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# ON CONTRACTIVE MAPPINGS IN $S$-MULTIPLICATIVE METRIC SPACES

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**Abstract.** In this paper, we establish existence and uniqueness results for fixed points of self maps in $S$-multiplicative metric spaces under various contractive conditions. An example is provided to illustrate the applicability of the result.

**Keywords:** $S$-metric space; multiplicative metric space; $S$-multiplicative metric space, fixed point.

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## 1. Introduction

Fixed point theory is considered one of the fundamental tools in nonlinear analysis. Its significance stems from its wide range of applications, not only within various areas of mathematics but also across numerous quantitative science disciplines. This theory plays a crucial role in establishing the existence and often uniqueness of solutions to nonlinear differential equations, integrodifferential equations and integral equations in diverse abstract spaces. The Banach Contraction Principle [4] was one of the first major results in fixed point theory, followed by Kannan's introduction of a new type of contractive mapping [11]. Since then, numerous researchers have explored fixed points for mappings satisfying various contractive conditions ([7], [19], [14], [15], [12], [2], [1], [13]).

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Bashirov et al. [5] played an important role in introducing multiplicative calculus to the scientific community and highlighting its practical value. In their study, they formulated the idea of a multiplicative metric space and proved the existence and uniqueness of solutions for multiplicative differential equations. This work motivated further research, leading to the development of several fixed point results in multiplicative metric spaces by different authors. Subsequently, Bashirov et al. [6] showed that multiplicative calculus provides a more suitable framework for modeling problems arising in various fields.

Recently, researchers have been studying fixed point theorems in new types of spaces. Examples include $S$-metric spaces by Sedghi et al. [16], modular $S$-metric spaces by Ege and Alaca [9] and controlled $S$-metric-type spaces by Ekiz et al. [8]. A particularly interesting development is the $S$-multiplicative metric space introduced by Adewale et al. [3], where distances follow a multiplicative version of the triangle inequality. This new way of measuring distance not only leads to novel fixed point results but also expands the theoretical framework and opens up avenues for further research.

In this paper, we prove some fixed point theorems in $S$-multiplicative metric spaces. Our findings build upon previous studies while introducing new insights to advance the field.

## 2. PRELIMINARIES

We review the fundamental definitions and key results that will be used in the subsequent sections. The notion of a metric space was first formalized by Fréchet [10] as described below:

**Definition 2.1.** [10] *Let $X$ be a non-empty set. A function $d : X \times X \to \mathbb{R}$ is called a metric on $X$, if it fulfills the following conditions:*

*(i)* $d(x,y) \geq 0$ *for all* $x,y \in X$;

*(ii)* $d(x,y) = 0$ *if and only if* $x = y$;

*(iii)* $d(x,y) = d(y,x)$, *for all* $x,y \in X$;

*(iv)* $d(x,y) \leq d(x,z) + d(z,y)$, *for all* $x,y,z \in X$.

*The pair $(X,d)$ is called a metric space.*

The notion of $S$-metric spaces was first proposed by Sedghi et al. [16] described as:

**Definition 2.2.** [16] *Let $X$ be a non empty set and $S : X \times X \times X \to [0,\infty)$ be a mapping satisfying following properties:*

(i) $S(x,y,z) = 0$ *if and only if* $x = y = z$*;*

(ii) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$ *for all* $a,x,y,z \in X$ *(rectangle inequality).*

*Then $(X,S)$ is called a S-metric space.*

The concept of multiplicative metric spaces was introduced by Bashirov et al. [5] and is defined as follows:

**Definition 2.3.** [5] *Let $X$ be a nonempty set and let $d : X \times X \to [0,\infty)$ be a function satisfying the following properties:*

*(i)* $d(x,y) \geq 1$ *for all* $x,y \in X$*;*

*(ii)* $d(x,y) = 1$ *if and only if* $x = y$*;*

*(iii)* $d(x,y) = d(y,x)$ *for all* $x,y \in X$ *;*

*(iv)* $d(x,y) \leq d(x,z)\ d(z,y)$ *for all* $x,y,z \in X$*.*

*Then $d$ is called multiplicative metric on $X$ and $(X,d)$ is called a multiplicative metric space.*

The concept of an $S$-multiplicative metric space was introduced by Adewale et al. [3] and is defined as follows:

**Definition 2.4.** [3] *Let $X$ be a nonempty set and $\overline{S} : X \times X \times X \to [0,\infty)$, a function satisfying the following properties:*

*(i)* $\overline{S}(x,y,z) \geq 1$*;*

*(ii)* $\overline{S}(x,y,z) = 1$ *if and only if* $x = y = z$*;*

*(iii)* $\overline{S}(x,y,z) \leq \overline{S}(x,x,a)\ \overline{S}(y,y,a)\ \overline{S}(z,z,a)$ *for all* $x,y,z,a \in X$*.*

*Then $(X,\overline{S})$ is called a S-multiplicative metric space.*

**Example 2.1.** [3] *Let $X = \mathbb{Z}$ and define $\overline{S} : X \times X \times X \to [0,\infty)$ by*

$$\overline{S}(x,y,z) = \begin{cases} 1, & \textit{if } x = y = z; \\ e^x, & \textit{otherwise}. \end{cases}$$

*Then $(X,\overline{S})$ is a S-multiplicative metric space.*

**Example 2.2.** [17] *Let $X=[1,\infty)$ and define the function $\overline{S}:X\times X\times X\to[0,\infty)$ by*

$$\overline{S}(x,y,z)=\max\left\{\frac{x}{y},\frac{y}{x},\frac{y}{z},\frac{z}{y},\frac{x}{z},\frac{z}{x}\right\}.$$

*Then $(X,\overline{S})$ is S-multiplicative metric space.*

**Example 2.3.** [18] *Let $X=\mathbb{N}$ and define $\overline{S}:X\times X\times X\to[0,\infty)$ by*

$$\overline{S}(x,y,z)=\begin{cases}1, & \text{if } x=y=z;\\ 1+xyz, & \text{otherwise}.\end{cases}$$

*Then $(X,\overline{S})$ is a S-multiplicative metric space.*

**Lemma 2.1.** [3] *Let $(X,\overline{S})$ be a S-multiplicative metric space and $\{x_n\}$ a sequence in $X$. Then, $\{x_n\}$ converges to $x$ if and only if*

$$\overline{S}(x_n,x,x)\to 1 \quad \text{as } n\to\infty.$$

**Lemma 2.2.** [3] *Let $(X,\overline{S})$ be a S-multiplicative metric space and $\{x_n\}$ a sequence in $X$. Then, $\{x_n\}$ is said to be a Cauchy sequence if and only if*

$$\overline{S}(x_n,x_m,x_l)\to 1 \quad \text{as } n,m,l\to\infty.$$

**Proposition 2.1.** [17] *Let $(X,\overline{S})$ be an S-multiplicative metric space. Then for all $x,y\in X$, the following hold:*

*(i)* $\overline{S}(x,x,y)=\overline{S}(y,y,x)$;
*(ii)* $\overline{S}(x,y,y)\leq\overline{S}(x,x,y)$;
*(iii)* $\overline{S}(x,y,x)\leq\overline{S}(y,y,x)$.

## 3. Main results

In this section, we establish the existence and uniqueness of a fixed point for different contractive conditions in $S$-multiplicative metric spaces.

**Theorem 3.1.** *Let $(X,\overline{S})$ be a complete S-multiplicative metric space and $T:X\to X$ be a mapping. Suppose there exist non-negative real numbers $\alpha_1,\alpha_2,\alpha_3$ and $\alpha_4$ such that* $2(\alpha_1+$

*$\alpha_2+\alpha_3)+\alpha_4<1$ and*

$$\overline{S}(Tx,Ty,Tz)\le\left(\overline{S}(x,y,z)\right)^{\alpha_1}\left(\overline{S}(x,x,Tx)\right)^{\alpha_2}\left(\overline{S}(y,y,Ty)\right)^{\alpha_3}\left(\overline{S}(z,z,Tz)\right)^{\alpha_4}\tag{3.1}$$

*for all $x,y,z\in X$. Then $T$ has a unique fixed point in $X$.*

*Proof.* Let $x_0\in X$ be an arbitrary element of $X$. Since $T:X\to X$, we can define a sequence $\{x_n\}$ in $X$ by $x_{n+1}=Tx_n$, for $n=0,1,2,\cdots$. Now, consider

$$\begin{aligned}
&\overline{S}(x_n,x_n,x_{n+1})\\
&\quad=\overline{S}(Tx_{n-1},Tx_{n-1},Tx_n)\\
&\quad\le\left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_1}\left(\overline{S}(x_{n-1},x_{n-1},Tx_{n-1})\right)^{\alpha_2}\left(\overline{S}(x_{n-1},x_{n-1},Tx_{n-1})\right)^{\alpha_3}\left(\overline{S}(x_n,x_n,Tx_n)\right)^{\alpha_4}\\
&\quad=\left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_1}\left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_2}\left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_3}\left(\overline{S}(x_n,x_n,x_{n+1})\right)^{\alpha_4}.
\end{aligned}$$

It gives that

$$\left(\overline{S}(x_n,x_n,x_{n+1})\right)^{1-\alpha_4}\le\left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_1+\alpha_2+\alpha_3}.$$

From this, we deduce that

$$\overline{S}(x_n,x_n,x_{n+1})\le\left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\left(\dfrac{\alpha_1+\alpha_2+\alpha_3}{1-\alpha_4}\right)}.$$

Let $k=\dfrac{\alpha_1+\alpha_2+\alpha_3}{1-\alpha_4}$. Then it follows that

$$\overline{S}(x_n,x_n,x_{n+1})\le\left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{k}.\tag{3.2}$$

By the same argument, we obtain

$$\overline{S}(x_{n-1},x_{n-1},x_n)\le\left(\overline{S}(x_{n-2},x_{n-2},x_{n-1})\right)^{k}.\tag{3.3}$$

Using inequalities (3.2) and (3.3), we have

$$\overline{S}(x_n,x_n,x_{n+1})\le\left(\overline{S}(x_{n-2},x_{n-2},x_{n-1})\right)^{k^2}.$$

Proceeding by induction, we obtain

$$\overline{S}(x_n,x_n,x_{n+1})\le\left(\overline{S}(x_0,x_0,x_1)\right)^{k^n}.\tag{3.4}$$

For any $m,n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}\overline{S}(x_n,x_m,x_m) &\leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_m,x_m,x_{n+1})\right)^2 \\ &\leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\left(\overline{S}(x_m,x_m,x_{n+2})\right)^2\ \overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2 \\ &= \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2\ \left(\overline{S}(x_m,x_m,x_{n+2})\right)^{2^2}.\end{aligned}$$

Continuing in the same way, we obtain

$$\begin{aligned}\overline{S}(x_n,x_m,x_m) &\leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2 \cdots \left(\overline{S}(x_{m-1},x_{m-1},x_m)\right)^{2^{m-n}} \\ &\leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2\ \left(\overline{S}(x_{n+2},x_{n+2},x_{n+3})\right)^{2^2} \cdots.\end{aligned}$$

Applying (3.4), we deduce

$$\begin{aligned}\overline{S}(x_n,x_m,x_m) &\leq \left(\overline{S}(x_0,x_0,x_1)\right)^{k^n}\ \left(\left(\overline{S}(x_0,x_0,x_1)\right)^{k^{n+1}}\right)^2\ \left(\left(\overline{S}(x_0,x_0,x_1)\right)^{k^{n+2}}\right)^{2^2} \cdots \\ &= \left(\overline{S}(x_0,x_0,x_1)\right)^{k^n+2k^{n+1}+2^2k^{n+2}+2^3k^{n+3}+\cdots} \\ &= \left(\overline{S}(x_0,x_0,x_1)\right)^{k^n\left(1+(2k)+(2k)^2+(2k)^3+\cdots\right)} \\ &= \left(\overline{S}(x_0,x_0,x_1)\right)^{\dfrac{k^n}{1-2k}} \quad \left(\because\ \ 0 \leq k < \frac{1}{2}\right).\end{aligned}$$

Therefore

$$\overline{S}(x_n,x_m,x_m) \leq \left(\overline{S}(x_0,x_0,x_1)\right)^{\dfrac{k^n}{1-2k}}.$$

Since, $0 \leq k < \frac{1}{2}$, taking the limit as $n,m \to \infty$, we obtain

$$\lim_{n,m\to\infty} \overline{S}(x_n,x_m,x_m) = 1.$$

Now, for $n,m,l \in \mathbb{N}$ with $n > m > l$, we have,

$$\overline{S}(x_n,x_m,x_l) \leq \overline{S}(x_n,x_n,x_{n-1})\ \overline{S}(x_m,x_m,x_{n-1})\ \overline{S}(x_l,x_l,x_{n-1}).$$

Letting $n,m,l \to \infty$, we obtain,

$$\lim_{n,m,l\to\infty} \overline{S}(x_n,x_m,x_l) = 1.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, there is $x \in X$ such that $\{x_n\}$ converges to $x$. That is

$$\lim_{n\to\infty} x_n = x.$$

Now, we show that $x$ is a fixed point of $T$. For this purpose, we consider

$$\begin{aligned}
&\overline{S}(x_n,x_n,Tx) = \overline{S}(Tx_{n-1},Tx_{n-1},Tx)\\
&\leq \left(\overline{S}(x_{n-1},x_{n-1},x)\right)^{\alpha_1}\left(\overline{S}(x_{n-1},x_{n-1},Tx_{n-1})\right)^{\alpha_2}\left(\overline{S}(x_{n-1},x_{n-1},Tx_{n-1})\right)^{\alpha_3}\left(\overline{S}(x,x,Tx)\right)^{\alpha_4}.
\end{aligned}$$

Therefore

$$\overline{S}(x_n,x_n,Tx) \leq \left(\overline{S}(x_{n-1},x_{n-1},x)\right)^{\alpha_1}\left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_2}\left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_3}\left(\overline{S}(x,x,Tx)\right)^{\alpha_4}.$$

Taking the limit as $n \to \infty$, we deduce that

$$\overline{S}(x,x,Tx) \leq \left(\overline{S}(x,x,x)\right)^{\alpha_1}\left(\overline{S}(x,x,x)\right)^{\alpha_2}\left(\overline{S}(x,x,x)\right)^{\alpha_3}\left(\overline{S}(x,x,Tx)\right)^{\alpha_4}.$$

Therefore, we have

$$\left(\overline{S}(x,x,Tx)\right)^{1-\alpha_4} \leq 1,$$

which implies

$$\overline{S}(x,x,Tx) \leq 1.$$

On the other hand, by the definition of $S$-multiplicative metric, we have $\overline{S}(x,x,Tx) \geq 1$. Combining these two inequalities, we conclude that $\overline{S}(x,x,Tx) = 1$. Consequently, $Tx = x$ and thus $x$ is a fixed point of $T$. To establish uniqueness, suppose that there exists $y \in X$ with $y \neq x$ such that $Ty = y$. Then

$$\begin{aligned}
\overline{S}(x,x,y) &= \overline{S}(Tx,Tx,Ty)\\
&\leq \left(\overline{S}(x,x,y)\right)^{\alpha_1}\left(\overline{S}(x,x,Tx)\right)^{\alpha_2}\left(\overline{S}(x,x,Tx)\right)^{\alpha_3}\left(\overline{S}(y,y,Ty)\right)^{\alpha_4}\\
&= \left(\overline{S}(x,x,y)\right)^{\alpha_1}\left(\overline{S}(x,x,x)\right)^{\alpha_2}\left(\overline{S}(x,x,x)\right)^{\alpha_3}\left(\overline{S}(y,y,y)\right)^{\alpha_4}.
\end{aligned}$$

Therefore,

$$\overline{S}(x,x,y) \leq \left(\overline{S}(x,x,y)\right)^{\alpha_1}.$$

It follows that $\left(\overline{S}(x,x,y)\right)^{1-\alpha_1} \leq 1$. Thus, we have $\overline{S}(x,x,y) \leq 1$. However $\overline{S}(x,x,y) \geq 1$. From these two inequalities, we conclude that $\overline{S}(x,x,y) = 1$. Hence, we must have $x = y$, which proves the uniqueness of a fixed point. □

**Theorem 3.2.** *Let $(X,\overline{S})$ be a complete S-multiplicative metric space and $T : X \to X$ be a mapping. Suppose there exist non negative real numbers $\alpha_1, \alpha_2, \alpha_3$ and $\alpha_4$ such that $2(\alpha_1 + \alpha_2) + 5\alpha_3 + \alpha_4 < 1$ and*

$$\overline{S}(Tx,Ty,Tz) \leq \left(\overline{S}(x,y,z)\right)^{\alpha_1} \left(\overline{S}(x,x,Ty)\right)^{\alpha_2} \left(\overline{S}(y,y,Tz)\right)^{\alpha_3} \left(\overline{S}(z,z,Tx)\right)^{\alpha_4} \tag{3.5}$$

*for all $x,y,z \in X$. Then $T$ possesses a unique fixed point in $X$.*

*Proof.* Let $x_0$ be any point $X$. Since $T$ maps $X$ into itself, we may construct a sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n$, for $n = 0,1,2,\cdots$. Now, consider

$$\begin{aligned}
&\overline{S}(x_n,x_n,x_{n+1}) \\
&\quad = \overline{S}(Tx_{n-1},Tx_{n-1},Tx_n) \\
&\quad \leq \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_1} \left(\overline{S}(x_{n-1},x_{n-1},Tx_{n-1})\right)^{\alpha_2} \left(\overline{S}(x_{n-1},x_{n-1},Tx_n)\right)^{\alpha_3} \left(\overline{S}(x_n,x_n,Tx_{n-1})\right)^{\alpha_4} \\
&\quad = \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_1} \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_2} \left(\overline{S}(x_{n-1},x_{n-1},x_{n+1})\right)^{\alpha_3} \left(\overline{S}(x_n,x_n,x_n)\right)^{\alpha_4} \\
&\quad \leq \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_1} \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_2} \left(\left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^2 \overline{S}(x_{n+1},x_{n+1},x_n)\right)^{\alpha_3} \\
&\quad = \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_1} \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_2} \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{2\alpha_3} \left(\overline{S}(x_n,x_n,x_{n+1})\right)^{\alpha_3}.
\end{aligned}$$

Therefore,

$$\left(\overline{S}(x_n,x_n,x_{n+1})\right)^{1-\alpha_3} \leq \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_1+\alpha_2+2\alpha_3}.$$

It implies that

$$\overline{S}(x_n,x_n,x_{n+1}) \leq \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\left(\frac{\alpha_1+\alpha_2+2\alpha_3}{1-\alpha_3}\right)}.$$

Put $k = \dfrac{\alpha_1+\alpha_2+2\alpha_3}{1-\alpha_3}$, we obtain

$$\overline{S}(x_n,x_n,x_{n+1}) \leq \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^k. \tag{3.6}$$

By an analogous argument, we obtain that

$$\overline{S}(x_{n-1},x_{n-1},x_n) \leq \left(\overline{S}(x_{n-2},x_{n-2},x_{n-1})\right)^k. \tag{3.7}$$

Inequalities (3.6) and (3.7) together imply that

$$\overline{S}(x_n,x_n,x_{n+1}) \leq \left(\overline{S}(x_{n-2},x_{n-2},x_{n-1})\right)^{k^2}.$$

Continuing in the same way, we obtain

$$\overline{S}(x_n,x_n,x_{n+1}) \leq \left(\overline{S}(x_0,x_0,x_1)\right)^{k^n}. \tag{3.8}$$

For any $m,n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
\overline{S}(x_n,x_m,x_m) &\leq \overline{S}(x_n,x_n,x_{n+1}) \ \left(\overline{S}(x_m,x_m,x_{n+1})\right)^2 \\
&\leq \overline{S}(x_n,x_n,x_{n+1}) \ \left(\left(\overline{S}(x_m,x_m,x_{n+2})\right)^2 \ \overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2 \\
&= \overline{S}(x_n,x_n,x_{n+1}) \ \left(\overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2 \ \left(\overline{S}(x_m,x_m,x_{n+2})\right)^{2^2}.
\end{aligned}$$

Continuing this procedure, we obtain

$$\begin{aligned}
\overline{S}(x_n,x_m,x_m) &\leq \overline{S}(x_n,x_n,x_{n+1}) \ \left(\overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2 \cdots \left(\overline{S}(x_{m-1},x_{m-1},x_m)\right)^{2^{m-n}} \\
&\leq \overline{S}(x_n,x_n,x_{n+1}) \ \left(\overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2 \ \left(\overline{S}(x_{n+2},x_{n+2},x_{n+3})\right)^{2^2} \cdots.
\end{aligned}$$

It follows from (3.8) that

$$\begin{aligned}
\overline{S}(x_n,x_m,x_m) &\leq \left(\overline{S}(x_0,x_0,x_1)\right)^{k^n} \ \left(\left(\overline{S}(x_0,x_0,x_1)\right)^{k^{n+1}}\right)^2 \ \left(\left(\overline{S}(x_0,x_0,x_1)\right)^{k^{n+2}}\right)^{2^2} \cdots \\
&= \left(\overline{S}(x_0,x_0,x_1)\right)^{k^n+2k^{n+1}+2^2k^{n+2}+2^3k^{n+3}+\cdots} \\
&= \left(\overline{S}(x_0,x_0,x_1)\right)^{k^n\left(1+(2k)+(2k)^2+(2k)^3+\cdots\right)} \\
&= \left(\overline{S}(x_0,x_0,x_1)\right)^{\dfrac{k^n}{1-2k}} \quad \left(\because \ 0 \leq k < \frac{1}{2}\right).
\end{aligned}$$

Therefore,

$$\overline{S}(x_n,x_m,x_m) \leq \left(\overline{S}(x_0,x_0,x_1)\right)^{\dfrac{k^n}{1-2k}}.$$

Since $0 \leq k < \dfrac{1}{2}$, letting $n,m \to \infty$, we obtain

$$\lim_{n,m\to\infty} \overline{S}(x_n,x_m,x_m) = 1.$$

Now, for $n,m,l \in \mathbb{N}$ with $n > m > l$, we have

$$\overline{S}(x_n,x_m,x_l) \leq \overline{S}(x_n,x_n,x_{n-1}) \ \overline{S}(x_m,x_m,x_{n-1}) \ \overline{S}(x_l,x_l,x_{n-1}).$$

Letting $n,m,l \to \infty$, we obtain

$$\lim_{n,m,l\to\infty} \overline{S}(x_n,x_m,x_l) = 1.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, there is $x \in X$ such that $\{x_n\}$ converges to $x$. That is

$$\lim_{n\to\infty} x_n = x.$$

We proceed to show that $x$ is fixed point of $T$. For this purpose, we consider

$$\begin{aligned}
&\overline{S}(x_n,x_n,Tx) = \overline{S}(Tx_{n-1},Tx_{n-1},Tx) \\
&\leq \left(\overline{S}(x_{n-1},x_{n-1},x)\right)^{\alpha_1} \left(\overline{S}(x_{n-1},x_{n-1},Tx_{n-1})\right)^{\alpha_2} \left(\overline{S}(x_{n-1},x_{n-1},Tx)\right)^{\alpha_3} \left(\overline{S}(x,x,Tx_{n-1})\right)^{\alpha_4} \\
&= \left(\overline{S}(x_{n-1},x_{n-1},x)\right)^{\alpha_1} \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_2} \left(\overline{S}(x_{n-1},x_{n-1},Tx)\right)^{\alpha_3} \left(\overline{S}(x,x,x_n)\right)^{\alpha_4}.
\end{aligned}$$

Therefore,

$$\overline{S}(x_n,x_n,Tx) \leq \left(\overline{S}(x_{n-1},x_{n-1},x)\right)^{\alpha_1} \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha_2} \left(\overline{S}(x_{n-1},x_{n-1},Tx)\right)^{\alpha_3} \left(\overline{S}(x,x,x_n)\right)^{\alpha_4}.$$

Letting $n \to \infty$, it follows that

$$\overline{S}(x,x,Tx) \leq \left(\overline{S}(x,x,x)\right)^{\alpha_1} \left(\overline{S}(x,x,x)\right)^{\alpha_2} \left(\overline{S}(x,x,Tx)\right)^{\alpha_3} \left(\overline{S}(x,x,x)\right)^{\alpha_4}.$$

That is,

$$\overline{S}(x,x,Tx) \leq \left(\overline{S}(x,x,Tx)\right)^{\alpha_3}.$$

Therefore,

$$\left(\overline{S}(x,x,Tx)\right)^{1-\alpha_3} \leq 1,$$

which implies that $\overline{S}(x,x,Tx) \leq 1$. On the other hand, $\overline{S}(x,x,Tx) \geq 1$. Combining these two inequalities, we conclude that $\overline{S}(x,x,Tx) = 1$,, which immediately yields $Tx = x$. Hence, $x$ is a fixed point of $T$. Now, to show uniqueness of a fixed point, suppose that $y \in X$ with $y \neq x$ satisfies $Ty = y$. Then

$$\begin{aligned}
\overline{S}(x,x,y) &= \overline{S}(Tx,Tx,Ty) \\
&\leq \left(\overline{S}(x,x,y)\right)^{\alpha_1} \left(\overline{S}(x,x,Tx)\right)^{\alpha_2} \left(\overline{S}(x,x,Ty)\right)^{\alpha_3} \left(\overline{S}(y,y,Tx)\right)^{\alpha_4} \\
&= \left(\overline{S}(x,x,y)\right)^{\alpha_1} \left(\overline{S}(x,x,x)\right)^{\alpha_2} \left(\overline{S}(x,x,y)\right)^{\alpha_3} \left(\overline{S}(y,y,x)\right)^{\alpha_4} \\
&= \left(\overline{S}(x,x,y)\right)^{\alpha_1} \left(\overline{S}(x,x,y)\right)^{\alpha_3} \left(\overline{S}(x,x,y)\right)^{\alpha_4}.
\end{aligned}$$

Therefore,

$$\overline{S}(x,x,y) \leq \left(\overline{S}(x,x,y)\right)^{\alpha_1+\alpha_3+\alpha_4}.$$

It follows that

$$\left(\overline{S}(x,x,y)\right)^{1-\alpha_1-\alpha_3-\alpha_4} \leq 1.$$

Therefore, we have $\overline{S}(x,x,y) \leq 1$. However $\overline{S}(x,x,y) \geq 1$. Using these two inequalities, we conclude that $\overline{S}(x,x,y) = 1$. Hence, we must have $x = y$, which proves the uniqueness of a fixed point. □

**Theorem 3.3.** *Let $(X,\overline{S})$ be a complete S-multiplicative metric space and $T : X \to X$ be a mapping. Suppose there exist a real number $0 \leq \alpha < \frac{1}{2}$ such that for all $x,y,z \in X$*

$$\overline{S}(Tx,Ty,Tz) \leq (M(x,y,z))^{\alpha} \tag{3.9}$$

*where*

$$M(x,y,z) = \max\left\{\overline{S}(x,y,z),\overline{S}(x,x,Tx),\overline{S}(y,y,Ty),\overline{S}(z,z,Tz)\right\}.$$

*Then $T$ has a unique fixed point in $X$.*

*Proof.* Let $x_0 \in X$ be an arbitrary element of $X$. Since $T : X \to X$, we can define a sequence $\{x_n\}$ in $X$ by $x_{n+1} = Tx_n$, for $n = 0,1,2,\cdots$. Now, consider

$$\begin{aligned}\overline{S}(x_n,x_n,x_{n+1}) &= \overline{S}(Tx_{n-1},Tx_{n-1},Tx_n)\\ &\leq (M(x_{n-1},x_{n-1},x_n))^{\alpha},\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}M(x_{n-1},x_{n-1},x_n) &= \max\Big\{\overline{S}(x_{n-1},x_{n-1},x_n),\overline{S}(x_{n-1},x_{n-1},Tx_{n-1}),\overline{S}(x_{n-1},x_{n-1},Tx_{n-1}),\\ &\qquad\qquad \overline{S}(x_n,x_n,Tx_n)\Big\}\\ &= \max\Big\{\overline{S}(x_{n-1},x_{n-1},x_n),\overline{S}(x_{n-1},x_{n-1},x_n),\overline{S}(x_{n-1},x_{n-1},x_n),\overline{S}(x_n,x_n,x_{n+1})\Big\}\\ &= \max\Big\{\overline{S}(x_{n-1},x_{n-1},x_n),\overline{S}(x_n,x_n,x_{n+1})\Big\}.\end{aligned}$$

**Case(i).** If $M(x_{n-1},x_{n-1},x_n) = \overline{S}(x_n,x_n,x_{n+1})$, then (3.10) implies that

$$\overline{S}(x_n,x_n,x_{n+1}) \leq \left(\overline{S}(x_n,x_n,x_{n+1})\right)^{\alpha},$$

which is not possible, as $0 \leq \alpha < \dfrac{1}{2}$.

**Case(ii).** If $M(x_{n-1},x_{n-1},x_n) = \overline{S}(x_{n-1},x_{n-1},x_n)$, then (3.10) gives that

$$\overline{S}(x_n,x_n,x_{n+1}) \leq \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha}.$$

Similarly, we can show that

$$\overline{S}(x_{n-1},x_{n-1},x_n) \leq \left(\overline{S}(x_{n-2},x_{n-2},x_{n-1})\right)^{\alpha}.$$

Using above two inequalities, we obtain

$$\overline{S}(x_n,x_n,x_{n+1}) \leq \left(\overline{S}(x_{n-2},x_{n-2},x_{n-1})\right)^{\alpha^2}.$$

Continuing in this manner, we obtain

$$\overline{S}(x_n,x_n,x_{n+1}) \leq \left(\overline{S}(x_0,x_0,x_1)\right)^{\alpha^n}. \tag{3.11}$$

For any $m,n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
\overline{S}(x_n,x_m,x_m) &\leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_m,x_m,x_{n+1})\right)^2 \\
&\leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\left(\overline{S}(x_m,x_m,x_{n+2})\right)^2\ \overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2 \\
&= \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2\ \left(\overline{S}(x_m,x_m,x_{n+2})\right)^{2^2}.
\end{aligned}$$

Continuing in the same way, we obtain

$$\begin{aligned}
\overline{S}(x_n,x_m,x_m) &\leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2 \cdots \left(\overline{S}(x_{m-1},x_{m-1},x_m)\right)^{2^{m-n}} \\
&\leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2\ \left(\overline{S}(x_{n+2},x_{n+2},x_{n+3})\right)^{2^2} \cdots.
\end{aligned}$$

It follows from (3.11) that

$$\begin{aligned}
\overline{S}(x_n,x_m,x_m) &\leq \left(\overline{S}(x_0,x_0,x_1)\right)^{\alpha^n}\ \left(\left(\overline{S}(x_0,x_0,x_1)\right)^{\alpha^{n+1}}\right)^2\ \left(\left(\overline{S}(x_0,x_0,x_1)\right)^{\alpha^{n+2}}\right)^{2^2} \cdots \\
&= \left(\overline{S}(x_0,x_0,x_1)\right)^{\alpha^n + 2\alpha^{n+1} + 2^2\alpha^{n+2} + 2^3\alpha^{n+3} + \cdots} \\
&= \left(\overline{S}(x_0,x_0,x_1)\right)^{\alpha^n \left(1 + (2\alpha) + (2\alpha)^2 + (2\alpha)^3 + \cdots\right)} \\
&= \left(\overline{S}(x_0,x_0,x_1)\right)^{\dfrac{\alpha^n}{1-2\alpha}}.
\end{aligned}$$

Therefore,

$$\overline{S}(x_n,x_m,x_m) \leq \left(\overline{S}(x_0,x_0,x_1)\right)^{\frac{\alpha^n}{1-2\alpha}}.$$

Since, $0 \leq \alpha < 1$, letting $n,m \to \infty$, we get

$$\lim_{n,m\to\infty} \overline{S}(x_n,x_m,x_m) = 1.$$

Now, for $n,m,l \in \mathbb{N}$ with $n > m > l$, we have,

$$\overline{S}(x_n,x_m,x_l) \leq \overline{S}(x_n,x_n,x_{n-1})\ \overline{S}(x_m,x_m,x_{n-1})\ \overline{S}(x_l,x_l,x_{n-1}).$$

Letting $n,m,l \to \infty$, we obtain

$$\lim_{n,m,l\to\infty} \overline{S}(x_n,x_m,x_l) = 1.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, there is $x \in X$ such that $\{x_n\}$ converges to $x$. That is

$$\lim_{n\to\infty} x_n = x.$$

We proceed to show that $x$ is fixed point of $T$. For this purpose, we consider

$$\begin{aligned}\overline{S}(x_n,x_n,Tx) &= \overline{S}(Tx_{n-1},Tx_{n-1},Tx)\\ &\leq \left(M(x_{n-1},x_{n-1},x)\right)^{\alpha},\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}M(x_{n-1},x_{n-1},x) &= \max\left\{\overline{S}(x_{n-1},x_{n-1},x),\overline{S}(x_{n-1},x_{n-1},Tx_{n-1}),\overline{S}(x_{n-1},x_{n-1},Tx_{n-1}),\right.\\ &\qquad \left.\overline{S}(x,x,Tx)\right\}\\ &= \max\left\{\overline{S}(x_{n-1},x_{n-1},x),\overline{S}(x_{n-1},x_{n-1},x_n),\overline{S}(x_{n-1},x_{n-1},x_n),\overline{S}(x,x,Tx)\right\}\\ &= \max\left\{\overline{S}(x_{n-1},x_{n-1},x),\overline{S}(x_{n-1},x_{n-1},x_n),\overline{S}(x,x,Tx)\right\}.\end{aligned}$$

**Case(i).** If $M(x_{n-1},x_{n-1},x) = \overline{S}(x_{n-1},x_{n-1},x)$, then (3.12) implies that

$$\overline{S}(x_n,x_n,Tx) \leq \left(\overline{S}(x_{n-1},x_{n-1},x)\right)^{\alpha}.$$

Taking the limit as $n \to \infty$, we get

$$\overline{S}(x,x,Tx) \leq \left(\overline{S}(x,x,x)\right)^{\alpha} = 1.$$

However, $\overline{S}(x,x,Tx) \geq 1$. Taking both inequalities into account, it follows that

$$\overline{S}(x,x,Tx) = 1.$$

Consequently, $Tx = x$.

**Case(ii).** If $M(x_{n-1},x_{n-1},x) = \overline{S}(x_{n-1},x_{n-1},x_n)$, then (3.12) implies that

$$\overline{S}(x_n,x_n,Tx) \leq \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha}.$$

Taking limit as $n \to \infty$, we get

$$\overline{S}(x,x,Tx) \leq \left(\overline{S}(x,x,x)\right)^{\alpha} = 1.$$

However, $\overline{S}(x,x,Tx) \geq 1$. Using these two inequalities, we obtain $\overline{S}(x,x,Tx) = 1$. Consequently, $Tx = x$.

**Case(iii).** If $M(x_{n-1},x_{n-1},x) = \overline{S}(x,x,Tx)$, then (3.12) implies that

$$\overline{S}(x_n,x_n,Tx) \leq \left(\overline{S}(x,x,Tx)\right)^{\alpha}.$$

Letting $n \to \infty$, it follows that

$$\overline{S}(x,x,Tx) \leq \left(\overline{S}(x,x,Tx)\right)^{\alpha}.$$

Therefore,

$$\left(\overline{S}(x,x,Tx)\right)^{1-\alpha} \leq 1.$$

This implies that $\overline{S}(x,x,Tx) \leq 1$. Since $\overline{S}(x,x,Tx) \geq 1$, implies that $\overline{S}(x,x,Tx) = 1$. Thus, $Tx = x$.

This shows that $x$ is a fixed point of $T$. To prove the uniqueness of a fixed point, suppose that there exists $y \in X$ with $y \neq x$ such that $Ty = y$. Consider

$$\begin{aligned}\overline{S}(x,x,y) &= \overline{S}(Tx,Tx,Ty)\\ &\leq \left(M(x,x,y)\right)^{\alpha},\end{aligned}$$

where

$$\begin{aligned}M(x,x,y) &= \max\left\{\overline{S}(x,x,y),\overline{S}(x,x,Tx),\overline{S}(x,x,Tx),\overline{S}(y,y,Ty)\right\}\\ &= \max\left\{\overline{S}(x,x,y),\overline{S}(x,x,x),\overline{S}(x,x,x),\overline{S}(y,y,y)\right\}\end{aligned}$$

$$= \max\left\{\overline{S}(x,x,y),1,1,1\right\}$$
$$= \overline{S}(x,x,y).$$

Therefore,

$$\overline{S}(x,x,y) \leq \left(\overline{S}(x,x,y)\right)^{\alpha}.$$

It gives that $\left(\overline{S}(x,x,y)\right)^{1-\alpha} \leq 1$. Thus, we have $\overline{S}(x,x,y) \leq 1$. However $\overline{S}(x,x,y) \geq 1$. From these two inequalities, we conclude that $\overline{S}(x,x,y) = 1$. Hence, we must have $x = y$. This proves the uniqueness of a fixed point. □

**Theorem 3.4.** *Let $(X,\overline{S})$ be a complete S-multiplicative metric space and $T : X \to X$ be a mapping. Suppose there exist a real number $0 \leq \alpha < \frac{1}{5}$ such that for all $x,y,z \in X$*

$$\overline{S}(Tx,Ty,Tz) \leq (M(x,y,z))^{\alpha}, \tag{3.13}$$

*where*

$$M(x,y,z) = \max\left\{\overline{S}(x,y,z),\overline{S}(x,x,Ty),\overline{S}(y,y,Tz),\overline{S}(z,z,Tx)\right\}.$$

*Then $T$ has a unique fixed point in $X$.*

*Proof.* Let $x_0 \in X$ be an arbitrary element of $X$. Since $T : X \to X$, we can define a sequence $\{x_n\}$ in $X$ by $x_{n+1} = Tx_n$, for $n = 0,1,2,\cdots$. Now, consider

$$\begin{aligned}\overline{S}(x_n,x_n,x_{n+1}) &= \overline{S}(Tx_{n-1},Tx_{n-1},Tx_n)\\ &\leq (M(x_{n-1},x_{n-1},x_n))^{\alpha}, \qquad (3.14)\end{aligned}$$

where

$$\begin{aligned}M(x_{n-1},x_{n-1},x_n) &= \max\Big\{\overline{S}(x_{n-1},x_{n-1},x_n),\overline{S}(x_{n-1},x_{n-1},Tx_{n-1}),\overline{S}(x_{n-1},x_{n-1},Tx_n),\\ &\qquad \overline{S}(x_n,x_n,Tx_{n-1})\Big\}\\ &= \max\Big\{\overline{S}(x_{n-1},x_{n-1},x_n),\overline{S}(x_{n-1},x_{n-1},x_n),\overline{S}(x_{n-1},x_{n-1},x_{n+1}),\overline{S}(x_n,x_n,x_n)\Big\}\\ &= \max\Big\{\overline{S}(x_{n-1},x_{n-1},x_n),\overline{S}(x_{n-1},x_{n-1},x_{n+1}),1\Big\}\\ &= \max\Big\{\overline{S}(x_{n-1},x_{n-1},x_n),\overline{S}(x_{n-1},x_{n-1},x_{n+1})\Big\}.\end{aligned}$$

**Case(i).** If $M(x_{n-1},x_{n-1},x_n)=\overline{S}(x_{n-1},x_{n-1},x_n)$, then inequality (3.14) implies that

$$\begin{aligned}\overline{S}(x_n,x_n,x_{n+1}) &\leq \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\alpha}\\ &\leq \left(\overline{S}(x_{n-2},x_{n-2},x_{n-1})\right)^{\alpha^2}.\end{aligned}$$

Continuing in this way, we obtain

$$\overline{S}(x_n,x_n,x_{n+1}) \leq \left(\overline{S}(x_0,x_0,x_1)\right)^{\alpha^n}.$$

For any $m,n\in\mathbb{N}$ with $m>n$, we have

$$\begin{aligned}\overline{S}(x_n,x_m,x_m) &\leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_m,x_m,x_{n+1})\right)^2\\ &\leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\left(\overline{S}(x_m,x_m,x_{n+2})\right)^2\ \overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2\\ &= \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2\ \left(\overline{S}(x_m,x_m,x_{n+2})\right)^{2^2}.\end{aligned}$$

Continuing in this, we deduce that

$$\begin{aligned}\overline{S}(x_n,x_m,x_m) &\leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2\cdots\left(\overline{S}(x_{m-1},x_{m-1},x_m)\right)^{2^{m-n}}\\ &\leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2\ \left(\overline{S}(x_{n+2},x_{n+2},x_{n+3})\right)^{2^2}\cdots.\end{aligned}$$

Hence,

$$\begin{aligned}\overline{S}(x_n,x_m,x_m) &\leq \left(\overline{S}(x_0,x_0,x_1)\right)^{\alpha^n}\ \left(\left(\overline{S}(x_0,x_0,x_1)\right)^{\alpha^{n+1}}\right)^2\ \left(\left(\overline{S}(x_0,x_0,x_1)\right)^{\alpha^{n+2}}\right)^{2^2}\cdots\\ &= \left(\overline{S}(x_0,x_0,x_1)\right)^{\alpha^n+2\alpha^{n+1}+2^2\alpha^{n+2}+2^3\alpha^{n+3}+\cdots}\\ &= \left(\overline{S}(x_0,x_0,x_1)\right)^{\alpha^n\left(1+(2\alpha)+(2\alpha)^2+(2\alpha)^3+\cdots\right)}\\ &= \left(\overline{S}(x_0,x_0,x_1)\right)^{\dfrac{\alpha^n}{1-2\alpha}}\quad\left(\because\ 0\leq\alpha<\frac{1}{5}<\frac{1}{2}\right).\end{aligned}$$

Therefore,

$$\overline{S}(x_n,x_m,x_m) \leq \left(\overline{S}(x_0,x_0,x_1)\right)^{\dfrac{\alpha^n}{1-2\alpha}}.$$

Since, $0\leq\alpha<\dfrac{1}{5}$, letting $n,m\to\infty$, we get

$$\lim_{n,m\to\infty}\overline{S}(x_n,x_m,x_m)=1.$$

Now, for $n,m,l \in \mathbb{N}$ with $n > m > l$, we have,

$$\overline{S}(x_n,x_m,x_l) \leq \overline{S}(x_n,x_n,x_{n-1})\ \overline{S}(x_m,x_m,x_{n-1})\ \overline{S}(x_l,x_l,x_{n-1}).$$

Letting $n,m,l \to \infty$, we obtain

$$\lim_{n,m,l\to\infty} \overline{S}(x_n,x_m,x_l) = 1.$$

**Case(ii).** If $M(x_{n-1},x_{n-1},x_n) = \overline{S}(x_{n-1},x_{n-1},x_{n+1})$, then inequality (3.14) implies that

$$\begin{aligned}
\overline{S}(x_n,x_n,x_{n+1}) &\leq \left(\overline{S}(x_{n-1},x_{n-1},x_{n+1})\right)^{\alpha} \\
&\leq \left(\left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^2 \overline{S}(x_{n+1},x_{n+1},x_n)\right)^{\alpha} \\
&= \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{2\alpha} \left(\overline{S}(x_n,x_n,x_{n+1})\right)^{\alpha}.
\end{aligned}$$

It implies that

$$\left(\overline{S}(x_n,x_n,x_{n+1})\right)^{1-\alpha} \leq \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{2\alpha}.$$

Therefore,

$$\overline{S}(x_n,x_n,x_{n+1}) \leq \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\frac{2\alpha}{1-\alpha}}.$$

Denote $\lambda = \dfrac{2\alpha}{1-\alpha}$. Then we have

$$\begin{aligned}
\overline{S}(x_n,x_n,x_{n+1}) &\leq \left(\overline{S}(x_{n-1},x_{n-1},x_n)\right)^{\lambda} \\
&\leq \left(\overline{S}(x_{n-2},x_{n-2},x_{n-1})\right)^{\lambda^2}.
\end{aligned}$$

By induction, we have

$$\overline{S}(x_n,x_n,x_{n+1}) \leq \left(\overline{S}(x_0,x_0,x_1)\right)^{\lambda^n}.$$

Now, for any $m,n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
\overline{S}(x_n,x_m,x_m) &\leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_m,x_m,x_{n+1})\right)^2 \\
&\leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\left(\overline{S}(x_m,x_m,x_{n+2})\right)^2\ \overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2 \\
&= \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2\ \left(\overline{S}(x_m,x_m,x_{n+2})\right)^{2^2}.
\end{aligned}$$

Continuing in this, we deduce that

$$\overline{S}(x_n,x_m,x_m) \leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2 \cdots \left(\overline{S}(x_{m-1},x_{m-1},x_m)\right)^{2^{m-n}}$$

$$\leq \overline{S}(x_n,x_n,x_{n+1})\ \left(\overline{S}(x_{n+1},x_{n+1},x_{n+2})\right)^2\ \left(\overline{S}(x_{n+2},x_{n+2},x_{n+3})\right)^{2^2}\cdots.$$

Hence,

$$\begin{aligned}
\overline{S}(x_n,x_m,x_m) &\leq \left(\overline{S}(x_0,x_0,x_1)\right)^{\lambda^n}\ \left(\left(\overline{S}(x_0,x_0,x_1)\right)^{\lambda^{n+1}}\right)^2\ \left(\left(\overline{S}(x_0,x_0,x_1)\right)^{\lambda^{n+2}}\right)^{2^2}\cdots \\
&= \left(\overline{S}(x_0,x_0,x_1)\right)^{\lambda^n+2\lambda^{n+1}+2^2\lambda^{n+2}+2^3\lambda^{n+3}+\cdots} \\
&= \left(\overline{S}(x_0,x_0,x_1)\right)^{\alpha^n\left(1+(2\lambda)+(2\lambda)^2+(2\lambda)^3+\cdots\right)} \\
&= \left(\overline{S}(x_0,x_0,x_1)\right)^{\frac{\lambda^n}{1-2\lambda}} \quad \left(\because\ 0\leq\lambda<\frac{1}{2}\right).
\end{aligned}$$

Therefore,

$$\overline{S}(x_n,x_m,x_m) \leq \left(\overline{S}(x_0,x_0,x_1)\right)^{\frac{\lambda^n}{1-2\lambda}}.$$

Since, $0\leq\lambda<\frac{1}{2}$, letting $n,m\to\infty$, we obtain

$$\lim_{n,m\to\infty}\overline{S}(x_n,x_m,x_m)=1.$$

Now, for $n,m,l\in\mathbb{N}$ with $n>m>l$, we have,

$$\overline{S}(x_n,x_m,x_l)\leq\overline{S}(x_n,x_n,x_{n-1})\ \overline{S}(x_m,x_m,x_{n-1})\ \overline{S}(x_l,x_l,x_{n-1}).$$

Letting $n,m,l\to\infty$, we obtain

$$\lim_{n,m,l\to\infty}\overline{S}(x_n,x_m,x_l)=1.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, there exists $x\in X$ such that $x_n\to x$. Now, we shall show that $x$ is a fixed point of $T$. For this, consider

$$\begin{aligned}
\overline{S}(x_n,x_n,Tx) &= \overline{S}(Tx_{n-1},Tx_{n-1},Tx) \\
&\leq \left(M(x_{n-1},x_{n-1},x)\right)^{\alpha},
\end{aligned}$$

where

$$\begin{aligned}
M(x_{n-1},x_{n-1},x_n) &= \max\left\{\overline{S}(x_{n-1},x_{n-1},x),\overline{S}(x_{n-1},x_{n-1},Tx_{n-1}),\overline{S}(x_{n-1},x_{n-1},Tx),\right. \\
&\qquad \left.\overline{S}(x,x,Tx_{n-1})\right\} \\
&= \max\left\{\overline{S}(x_{n-1},x_{n-1},x),\overline{S}(x_{n-1},x_{n-1},x_n),\overline{S}(x_{n-1},x_{n-1},Tx),\overline{S}(x,x,x_n)\right\}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}\overline{S}(x,x,Tx) &\leq \left(\max\left\{\overline{S}(x,x,x),\overline{S}(x,x,x),\overline{S}(x,x,Tx),\overline{S}(x,x,x)\right\}\right)^{\alpha}\\ &= \left(\max\left\{1,1,\overline{S}(x,x,Tx),1)\right\}\right)^{\alpha}\\ &= \left(\overline{S}(x,x,Tx)\right)^{\alpha}.\end{aligned}$$

Therefore,

$$\overline{S}(x,x,Tx) \leq \left(\overline{S}(x,x,Tx)\right)^{\alpha}.$$

This implies that

$$\left(\overline{S}(x,x,Tx)\right)^{\alpha-1} \leq 1.$$

Hence, $\overline{S}(x,x,Tx) \leq 1$. On the other hand, by definition,

$$\overline{S}(x,x,Tx) \geq 1.$$

Combining the above inequalities, we obtain $\overline{S}(x,x,Tx) = 1$. Consequently, $Tx = x$ and therefore $x$ is a fixed point of $T$. To prove the uniqueness of the fixed point, suppose that there exists $y \in X$ with $y \neq x$ be such that $Ty = y$. Consider

$$\begin{aligned}\overline{S}(x,x,y) &= \overline{S}(Tx,Tx,Ty)\\ &\leq (M(x,x,y))^{\alpha},\end{aligned}$$

where

$$\begin{aligned}M(x,x,y) &= \max\left\{\overline{S}(x,x,y),\overline{S}(x,x,Tx),\overline{S}(x,x,Ty),\overline{S}(y,y,Tx)\right\}\\ &= \max\left\{\overline{S}(x,x,y),\overline{S}(x,x,x),\overline{S}(x,x,y),\overline{S}(y,y,x)\right\}\\ &= \max\left\{\overline{S}(x,x,y),1,\overline{S}(x,x,y)\right\}\\ &= \overline{S}(x,x,y).\end{aligned}$$

Therefore,

$$\overline{S}(x,x,y) \leq \left(\overline{S}(x,x,y)\right)^{\alpha}.$$

Consequently,

$$\left(\overline{S}(x,x,y)\right)^{1-\alpha} \leq 1,$$

which implies that $\overline{S}(x,x,y) \leq 1$. On the other hand, since $\overline{S}(x,x,y) \geq 1$, we conclude that

$$\overline{S}(x,x,y) = 1.$$

Hence, $x = y$. This establishes the uniqueness of a fixed point $\square$

**Example 3.1.** *Let $X = \mathbb{Z}$ and consider S-multiplicative metric $\overline{S} : X \times X \times X \to [0,\infty)$ on $X$ defined by*

$$\overline{S}(x,y,z) = \begin{cases} 1, & \text{if } x = y = z, \\ e^{x+y+z}, & \text{otherwise}. \end{cases}$$

*Then $(X,\overline{S})$ is a complete S-multiplicative metric space. Define a mapping $T : X \to X$ by*

$$T(x) = \begin{cases} 0, & \text{if } x \geq 0, \\ \dfrac{x}{2}, & \text{if } x < 0. \end{cases}$$

*Choose constants*

$$\alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 0,\ \alpha_4 = \frac{1}{5}.$$

*Then*

$$2(\alpha_1 + \alpha_2 + \alpha_3) + \alpha_4 = 2\,(0+0+0) + \frac{1}{5} = \frac{1}{5} < 1.$$

*For any $x,y,z \in X$, we verify the contractive condition*

$$\overline{S}(Tx,Ty,Tz) \leq \left(\overline{S}(x,y,z)\right)^{\alpha_1} \left(\overline{S}(x,x,Tx)\right)^{\alpha_2} \left(\overline{S}(y,y,Ty)\right)^{\alpha_3} \left(\overline{S}(z,z,Tz)\right)^{\alpha_4}. \tag{3.15}$$

***Case(i).*** *If all $x,y,z \geq 0$, then $Tx = Ty = Tz = 0$ and hence $\overline{S}(Tx,Ty,Tz) = 1$. Therefore, the inequality (3.15) holds trivially.*

***Case(ii).*** *If all $x,y,z < 0$, then $Tx = \frac{x}{2}$, $Ty = \frac{y}{2}$, $Tz = \frac{z}{2}$. Therefore,*

$$\overline{S}(Tx,Ty,Tz) = \overline{S}\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right) = e^{\left(\frac{x+y+z}{2}\right)}.$$

*On the other hand*

$$\overline{S}(x,y,z) = e^{x+y+z}, \quad \overline{S}(x,x,Tx) = \overline{S}(x,x,\frac{x}{2}) = e^{\frac{5x}{2}}.$$

*Similarly,*

$$\overline{S}(y,y,Ty) = e^{\frac{5y}{2}} \quad \text{and} \quad \overline{S}(z,z,Tz) = e^{\frac{5z}{2}}.$$

*Therefore,*

$$\begin{aligned}\left(\overline{S}(x,y,z)\right)^{\alpha_1}\left(\overline{S}(x,x,Tx)\right)^{\alpha_2}\left(\overline{S}(y,y,Ty)\right)^{\alpha_3}\left(\overline{S}(z,z,Tz)\right)^{\alpha_4} &= \left(e^{\frac{5z}{2}}\right)^{\frac{1}{5}}\\ &= e^{\frac{z}{2}}\\ &\geq e^{\frac{x+y+z}{2}}\\ &= \overline{S}(Tx,Ty,Tz).\end{aligned}$$

*Thus,*

$$\overline{S}(Tx,Ty,Tz)\leq\left(\overline{S}(x,y,z)\right)^{\alpha_1}\left(\overline{S}(x,x,Tx)\right)^{\alpha_2}\left(\overline{S}(y,y,Ty)\right)^{\alpha_3}\left(\overline{S}(z,z,Tz)\right)^{\alpha_4}.$$

***Case(iii).*** *Let any two of $x,y,z$ are greater than or equal to 0 and let other be less than 0. Without loss of generality, assume that $x,y\geq 0$ and $z<0$. Then $Tx=0,\ Ty=0$ and $Tz=\dfrac{z}{2}$. Therefore,*

$$\overline{S}(Tx,Ty,Tz)=\overline{S}\left(0,0,\frac{z}{2}\right)=e^{\frac{z}{2}}.$$

*On the other hand*

$$\overline{S}(x,y,z)=e^{x+y+z},\ \ \overline{S}(x,x,Tx)=\overline{S}(x,x,0)=e^{2x}.$$

*Similarly,*

$$\overline{S}(y,y,Ty)=\overline{S}\left(y,y,\frac{y}{2}\right)=e^{\frac{5y}{2}},\ \text{ and }\ \overline{S}(z,z,Tz)=\overline{S}\left(z,z,\frac{z}{2}\right)=e^{\frac{5z}{2}}.$$

*Therefore,*

$$\begin{aligned}\left(\overline{S}(x,y,z)\right)^{\alpha_1}\left(\overline{S}(x,x,Tx)\right)^{\alpha_2}\left(\overline{S}(y,y,Ty)\right)^{\alpha_3}\left(\overline{S}(z,z,Tz)\right)^{\alpha_4} &= \left(e^{\frac{5z}{2}}\right)^{\frac{1}{5}}\\ &= e^{\frac{z}{2}}\\ &= \overline{S}(Tx,Ty,Tz).\end{aligned}$$

*Thus,*

$$\overline{S}(Tx,Ty,Tz)\leq\left(\overline{S}(x,y,z)\right)^{\alpha_1}\left(\overline{S}(x,x,Tx)\right)^{\alpha_2}\left(\overline{S}(y,y,Ty)\right)^{\alpha_3}\left(\overline{S}(z,z,Tz)\right)^{\alpha_4}.$$

***Case(iv).*** *Let any two of $x,y,z$ are less than 0 and let other be greater than or equal to 0. Without loss of generality, assume that $x\geq 0$ and $y,z<0$. Then $Tx=0,\ Ty=\frac{y}{2}$ and $Tz=\dfrac{z}{2}$. Therefore,*

$$\overline{S}(Tx,Ty,Tz)=\overline{S}\left(0,\frac{y}{2},\frac{z}{2}\right)=e^{\left(\frac{y+z}{2}\right)}.$$

*On the other hand*

$$\overline{S}(x,y,z) = e^{x+y+z}, \ \ \overline{S}(x,x,Tx) = \overline{S}(x,x,0) = e^{2x}.$$

*Similarly,*

$$\overline{S}(y,y,Ty) = \overline{S}\left(y,y,\frac{y}{2}\right) = e^{\frac{5y}{2}}, \ \ \textit{and} \ \ \overline{S}(z,z,Tz) = \overline{S}\left(z,z,\frac{z}{2}\right) = e^{\frac{5z}{2}}.$$

*Therefore,*

$$\begin{aligned}\left(\overline{S}(x,y,z)\right)^{\alpha_1}\left(\overline{S}(x,x,Tx)\right)^{\alpha_2}\left(\overline{S}(y,y,Ty)\right)^{\alpha_3}\left(\overline{S}(z,z,Tz)\right)^{\alpha_4} &= \left(e^{\frac{5z}{2}}\right)^{\frac{1}{5}} \\ &= e^{\frac{z}{2}} \\ &\geq e^{\left(\frac{y+z}{2}\right)} \\ &= \overline{S}(Tx,Ty,Tz).\end{aligned}$$

*Hence,*

$$\overline{S}(Tx,Ty,Tz) \leq \left(\overline{S}(x,y,z)\right)^{\alpha_1}\left(\overline{S}(x,x,Tx)\right)^{\alpha_2}\left(\overline{S}(y,y,Ty)\right)^{\alpha_3}\left(\overline{S}(z,z,Tz)\right)^{\alpha_4}.$$

*Therefore, the inequality (3.15) is satisfied. Hence, all assumptions of Theorem3.1 are fulfilled. Consequently, by Theorem 3.1 the mapping $T$ has a unique fixed point in $x = 0 \in X$.*

## 4. Conclusion

In this paper, we have established existence and uniqueness results for fixed points of self-maps in $S$-multiplicative metric spaces under various contractive conditions. An illustrative example is provided to demonstrate the applicability of the result.

## Conflict of Interests

The author(s) declare that there is no conflict of interests.

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