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SUM PROPERTIES FOR THE K-LUCAS NUMBERS WITH ARITHMETIC INDEXES

BIJENDRA SINGH[1], POOJA BHADOURIA[1], AND OMPRAKASH SIKHWAL[2,*]

[1]School of Studies in Mathematics, Vikram University, Ujjain, India

[2]Department of Mathematics, Mandsaur Institute of Technology Mandsaur, India

Abstract: Fibonacci numbers are fascinating and their impact on the field of mathematics has been great. In this paper, mainly present formulas for the sums of k-Lucas numbers with indexes in an arithmetic sequence, say $an+r$, for fixed integers a and r $(0 \le r \le a-1)$. Also the generating function evaluates and presents the alternating sum for the k-Lucas numbers with arithmetic index.

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1. INTRODUCTION:

The field of Fibonacci is about more than just a sequence. Like any sequence one can analyze its starting values and the ratio of its terms, from which we obtain the generalized Fibonacci sequence and the golden ratio respectively. Fibonacci and Lucas numbers and their generalization have many interesting properties and applications to almost every field of science and art. Besides the usual Fibonacci numbers many kind of generalizations of these numbers have been presented in the literature [9, 10, 11]. In [1, 4] a new generalization of the classical

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*Corresponding author

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Fibonacci sequence is introduced. It should be noted that the recurrence formula of these numbers depends on one real parameter k.

Definition 1: For any integer number $k \geq 1$, the k^{th} Fibonacci sequence, say $\{F_{k,n}\}_{n \in N}$ is defined recurrently by:

$$F_{k,0} = 0, F_{k,1} = 1, and \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1. \qquad (1.1)$$

Particular cases of definition (1.1)

If $k = 1,$ we obtain the classical Fibonacci sequence $\{0,1,1,2,3,5,8,...\}$.

If $k = 2,$ we obtain the Pell sequence $\{0,1,2,5,12,29,70,...\}$.

If $k = 3,$ we obtain the sequence $\{\mathrm{F}_{3,n}\}_{n \in N} = \{0,1,3,10,33,109,...\}$.

Few terms of the k-Fibonacci numbers are

$$F_{k,0} = 0$$
$$F_{k,1} = 1$$
$$F_{k,2} = k$$
$$F_{k,3} = k^2 + 1$$
$$F_{k,4} = k^3 + 2k$$
$$F_{k,5} = k^4 + 3k^2 + 1$$

Many properties of k- Fibonacci numbers are obtained directly from elementary matrix algebra. In [4, 5] several properties of these numbers are deduced that are related with their derivatives and the so called Pascal triangle. Falcon and Plaza [3], gives several formulas for the sum of these numbers with indexes in an arithmetic sequence. Also in [6], authors apply the binomial and the k-binomial transforms to the k-Fibonacci sequences and derives many formulas like generating function and Binet’s formula. In [7] authors defined k- Fibonacci hyperbolic functions and deduced some properties of k- Fibonacci hyperbolic functions related with the analogous identities for the k- Fibonacci numbers. In [2], Falcon studied the k- Lucas numbers and proved various properties related with the k- Fibonacci numbers.

Some of the interesting properties that the k-Fibonacci sequence satisfies are summarized as below [1], [2], [4]:

1. Binet's formula: The Binet's formula for the k-Fibonacci numbers is

$$F_{k,n} = \frac{\mu_1^n - \mu_2^n}{\mu_1 - \mu_2}, \tag{1.2}$$

where $\mu_1 = \frac{k + \sqrt{k^2 + 4}}{2}$ is the positive root of the characteristic equation $r^2 - kr - 1 = 0$ associated to the recurrence relation defined in (1.1).

2. Catalan's identity: $F_{k,n-r}F_{k,n+r} - F_{k,n}^2 = (-1)^{n+1-r} F_{k,r}^2$ (1.3)

3. D'Ocagne Identity: $F_{k,m}F_{k,n+1} - F_{k,m+1}F_{k,n} = (-1)^n F_{k,m-n}$ (1.4)

4. Convolution product: $F_{k,n+m} = F_{k,n+1}F_{k,m} + F_{k,n}F_{k,m-1}$ (1.5)

5. Sum of the first n terms: $\sum_{i=1}^{n} F_{k,i} = \frac{\left(F_{k,n+1} + F_{k,n} - 1\right)}{k}$ (1.6)

6. Sum of the first n even terms: $\sum_{i=1}^{n} F_{k,2i} = \frac{\left(F_{k,2n+1} - 1\right)}{k}$ (1.7)

7. Sum of the first n odd terms: $\sum_{i=1}^{n} F_{k,2i+1} = \frac{F_{k,2n+2}}{k}$ (1.8)

8. Generating function: $f_k(x) = \frac{x}{1 - kx - x^2}$ (1.9)

In [2] Falcon generates the k-Lucas numbers from the k-Fibonacci numbers whose recurrence relation is given below:

Definition 2: For any integer number $k \geq 1$, the k^{th} Lucas sequence, say $\{L_{k,n}\}_{n \in N}$ is given by:

$L_{k,n+1} = kL_{k,n} + L_{k,n-1},\ for\ \ n \geq 1$ with initial conditions $L_{k,0} = 2, and L_{k,1} = k$. (1.10)

As particular cases:

If $k = 1$, we obtain the Lucas sequence $\{2,1,3,4,7,11,18,29,...\}$,

if $k = 2$, we obtain the Pell-Lucas sequence $\{2,2,6,14,34,82,198,...\}$,

if $k = 3$, we obtain the sequence $\{L_{3,n}\}_{n\in N} = \{2,3,11,36,119,393,1298,...\}$.

Few terms of the k-Lucas numbers (1.10) are

$$L_{k,0} = 2$$
$$L_{k,1} = k$$
$$L_{k,2} = k^2 + 2$$
$$L_{k,3} = k^3 + 3k$$
$$L_{k,4} = k^4 + 4k^2 + 2$$
$$L_{k,5} = k^5 + 5k^3 + 5k$$

2. PRELIMINARIES:

Now, some basic properties of k-Lucas numbers are describe.

1. Binet's formula:

$$L_{k,n} = \mu_1^n + \mu_2^n, where \mu_1 = \frac{k+\sqrt{k^2+4}}{2} \text{ and } \mu_2 = \frac{k-\sqrt{k^2+4}}{2}. \tag{1.11}$$

These roots verify $\mu_1 + \mu_2 = k, \mu_1.\mu_2 = -1 and \mu_1 - \mu_2 = \sqrt{k^2+4}$.

(See [2] for the proof)

2. $L_{k,m+n} = L_{k,n+1}F_{k,m} + L_{k,n}F_{k,m\text{-}1}$ (1.12)

Proof: Applying the Binet's formula on R. H. S., obtain

$$L_{k,n+1}F_{k,m} + L_{k,n}F_{k,m-1} = \left(\mu_1^{n+1} + \mu_2^{n+1}\right)\left(\frac{\mu_1^m - \mu_2^m}{\mu_1 - \mu_2}\right) + \left(\mu_1^n + \mu_2^n\right)\left(\frac{\mu_1^{m-1} - \mu_2^{m-1}}{\mu_1 - \mu_2}\right)$$

$$= \frac{1}{\mu_1 - \mu_2}\left[\left(\mu_1^{n+1} + \mu_2^{n+1}\right)\left(\mu_1^m - \mu_2^m\right) + \left(\mu_1^n + \mu_2^n\right)\left(\mu_1^{m-1} - \mu_2^{m-1}\right)\right]$$

$$L_{k,n+1}F_{k,m}+L_{k,n}F_{k,m-1}=\frac{1}{\mu_1-\mu_2}\left[\mu_1^{m+n}\left(\mu_1-\mu_2\right)-\mu_2^{m+n}\left(\mu_2-\mu_1\right)\right] \quad \left(\because \mu_1.\mu_2=-1\right)$$

$$=\left(\mu_1^{m+n}+\mu_2^{m+n}\right)=L_{k,m+n}. \qquad \text{[From (1.11)]}$$

2. $F_{k,m}L_{k,n+1}-F_{k,m+1}L_{k,n}=\left(-1\right)^{n+1}L_{k,m-n}$ (1.13)

Proof: Mathematical induction method use for proof of this. If $n=0$, then

$$F_{k,m}L_{k,n+1}-F_{k,m+1}L_{k,n}=F_{k,m}L_{k,1}-F_{k,m+1}L_{k,0}$$

$$=F_{k,m}.k-F_{k,m+1}.2=-\left(F_{k,m-1}+F_{k,m+1}\right) \qquad \text{[From (1.1)]}$$

$$=-L_{k,m} \qquad [\because L_{k,n}=F_{k,n+1}+F_{k,n-1}]$$

If $n=1$, then

$$F_{k,m}L_{k,2}-F_{k,m+1}L_{k,1}=\left(k^2+2\right)F_{k,m}-kF_{k,m+1}=-kF_{k,m-1}+2F_{k,m} \qquad \text{[From (1.1)]}$$

$$=F_{k,m-2}-F_{k,m}+2F_{k,m}=\left(F_{k,m-2}+F_{k,m}\right)$$

$$=L_{k,m-1} \qquad [\because L_{k,n}=F_{k,n+1}+F_{k,n-1}]$$

Now, suppose the formula is true until $(n-1)$:

$$\left.\begin{aligned}F_{k,m}L_{k,n-1}-F_{k,m+1}L_{k,n-2}&=\left(-1\right)^{n-1}L_{k,m-(n-2)},\\ F_{k,m}L_{k,n}-F_{k,m+1}L_{k,n-1}&=\left(-1\right)^{n}L_{k,m-(n-1)}\end{aligned}\right\}\text{Then,}$$

$$F_{k,m}L_{k,n+1}-F_{k,m+1}L_{k,n}=F_{k,m}\left(kL_{k,n}+L_{k,n-1}\right)-F_{k,m+1}\left(kL_{k,n-1}+L_{k,n-2}\right) \text{[From (1.10)]}$$

$$=k\left(F_{k,m}L_{k,n}-F_{k,m+1}L_{k,n-1}\right)+\left(F_{k,m}L_{k,n-1}-F_{k,m+1}L_{k,n-2}\right)$$

$$=\left(-1\right)^{n}\left(kL_{k,m-(n-1)}-L_{k,m-(n-2)}\right)=\left(-1\right)^{n+1}L_{k,m-n}.$$

3. ON THE k-LUCAS NUMBERS OF KIND $an+r$:

In this section, some interesting formulas are derive for the sums of the k-Lucas numbers with index in an arithmetic sequence, say $an+r$ for fixed integer a and r such that $0 \le r \le a-1$. Also, describe generating function for these numbers with index in an arithmetic sequence.

Let us prove following lemmas that we will need after some time:

Lemma 1: For all integers $n \ge 1$,

$$\mu_1^n + \mu_2^n = F_{k,n+1} + F_{k,n-1} \quad \text{(see [3] for the detail of proof)} \tag{2.1}$$

Lemma 2: $L_{k,a(n+2)+r} = \left(F_{k,a-1} + F_{k,a+1}\right) L_{k,a(n+1)+r} - (-1)^a L_{k,an+r}$.

Proof: Applying Binet's formula and Lemma 1 on R. H. S., obtain

$$\left(F_{k,a-1} + F_{k,a+1}\right) L_{k,a(n+1)+r} - (-1)^a L_{k,an+r}$$

$$= (\mu_1^a + \mu_2^a)\left(\mu_1^{a(n+1)+r} + \mu_2^{a(n+1)+r}\right) - (-1)^a L_{k,an+r}$$

$$= \mu_1^{a(n+2)+r} + \mu_2^{a(n+2)+r} + (-1)^a \left(\mu_1^{an+r} + \mu_2^{an+r}\right) - (-1)^a L_{k,an+r}$$

$$= L_{k,a(n+2)+r} \qquad \text{[From (1.11)]}$$

Since $L_{k,n} = F_{k,n+1} + F_{k,n-1}$, then the above formula can be rewritten as

$$L_{k,a(n+2)+r} = L_{k,a} L_{k,a(n+1)+r} - (-1)^a L_{k,an+r} . \tag{2.2}$$

Formula (2.2) gives the general term of the k-Lucas sequence $\left\{L_{k,an+r}\right\}_{n=0}^{\infty}$ as a linear combination of the two preceding terms.

2.1 GENERATING FUNCTION FOR THE SEQUENCE $\left\{L_{k,an+r}\right\}_{n=0}^{\infty}$:

Suppose that $l_{a,r}(\mathrm{k,x})$ be the generating function of the sequence $\left\{L_{k,an+r}\right\}$ with $0 \le r \le a-1$.

That is

$$l_{a,r}(\mathrm{k},\mathrm{x}) = L_{k,r} + L_{k,a+r}x + L_{k,2a+r}x^2 + L_{k,3a+r}x^3 + \ldots. \tag{2.3}$$

Now multiplying both sides by the algebraic expression $?1 - L_{(k,a)}x + (-1)^a x^2)$, obtain

$$\left(1 - L_{k,a}x + (-1)^a x^2\right) l_{a,r}(k,x)$$

$$= \left(1 - L_{k,a}x + (-1)^a x^2\right)\left(L_{k,r} + L_{k,a+r}x + L_{k,2a+r}x^2 + L_{k,3a+r}x^3 + \ldots\right)$$

$$= L_{k,r} + \left(L_{k,a+r} - L_{k,a}L_{k,r}\right)x + \left(L_{k,2a+r} - L_{k,a}L_{k,a+r} + (-1)^a L_{k,r}\right)x^2 + \ldots$$

$$= L_{k,r} + \left(L_{k,a+r} - L_{k,a}L_{k,r}\right)x + \sum_{n\geq 2}\left(L_{k,a(n+2)+r} - L_{k,a}L_{k,a(n+1)+r} + (-1)^a L_{k,an+r}\right)x^n$$

Now, from equation (2.2), the summation of right hand side of the above equation vanishes.That is

$$\left(1 - L_{k,a}x + (-1)^a x^2\right) l_{a,r}(k,x) = L_{k,r} + \left(L_{k,a+r} - L_{k,a}L_{k,r}\right)x \tag{2.4}$$

From (1.12), we have

$$L_{k,\mathrm{a+r}} = L_{k,\mathrm{r+1}}F_{k,\mathrm{a}} + L_{k,\mathrm{r}}F_{k,\mathrm{a-1}}$$

$$L_{k,a+r} - L_{k,r}L_{k,a} = L_{k,r+1}F_{k,a} - L_{k,r}F_{k,a+1}$$

$$L_{k,a+r} - L_{k,r}L_{k,a} = (-1)^{r+1} L_{k,a-r} \qquad \text{[From (1.13)]}$$

Hence equation (2.4) becomes,

$$\left(1 - L_{k,a}x + (-1)^a x^2\right) l_{a,r}(k,x) = L_{k,r} + (-1)^{r+1} L_{k,a-r}x$$

$$l_{a,r}(k,x) = \frac{L_{k,r} + (-1)^{r+1} L_{k,a-r}x}{\left(1 - L_{k,a}x + (-1)^a x^2\right)} \tag{2.5}$$

Particular cases:

The generating function of sequence $\{L_{k,an+r}\}$ for values of a and r are:

1) If $a = 1$ and then $r = 0$:

$l_{1,0}(k,x) = \frac{L_{k,0} + (-1)L_{k,1}x}{(1 - L_{k,1}x + (-1)x^2)} = \frac{2 - kx}{1 - kx - x^2}$, which is the generating function of the k-Lucas sequence as it is given in [2].

2) If $a = 2$, then

a) At $r = 0$, then $l_{2,0}(k,x) = \frac{L_{k,0} + (-1)L_{k,2}x}{1 - L_{k,2}x + (-1)^2 x^2} = \frac{2 - (k^2 + 2)x}{1 - (k^2 + 2)x + x^2}$

b) At $r = 1$, then $l_{2,1}(k,x) = \frac{L_{k,1} + (-1)^2 L_{k,1}x}{1 - L_{k,2}x + (-1)^2 x^2} = \frac{k(1 + x)}{1 - (k^2 + 2)x + x^2}$

3) If $a = 3$, then

a) At $r = 0$, then $l_{3,0}(k,x) = \frac{L_{k,0} + (-1)L_{k,3}x}{1 - L_{k,3}x + (-1)^3 x^2} = \frac{2 - (k^3 + 3k)x}{1 - (k^3 + 3k)x - x^2}$

b) At $r = 1$, then $l_{3,1}(k,x) = \frac{L_{k,1} + (-1)^2 L_{k,2}x}{1 - L_{k,3}x + (-1)^3 x^2} = \frac{k + (k^2 + 2)x}{1 - (k^3 + 3k)x - x^2}$

c) At $r = 1$, then $l_{3,2}(k,x) = \frac{L_{k,2} + (-1)^3 L_{k,1}x}{1 - L_{k,3}x + (-1)^3 x^2} = \frac{(k^2 + 2) - kx}{1 - (k^3 + 3k)x - x^2}$.

2.2 SUM OF k-LUCAS NUMBERS WITH ARITHMETIC INDEX $an + r$:

In this section, present and derive the sums of k-Lucas numbers with arithmetic index $an + r$, where a and r are fixed integer such that $0 \le r \le a - 1$.

Theorem 3: Sum of k-Lucas number of kind $an + r$, is

$$\sum_{i=0}^{n} L_{k,ai+r} = \frac{L_{k,a(n+1)+r} - (-1)^a L_{k,an+r} - L_{k,r} + (-1)^r L_{k,a-r}}{L_{k,a} - (-1)^a - 1} \tag{2.6}$$

Proof: Applying Binet's formula for the k-Lucas numbers, we have

$$\sum_{i=0}^{n} L_{k,ai+r} = \sum_{i=0}^{n} \left(\mu_1^{ai+r} + \mu_2^{ai+r}\right) = \sum_{i=0}^{n} \mu_1^{ai+r} + \sum_{i=0}^{n} \mu_2^{ai+r}$$

$$=\frac{\mu_1^r(\mu_1^{a(n+1)}-1)}{\mu_1^a-1}+\frac{\mu_2^r(\mu_2^{a(n+1)}-1)}{\mu_2^a-1}$$

$$=\frac{(\mu_1\mu_2)^a\,\mu_1^{an+r}-\mu_1^{an+a+r}-\mu_1^r\mu_2^a+\mu_1^r+(\mu_1\mu_2)^a\,\mu_2^{an+r}-\mu_2^{an+a+r}-\mu_2^r\mu_1^a+\mu_2^r}{(\mu_1^a-1)(\mu_2^a-1)}$$

$$=\frac{(-1)^a(\mu_1^{an+r}+\mu_2^{an+r})-(\mu_1^{an+a+r}+\mu_2^{an+a+r})+(\mu_1^r+\mu_2^r)-(\mu_1\mu_2)^r(\mu_1^{a-r}+\mu_2^{a-r})}{(-1)^a-(\mu_1^a+\mu_2^a)+1}$$

$$=\frac{L_{k,a(n+1)+r}-(-1)^a L_{k,an+r}-L_{k,r}+(-1)^r L_{k,a-r}}{L_{k,a}-(-1)^a-1}$$

Corollary 4: Formula for sum of odd k-Lucas numbers:

If $a=2s+1$, then equation (2.6) gives

$$\sum_{i=0}^{n}L_{k,(2s+1)i+r}=\frac{L_{k,(2s+1)(n+1)+r}+L_{k,(2s+1)n+r}-L_{k,r}+(-1)^r L_{k,(2s+1)-r}}{L_{k,2s+1}}$$

For example: (1) If $s=0$ then $a=1, r=0$ and

$$\sum_{i=0}^{n}L_{k,i}=\frac{L_{k,n+1}+L_{k,n}-L_{k,0}+L_{k,1}}{L_{k,1}}=\frac{L_{k,n+1}+L_{k,n}-2+k}{k}$$

a) If k=1, then it is sum formula for Lucas sequence, that is

$$\sum_{i=0}^{n}L_i=L_{n+1}+L_n-1=L_{n+2}-1\ [2]$$

b) If k= 2, then the sum formula for the Pell-Lucas sequence, that is

$$\sum_{i=0}^{n}P_i=\frac{P_{n+1}+P_n}{2}$$

(2) If $s=1 \rightarrow a=3$, then

$$\sum_{i=0}^{n}L_{k,3i+r}=\frac{L_{k,3(n+1)+r}+L_{k,3n+r}-L_{k,r}+(-1)^r L_{k,3-r}}{L_{k,3}}$$

a) If r = 0, then

$$\sum_{i=0}^{n} L_{k,3i} = \frac{L_{k,3(n+1)} + L_{k,3n} - L_{k,0} + L_{k,3}}{L_{k,3}} = \frac{L_{k,3n+3} + L_{k,3n} - 2 + (k^3 + 3k)}{(k^3 + 3k)}$$

If k=1, then

$$\sum_{i=0}^{n} L_{3i} = \frac{L_{3n+3} + L_{3n} + 2}{4}$$

b) If r = 1, then

$$\sum_{i=0}^{n} L_{k,3i+1} = \frac{L_{k,3n+4} + L_{k,3n+1} - L_{k,1} + L_{k,2}}{L_{k,3}} = \frac{L_{k,3n+4} + L_{k,3n+1} - k^2 - k - 2}{(k^3 + 3k)}$$

If k=1, then

$$\sum_{i=0}^{n} L_{3i+1} = \frac{L_{3n+4} + L_{3n+1} - 4}{4}$$

c) If r = 2, then

$$\sum_{i=0}^{n} L_{k,3i+2} = \frac{L_{k,3n+5} + L_{k,3n+2} - L_{k,2} + L_{k,1}}{L_{k,3}} = \frac{L_{k,3n+5} + L_{k,3n+2} - k^2 + k - 2}{(k^3 + 3k)}$$

If k=1, then

$$\sum_{i=0}^{n} L_{3i+2} = \frac{L_{3n+5} + L_{3n+2} - 2}{4}$$

(3) If s = 2 $\rightarrow$ a = 5, then

$$\sum_{i=0}^{n} L_{k,5i+r} = \frac{L_{k,5(n+1)+r} + L_{k,5n+r} - L_{k,r} + (-1)^r L_{k,5-r}}{L_{k,5}}$$

a) If r = 0, then $\displaystyle\sum_{i=0}^{n} L_{k,5i} = \frac{L_{k,5n+5} + L_{k,5n} + k^5 + 5k^3 + 5k - 2}{k^5 + 5k^3 + 5k}$

b) If r = 1, then $\displaystyle\sum_{i=0}^{n} L_{k,5i+1} = \frac{L_{k,5n+6} + L_{k,5n+1} - k^4 - 4k^2 - k - 2}{k^5 + 5k^3 + 5k}$

c) If r = 2, then $\displaystyle\sum_{i=0}^{n} L_{k,5i+2} = \frac{L_{k,5n+7} + L_{k,5n+2} + k^3 - k^2 + 3k - 2}{k^5 + 5k^3 + 5k}$

d) If r = 3, then $\displaystyle\sum_{i=0}^{n} L_{k,5i+3} = \frac{L_{k,5n+8} + L_{k,5n+3} - k^3 - k^2 - 3k - 2}{k^5 + 5k^3 + 5k}$

e) If $r = 4$, then $\sum_{i=0}^{n} L_{k,5i+4} = \frac{L_{k,5n+9} + L_{k,5n+4} - k^4 - 4k^2 + k - 2}{k^5 + 5k^3 + 5k}$

Corollary 5: Sum of even k-Lucas numbers

If $a = 2s$, then equation (2.6) is

$$\sum_{i=0}^{n} L_{k,2si+r} = \frac{L_{k,2s(n+1)+r} - L_{k,2sn+r} - L_{k,r} + (-1)^r L_{k,2s-r}}{L_{k,2s} - 2}$$

For example: (1) If $s = 1 \rightarrow a = 2$, then

$$\sum_{i=0}^{n} L_{k,2i+r} = \frac{L_{k,2(n+1)+r} - L_{k,2n+r} - L_{k,r} + (-1)^r L_{k,2-r}}{L_{k,2} - 2}$$

a) If $r = 0$: $\sum_{i=0}^{n} L_{k,2i} = \frac{kL_{k,2n+1} + k^2 - k + 2}{k^2}$

For the Lucas sequence, k=1, it is $\sum_{i=0}^{n} L_{2i} = L_{2n+1} + 2$

b) If $r = 1$: $\sum_{i=0}^{n} L_{k,2i+1} = \frac{L_{k,2n+2} - 2}{k}$

For the Lucas sequence, k=1, it is $\sum_{i=0}^{n} L_{2i+1} = L_{2n+2} - 2$

(2) If $s = 2 \rightarrow a = 4$, then

$$\sum_{i=0}^{n} L_{k,4i+r} = \frac{L_{k,4(n+1)+r} - L_{k,4n+r} - L_{k,r} + (-1)^r L_{k,4-r}}{L_{k,4} - 2}$$

a) If $r = 0$, then $\sum_{i=0}^{n} L_{k,4i} = \frac{L_{k,4n+4} - L_{k,4n} + k^4 + 4k^2}{k^4 + 4k^2}$

b) If $r = 1$, then $\sum_{i=0}^{n} L_{k,4i+1} = \frac{L_{k,4n+5} - L_{k,4n+1} - k^3 - 4k}{k^4 + 4k^2}$

c) If $r = 2$, then $\sum_{i=0}^{n} L_{k,4i+2} = \frac{L_{k,4n+6} - L_{k,4n+2}}{k^4 + 4k^2}$

d) If $r = 3$, then $\sum_{i=0}^{n} L_{k,4i+3} = \frac{L_{k,4n+7} - L_{k,4n+3} - k^3 - 4k}{k^4 + 4k^2}$.

Now, we consider the alternating sequence $\{(-1)^n L_{k,an+r}\}$. By the previous method we can also find the sum formula for this sequence. Moreover, generating function for this alternating sequence has been also proved by the previous method.

Theorem 6: Alternating sum of the k-Lucas numbers with index $an+r$ is given by:

$$\sum_{i=0}^{n}(-1)^i L_{k,ai+r} = \frac{(-1)^n L_{k,a(n+1)+r} + (-1)^{n+a} L_{k,an+r} + L_{k,r} - (-1)^{r+1} L_{k,a-r}}{L_{k,a} + (-1)^a + 1}$$

For different values of *a* and *r* the above sum can be written as:

(1) For a=1, r=0 and $\sum_{i=0}^{n}(-1)^i L_{k,\mathrm{i}} = \frac{(-1)^n L_{k,n+1} - (-1)^n L_{k,n} + k + 2}{k}$

(2) For a=2, $\sum_{i=0}^{n}(-1)^i L_{k,2i+r} = \frac{(-1)^n L_{k,2(n+1)+r} + (-1)^n L_{k,2n+r} + L_{k,r} - (-1)^{r+1} L_{k,2-r}}{k^2+4}$

a) If r =0, then $\sum_{i=0}^{n}(-1)^i L_{k,2i} = \frac{(-1)^n L_{k,2n+2} + (-1)^n L_{k,2n} + k^2 + 4}{k^2+4}$

b) If r = 1, then $\sum_{i=0}^{n}(-1)^i L_{k,2i+1} = \frac{(-1)^n L_{k,2n+3} + (-1)^n L_{k,2n+1}}{k^2+4}$

(3) For a=3, $\sum_{i=0}^{n}(-1)^i L_{k,3i+r} = \frac{(-1)^n L_{k,3(n+1)+r} - (-1)^n L_{k,3n+r} + L_{k,r} - (-1)^{r+1} L_{k,3-r}}{k^3+3k}$

a) If r =0, then $\sum_{i=0}^{n}(-1)^i L_{k,3i} = \frac{(-1)^n L_{k,3n+3} - (-1)^n L_{k,3n} + k^3 + 3k + 2}{k^3+3k}$

b) If r = 1, then $\sum_{i=0}^{n}(-1)^i L_{k,3i+1} = \frac{(-1)^n L_{k,3n+4} - (-1)^n L_{k,3n+1} - k^2 + k - 2}{k^3+3k}$

c) If r = 2, then $\sum_{i=0}^{n}(-1)^i L_{k,3i+2} = \frac{(-1)^n L_{k,3n+5} - (-1)^n L_{k,3n+2} + k^2 + k + 2}{k^3+3k}$

CONCLUSION:

The k-Lucas numbers are generated by the k- Fibonacci numbers. In this paper, the sum properties are presented for k- Lucas numbers with index in an arithmetic sequence. In additional, generating function and alternated sum formula for the k- Lucas numbers presented and derived.

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Conflict of Interests

The author declares that there is no conflict of interests.

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