ is arbitrary, every $u \in TS_k^{a,\delta}(\hbar, l)$ is starlike of order α in E. □

Corollary 2.4. *Let $u \in TS_k^{a,\delta}(\hbar, l)$. Then u is starlike in the disk $|z| < R^*$, where*

$$R^* = R^*(0) = \inf_{n \geq 2} \left(\frac{[1+\hbar(n-1)]\mathfrak{F}_k^{a,\delta}(n)}{n(1-l)} \right)^{\frac{1}{n-1}}. \tag{21}$$

The result is sharp in the sense described in Theorem 2.3.

Remark 2.5. *Theorem 2.3 corrects the usual but insufficient disk estimate $\left|zu'(z)/u(z) - 1\right| < 1$ by working instead with Silverman's exact coefficient criterion for negative coefficients. This is the key point that makes the proof both rigorous and sharp.*

3. Radius of Convexity

The convex case is handled in exactly the same spirit, now using Silverman's coefficient characterization for $C(\alpha)$ (see [23, Corollary 2]).

Lemma 3.1 (Silverman)**.** *Let $0 \le \alpha < 1$ and let*

$$f(z) = z - \sum_{n=2}^{\infty} b_n z^n, \qquad b_n \ge 0.$$

Then $f \in C(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} n(n-\alpha) b_n \le 1-\alpha. \tag{22}$$

Definition 3.2 (Radius of Convexity of Order α)**.** *For $0 \le \alpha < 1$, the radius of convexity of order α for the class $TS_k^{a,\delta}(\hbar, l)$, denoted by $R_c(\alpha)$, is the largest number $r \in (0,1]$ such that every function in $TS_k^{a,\delta}(\hbar, l)$ is convex of order α in the disk $|z| < r$.*

Theorem 3.3. *Let $0 \le \alpha < 1$ and let $u \in TS_k^{a,\delta}(\hbar, l)$. Then u is convex of order α in the disk $|z| < R_c(\alpha)$, where*

$$R_c(\alpha) = \inf_{n \ge 2} \left(\frac{(1-\alpha)[1+\hbar(n-1)]\mathfrak{F}_k^{a,\delta}(n)}{n(n-\alpha)(1-l)} \right)^{\frac{1}{n-1}}. \tag{23}$$

Moreover,

$$\lim_{n \to \infty} \left(\frac{(1-\alpha)[1+\hbar(n-1)]\mathfrak{F}_k^{a,\delta}(n)}{n(n-\alpha)(1-l)} \right)^{\frac{1}{n-1}} = 1. \tag{24}$$

Hence $0 < R_c(\alpha) \le 1$. If $R_c(\alpha) < 1$, then the infimum in (23) *is attained at some index $n_0 \ge 2$, and the result is sharp for the function*

$$u_{n_0}(z) = z - \frac{1-l}{[1+\hbar(n_0-1)]\mathfrak{F}_k^{a,\delta}(n_0)} z^{n_0}. \tag{25}$$

If $R_c(\alpha) = 1$, then every function in $TS_k^{a,\delta}(\hbar, l)$ is convex of order α in the whole unit disk E.

Proof. Let $W_n = [1+\hbar(n-1)]\mathfrak{F}_k^{a,\delta}(n)$ as in the proof of Theorem 2.3, and define

$$u_r(z) = \frac{u(rz)}{r} = z - \sum_{n=2}^{\infty} a_n r^{n-1} z^n.$$

Since multiplication by a positive constant does not change the quantity $1 + zu''(z)/u'(z)$, the function u is convex of order α in $|z| < r$ if and only if $u_r \in C(\alpha)$.

Assume that

$$r \le \left(\frac{(1-\alpha)W_n}{n(n-\alpha)(1-l)} \right)^{\frac{1}{n-1}} \qquad (n \ge 2). \tag{26}$$

Then

$$n(n-\alpha)r^{n-1} \le \frac{(1-\alpha)W_n}{1-l} \qquad (n \ge 2).$$

Multiplying by a_n, summing, and using Lemma 1.7, we obtain

$$\sum_{n=2}^{\infty} n(n-\alpha)a_n r^{n-1} \le \frac{1-\alpha}{1-l}\sum_{n=2}^{\infty} W_n a_n \le 1-\alpha.$$

Hence, by Lemma 3.1, $u_r \in C(\alpha)$. Therefore u is convex of order α in $|z| < r$, and (23) follows.

The limit (24) is proved exactly as in Theorem 2.3: the logarithm of the quantity inside the parentheses is $O(\log n)$, so division by $n-1$ tends to 0.

If $R_c(\alpha) < 1$, the limit (24) implies that the infimum is attained by some finite index $n_0 \ge 2$. For the one-term function (25), Lemma 1.7 gives $u_{n_0} \in TS_k^{a,\delta}(\hbar, l)$, and Lemma 3.1 shows that $(u_{n_0})_r \in C(\alpha)$ if and only if

$$n_0(n_0-\alpha)\frac{1-l}{W_{n_0}} r^{n_0-1} \le 1-\alpha,$$

that is,

$$r \le \left(\frac{(1-\alpha)W_{n_0}}{n_0(n_0-\alpha)(1-l)}\right)^{\frac{1}{n_0-1}} = R_c(\alpha).$$

This proves sharpness. The case $R_c(\alpha) = 1$ is handled exactly as in Theorem 2.3. □

Corollary 3.4. *Let $u \in TS_k^{a,\delta}(\hbar, l)$. Then u is convex in the disk $|z| < R_c$, where*

$$R_c = R_c(0) = \inf_{n \ge 2}\left(\frac{[1+\hbar(n-1)]\mathfrak{F}_k^{a,\delta}(n)}{n^2(1-l)}\right)^{\frac{1}{n-1}}. \tag{27}$$

The result is sharp in the sense described in Theorem 3.3.

Remark 3.5. *From* (17) *and* (23)*, we immediately obtain*

$$R_c(\alpha) \le R^*(\alpha) \qquad (0 \le \alpha < 1),$$

since each term in the infimum for $R_c(\alpha)$ does not exceed the corresponding term in the infimum for $R^*(\alpha)$. In general, equality may occur, so one cannot replace "$\le$" by a strict inequality without additional hypotheses.

4. Special Cases and Applications

In this section, we record some consequences of the main theorems. The first one gives a convenient criterion for the full unit disk to be the starlikeness or convexity region.

Corollary 4.1. *Let* $0 \le \alpha < 1$.

(1) If

$$(1-\alpha)[1+\hbar(n-1)]\mathfrak{F}_k^{a,\delta}(n) \ge (n-\alpha)(1-l) \qquad (n \ge 2), \tag{28}$$

then every function in $TS_k^{a,\delta}(\hbar,l)$ *is starlike of order* α *in* E.

(2) If

$$(1-\alpha)[1+\hbar(n-1)]\mathfrak{F}_k^{a,\delta}(n) \ge n(n-\alpha)(1-l) \qquad (n \ge 2), \tag{29}$$

then every function in $TS_k^{a,\delta}(\hbar,l)$ *is convex of order* α *in* E.

Proof. Both assertions follow immediately from Theorems 2.3 and 3.3: under (28) or (29), each term appearing in the corresponding infimum is at least 1, while the limit of the sequence is 1, so the radius is exactly 1. □

4.1. A Close-to-Convexity Corollary.

The same argument also gives a sharp radius for the bounded-turning condition $\operatorname{Re} u'(z) > \alpha$, which is the standard close-to-convexity condition with respect to the identity map.

Corollary 4.2. *Let* $0 \le \alpha < 1$ *and let* $u \in TS_k^{a,\delta}(\hbar,l)$. *Then*

$$\operatorname{Re} u'(z) > \alpha \qquad (|z| < R_{cc}(\alpha)),$$

where

$$R_{cc}(\alpha) = \inf_{n\ge 2}\left(\frac{(1-\alpha)[1+\hbar(n-1)]\mathfrak{F}_k^{a,\delta}(n)}{n(1-l)}\right)^{\frac{1}{n-1}}. \tag{30}$$

The result is sharp whenever $R_{cc}(\alpha) < 1$.

Proof. Let $0 < r < 1$ and write $u_r(z) = u(rz)/r = z - \sum_{n=2}^{\infty} a_n r^{n-1} z^n$. If

$$r \le \left(\frac{(1-\alpha)[1+\hbar(n-1)]\mathfrak{F}_k^{a,\delta}(n)}{n(1-l)}\right)^{\frac{1}{n-1}} \qquad (n \ge 2),$$

then

$$\sum_{n=2}^{\infty} na_n r^{n-1} \leq \frac{1-\alpha}{1-l}\sum_{n=2}^{\infty}[1+\hbar(n-1)]\mathfrak{F}_k^{a,\delta}(n)a_n \leq 1-\alpha.$$

Hence, for $|z|<1$,

$$\operatorname{Re} u_r'(z) = \operatorname{Re}\left(1-\sum_{n=2}^{\infty} na_n r^{n-1}z^{n-1}\right) \geq 1-\sum_{n=2}^{\infty} na_n r^{n-1} > \alpha$$

whenever $r < R_{cc}(\alpha)$. Therefore $\operatorname{Re} u'(z) > \alpha$ for $|z| < R_{cc}(\alpha)$. Sharpness follows from the one-term extremal function exactly as in Theorems 2.3 and 3.3. □

4.2. The Salagean Operator Case ($\delta = 0$).

When $\delta = 0$, the multiplier reduces to $\mathfrak{F}_k^{a,\delta}(n) = n^k$, and the operator $D^k L_a^\delta$ becomes the Salagean operator D^k.

Corollary 4.3. *For the class $TS_k^{a,0}(\hbar,l)$, the radii of starlikeness and convexity of order α are given by*

$$R^*(\alpha) = \inf_{n\geq 2}\left(\frac{(1-\alpha)[1+\hbar(n-1)]n^k}{(n-\alpha)(1-l)}\right)^{\frac{1}{n-1}}, \tag{31}$$

$$R_c(\alpha) = \inf_{n\geq 2}\left(\frac{(1-\alpha)[1+\hbar(n-1)]n^k}{n(n-\alpha)(1-l)}\right)^{\frac{1}{n-1}}. \tag{32}$$

4.3. The Komatu Operator Case ($k = 0$).

When $k = 0$, the multiplier becomes $\mathfrak{F}_k^{a,\delta}(n) = \left(\frac{a}{a+n-1}\right)^\delta$, and the operator reduces to the Komatu integral operator L_a^δ.

Corollary 4.4. *For the class $TS_0^{a,\delta}(\hbar,l)$, the radii of starlikeness and convexity of order α are given by*

$$R^*(\alpha) = \inf_{n\geq 2}\left(\frac{(1-\alpha)[1+\hbar(n-1)]}{(n-\alpha)(1-l)}\left(\frac{a}{a+n-1}\right)^\delta\right)^{\frac{1}{n-1}}, \tag{33}$$

$$R_c(\alpha) = \inf_{n\geq 2}\left(\frac{(1-\alpha)[1+\hbar(n-1)]}{n(n-\alpha)(1-l)}\left(\frac{a}{a+n-1}\right)^\delta\right)^{\frac{1}{n-1}}. \tag{34}$$

4.4. The Identity Operator Case ($k = 0$, $\delta = 0$).

When both $k = 0$ and $\delta = 0$, we have $\mathfrak{F}_k^{a,\delta}(n) = 1$ for all n, and the operator is the identity.

Corollary 4.5. *For the class $TS_0^{a,0}(\hbar,l)$, the radii of starlikeness and convexity of order α are given by*

$$R^*(\alpha)=\inf_{n\geq 2}\left(\frac{(1-\alpha)(1+\hbar(n-1))}{(n-\alpha)(1-l)}\right)^{\frac{1}{n-1}}, \tag{35}$$

$$R_c(\alpha)=\inf_{n\geq 2}\left(\frac{(1-\alpha)(1+\hbar(n-1))}{n(n-\alpha)(1-l)}\right)^{\frac{1}{n-1}}. \tag{36}$$

4.5. The Case $\hbar=0$.

When $\hbar=0$, the defining condition (14) simplifies to $\mathrm{Re}(D^kL_a^\delta u(z)/z)\geq l$.

Corollary 4.6. *For the class $TS_k^{a,\delta}(0,l)$, the radii of starlikeness and convexity of order α are given by*

$$R^*(\alpha)=\inf_{n\geq 2}\left(\frac{(1-\alpha)\mathfrak{F}_k^{a,\delta}(n)}{(n-\alpha)(1-l)}\right)^{\frac{1}{n-1}}, \tag{37}$$

$$R_c(\alpha)=\inf_{n\geq 2}\left(\frac{(1-\alpha)\mathfrak{F}_k^{a,\delta}(n)}{n(n-\alpha)(1-l)}\right)^{\frac{1}{n-1}}. \tag{38}$$

Example 4.7. *Consider the specific case $k=1$, $\delta=1$, $a=2$, $\hbar=1$, and $l=0$. Then*

$$\mathfrak{F}_k^{a,\delta}(n)=n\cdot\frac{2}{n+1}=\frac{2n}{n+1},$$

and therefore

$$[1+\hbar(n-1)]\mathfrak{F}_k^{a,\delta}(n)=n\cdot\frac{2n}{n+1}=\frac{2n^2}{n+1}.$$

For $\alpha=0$, Theorems 2.3 and 3.3 give

$$R^*=\inf_{n\geq 2}\left(\frac{2n}{n+1}\right)^{\frac{1}{n-1}},$$

$$R_c=\inf_{n\geq 2}\left(\frac{2}{n+1}\right)^{\frac{1}{n-1}}.$$

Now

$$\frac{2n}{n+1}\geq 1 \qquad (n\geq 2),$$

so Corollary 4.1 implies that $R^*=1$. On the other hand,

$$R_c=\min\left\{\frac{2}{3},\ \sqrt{\frac{1}{2}},\ \sqrt[3]{\frac{2}{5}},\dots\right\}=\frac{2}{3},$$

so the radius of convexity equals $2/3$.

5. Concluding Remarks

In this paper, we have established sharp radii of starlikeness and convexity for the class $TS_k^{a,\delta}(\hbar, l)$ of analytic functions with negative coefficients defined via the composite Salagean-Komatu operator. More precisely, we proved the stronger order-α results in Theorems 2.3 and 3.3, from which the classical radii follow as immediate corollaries. The proofs rely on the exact coefficient characterization of the class $TS_k^{a,\delta}(\hbar, l)$ and on Silverman's exact coefficient criteria for negative coefficients, which makes the sharpness argument transparent.

The results obtained here may be viewed as the radius counterpart of the coefficient theory developed in [17]. In addition, Corollary 4.2 gives a sharp close-to-convexity radius of order α, while Corollary 4.1 isolates a simple criterion guaranteeing that the full unit disk is already the starlikeness or convexity region.

Several directions for further work remain open. One may study radii associated with other geometric subclasses such as uniformly starlike or uniformly convex functions, determine inclusion relations between operator-defined negative coefficient classes, or investigate whether the infima in the special cases from Section 4 admit closed forms under additional parameter restrictions.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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