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SOME REFINEMENTS OF THE HERMITE-HADAMARD INEQUALITY CONCERNING PRODUCTS OF CONVEX FUNCTIONS

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Abstract. In this paper, refinements and new results concerning the Hermite-Hadamard's inequality concerning products of convex functions are presented.

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1. Introduction

A real-valued function $f$is said to be convex on a closed interval I if

$f\,(tx + (1-t)y) \,\le\, tf(x) + (1-t)f(y)$, for all $x,\, y \in I,\;\; 0 \le t \le 1$. If the inequality is reversed, the f is called concave. It is known that $f$is convex if $f''(x)\,\ge 0$.

The inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \,\le \frac{f(a)+f(b)}{2} \tag{1}$$

which holds for all convex mapping $f : [a, b] \to \Re$, is known in the literature as Hadamard's inequality. In [2], Fejér generalized Hadamard's inequality by giving the following :

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Theorem 1.1. *If $g : [a,b] \to \Re$ is non-negative integrable and symmetric to $x = \frac{a+b}{2}$, and if f is convex on [a,b], then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \le \int_a^b f(x)\,g(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{2}$$

2. Main Results

Lemma 2.1. *If $f, g : I \to \Re$ are positive convex functions such that*

$$(f(a)-f(b))\,(g(a)-g(b)) > 0, \quad \forall a, b \in I, \tag{3}$$

then $fg : I^2 \to \Re$ is convex.

Proof. *By the hypothesis, we have for all $a,\ b \in I$,*

$$\begin{aligned} f(a)g(b)+f(b)g(a) &\le f(a)g(a)+f(b)g(b) \\ &\Rightarrow \quad f(a)g(b)+f(b)g(a)+f(a)g(a)+f(b)g(b) \le 2\,(f(a)g(a)+f(b)g(b)) \\ &\Rightarrow \frac{(f(a)+f(b))}{2}\frac{g(a)+g(b)}{2} \le \frac{f(a)g(a)+f(b)g(b)}{2}. \end{aligned}$$

Since $f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2}$, *then*

$$\begin{aligned} (fg)\left(\frac{a+b}{2}\right) &= f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2} \\ &\le \frac{(fg)(a)+(fg)(b)}{2}. \end{aligned}$$

Theorem 2.2. *Let $f, g, h\ : I \supset [x,y] \to \Re$ be positive convex functions such that (3) is satisfied, his integrable and symmetric to $t = (x+y)/2$. Then the following inequalities hold*

$$(fg)\left(\frac{x+y}{2}\right) \le \frac{1}{y-x}\int_x^y (fg)(u)\,du \le \frac{(fg)(x)+(fg)(y)}{2}, \tag{4}$$

$$(fg)\left(\frac{x+y}{2}\right)\int_x^y h(u)\,du \le \int_x^y (fgh)(u)\ dx \le \frac{(fg)(x)+(fg)(y)}{2}\int_a^b h(u)\,du. \tag{5}$$

Proof. *The proof follows from (1) and (2) .*

Theorem 2.3. *Assume that* $f : I \to \Re$ *is a convex function on* $I = [a, b]$. *Then for all* $c \in [a, b],\ \ c = (1 - \lambda)a + \lambda b,\ \ \lambda \in [0, 1]$, *we have*

$$f\left(\frac{a+b}{2}\right) \leq l(c) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \ \leq L(c) \leq \frac{f(a)+f(b)}{2} \tag{6}$$

where

$$l(c) \ = \ \lambda\, f\left(\frac{a+c}{2}\right) \ + \ (1-\lambda)\, f\left(\frac{b+c}{2}\right),$$

$$L(c) \ = \ \frac{1}{2}\left(\,\lambda\, f(a) \ + f(c) \ + \ (1-\lambda)\, f(b)\right)\, .$$

Proof. We have

$$c - a = \lambda(b-a), \quad b - c = (1-\lambda)(b-a)\, .$$

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &= f\left(\frac{\lambda(a+c)+(1-\lambda)(c+b)}{2}\right) \\
&\leq \lambda\, f\left(\frac{a+c}{2}\right) \ + \ (1-\lambda)\, f\left(\frac{c+b}{2}\right) \qquad (= l(c)) \\
&\leq \frac{\lambda}{c-a}\int_a^c f(t)\,dt \ + \ \frac{1-\lambda}{b-c}\int_c^b f(t)\,dt \\
&= \frac{1}{b-a}\int_a^c f(t)\,dt \ + \ \frac{1}{b-a}\int_c^b f(t)\,dt \qquad \left(= \frac{1}{b-a}\int_a^b f(t)\,dt\right) \\
&\leq \lambda\, \frac{f(a)+f(c)}{2} \ + \ (1-\lambda)\, \frac{f(c)+f(b)}{2} \\
&= \frac{1}{2}\left(\,\lambda\, f(a) \ + f(c) \ + \ (1-\lambda)\, f(b)\right) \quad (= \ L(c)) \\
&\leq \frac{1}{2}\left(\,\lambda\, f(a) \ + (1-\lambda)\, f(a) \ + \ \lambda\, f(b) \ + \ (1-\lambda)\, f(b)\right) \\
&= \frac{f(a)+f(b)}{2}\, .
\end{aligned}$$

Theorem 2.4. *Assume that $f, g : I \to \Re$ be positive convex functions on $I = [a, b]$. Then for all $c \in [a, b]$, $c = (1 - \lambda)a + \lambda b$, $\lambda \in [0, 1]$, we have*

$$(fg)\left(\frac{a+b}{2}\right) \leq l_f(c)l_g(c) \leq \frac{1}{b-a}\int_a^b f(x)g(x)\,dx \leq L_f(c)L_g(c)$$

$$\leq \frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}. \tag{7}$$

where

$$l_f(c) = \lambda f\left(\frac{a+c}{2}\right) + (1-\lambda) f\left(\frac{b+c}{2}\right), \quad l_g(c) = \lambda g\left(\frac{a+c}{2}\right) + (1-\lambda) g\left(\frac{b+c}{2}\right)$$

$$L_f(c) = \frac{1}{2}\left(\lambda f(a) + f(c) + (1-\lambda) f(b)\right), \quad L_g(c) = \frac{1}{2}\left(\lambda g(a) + f(c) + (1-\lambda) g(b)\right).$$

Proof. Applying Theorem 2.3 twice, we have

$$f\left(\frac{a+b}{2}\right) \leq l_f(c) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq L_f(c) \leq \frac{f(a)+f(b)}{2}, \tag{8}$$

$$g\left(\frac{a+b}{2}\right) \leq l_g(c) \leq \frac{1}{b-a}\int_a^b g(x)\,dx \leq L_g(c) \leq \frac{g(a)+g(b)}{2}. \tag{9}$$

The proof follows by multiplying (8) and (9).

Theorem 2.5. *Assume that $f, g : I \to \Re$ be positive convex functions on $I = [a, b]$ such that f and g are both non-increasing or non-decreasing. Then for all $c \in [a, b]$, $c = (1 - \lambda)a + \lambda b$, $\lambda \in [0, 1]$, we have*

$$(fg)\left(\frac{a+b}{2}\right) \leq l_f(c)l_g(c) \leq \frac{1}{b-a}\int_a^b f(x)\,dx\,\frac{1}{b-a}\int_a^b g(x)\,dx \leq \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx$$

$$\leq F(a,b), \tag{10}$$

where $l_f(c)$, $l_g(c)$ are as defined in Theorem 2.4 and

$$F(a,b) = \min\left\{\left(\frac{f^p(a)+f^p(b)}{2}\right)^{1/p}\left(\frac{g^q(a)+g^g(b)}{2}\right)^{1/q}, \frac{1}{3}\left(f(a)+f(b)\right)\left(g(a)+g(b)\right)\right\}.$$

Proof. As $p, q > 1$, then f^p, g^q are both convex. Hence via Chebyshev inequality and Theorem 2.4, we have

$$(fg)\left(\frac{a+b}{2}\right) \le l_f(c)l_g(c) \le \frac{1}{b-a}\int_a^b f(x)\,dx\,\frac{1}{b-a}\int_a^b g(x)\,dx \le \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx.$$

Also,

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx &\le \left(\frac{1}{b-a}\int_a^b f^p(x)\,dx\right)^{1/p}\left(\frac{1}{b-a}\int_a^b g^q(x)\,dx\right)^{1/q} \\ &\le \left(\frac{f^p(a)+f^p(b)}{2}\right)^{1/p}\left(\frac{g^q(a)+g^g(b)}{2}\right)^{1/q}. \end{aligned}$$

and,

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx &= \int_0^1 f\left((1-\lambda)a+\lambda b\right) g\left((1-\lambda)a+\lambda b\right) d\lambda \\ &\le \int_0^1 \left((1-\lambda)f(a)+\lambda f(b)\right)\left((1-\lambda)g(a)+\lambda g(b)\right)\,d\lambda \\ &= \frac{1}{3}\left(f(a)g(a)+f(b)g(b)+f(a)g(b)+f(b)g(a)\right) \\ &= \frac{1}{3}\left(f(a)+f(b)\right)\left(g(a)+g(b)\right). \end{aligned}$$

The Theorem follows.

A positive function f is said to be log-convex if $\log f$ is convex function. Concerning such function, we have

Theorem 2.6. *Let $f,\, g : [a,b] \to \Re$ such that f is convex and g is log-convex, both f and $\log g$ are positive, and that one of these functions are increasing and the other decreasing. Then the following inequality holds*

$$\frac{1}{b-a}\int_a^b g^{f(x)}(x)\,dx \le \left(g(a)\,g(b)\right)^{\frac{f(a)+f(b)}{4}}. \tag{11}$$

Proof. Applying Chebyshev inequality, we have

$$\begin{aligned}
\frac{1}{b-a}\int_a^b g^{f(x)}(x)\,dx &= \exp\left(\ln\left(\frac{1}{b-a}\int_a^b g^{f(x)}(x)\,dx\right)\right) \\
&\leq \exp\left(\frac{1}{b-a}\int_a^b \ln\left(g^{f(x)}(x)\right)dx\right) \\
&= \exp\left(\frac{1}{b-a}\int_a^b f(x)\ln g(x)\,dx\right) \\
&\leq \exp\left(\frac{1}{b-a}\int_a^b f(x)\,dx\,.\frac{1}{b-a}\int_a^b \ln g(x)\,dx\right) \\
&\leq \exp\left(\frac{f(a)+f(b)}{2}\right)\left(\frac{\ln g(a)+\ln g(b)}{2}\right) \\
&= (g(a)\,g(b))^{\frac{f(a)+f(b)}{4}}.
\end{aligned}$$

Theorem 2.7. *Let $f,g : I \supset [a,b] \to \Re$ be convex functions such that (3) is satisfied, $a \geq 0$. Let $c \in [a,b]\,,\ c \neq (b-a)/2$. Then*

$$\frac{b-a-3c}{b-a-2c}\int_{a+c}^{b-c}(fg)(t)\,dt \leq \int_a^b (fg)(t)\,dt - 3c\,(fg)\left(\frac{a+b}{2}\right). \tag{12}$$

Proof. As fg is convex, then

$$\begin{aligned}
(fg)\left(\frac{a+b}{2}\right) &= (fg)\left(\frac{a+a+c \quad + \quad a+c+b-c \quad + \quad b-c+c}{6}\right) \\
&\leq \frac{1}{3}\left((fg)\left(\frac{a+a+c}{2}\right) + (fg)\left(\frac{a+c+b-c}{2}\right) + (fg)\left(\frac{b-c+b}{2}\right)\right) \\
&\leq \frac{1}{3}\left(\frac{1}{c}\int_a^{a+c}(fg)(t)\,dt + \frac{1}{b-a-2c}\int_{a+c}^{b-c}(fg)(t)\,dt + \frac{1}{c}\int_{b-c}^{b}(fg)(t)\,dt\right) \\
&= \frac{1}{3c}\left(\int_a^b (fg)(t)\,dt + \frac{3c+a-b}{b-a-2c}\int_{a+c}^{b-c}(fg)(t)\,dt\right),
\end{aligned}$$

which implies

$$\frac{b-a-3c}{b-a-2c}\int\limits_{a+c}^{b-c}(fg)(t)\,dt\ \leq\ \int\limits_{a}^{b}(fg)(t)\,dt-3c\,(fg)\left(\frac{a+b}{2}\right).$$

Corollary 2.8. *Let* $f_1, f_2, g_1, g_2\ :[a,b]\to\Re$ *be positive convex functions such that (3) is satisfied for* f_1, g_1 *and* f_2, g_2*, and* $h:[a,b]\to\Re$ *is positive, integrable and symmetric to* $x=(a+b)/2$*. Then the following inequalities hold*

$$\frac{1}{(f_1g_1)(a)+(f_1g_1)(b)}\int\limits_{a}^{b}(f_1g_1)(x)+\frac{1}{(f_2g_2)(a)+(f_2g_2)(b)}\int\limits_{a}^{b}(f_2g_2)(x)$$

$$\leq\int\limits_{a}^{b}h(x)\,dx \tag{13}$$

Proof. The proof follows from Theorem 1.1(the right inequality) by replacing $f(x)$ by $\dfrac{(f_1g_1)(x)}{(f_1g_1)(a)+(f_1g_1)(b)}+\dfrac{(f_2g_2)(x)}{(f_2g_2)(a)+(f_2g_2)(b)}$ and $g(x)$ by $h(x)$.

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