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ON SOME CALCULUS OF VARIATIONS RESULTS IN NON STANDARD SETTING

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Abstract. In the present work, we prove an approximation result in weighted Orlicz-Sobolev spaces and we give an application of this approximation result to a necessary condition in the calculus of variations.

Keywords: Weighted Orlicz-Sobolev spaces, level set, calculus of variations.

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1. Introduction

The calculus of variations may be said to begin with the brachistochrone curve problem raised by Johann Bernoulli(1696). It immediatly occupied of Jakob Bernoulli and the Marquis de l'Hôpital, but Leonhard Euler first elaborated the subject. His contributions began in 1733, and his Elementa Calculi Variationum gave to the science its name. Lagrange contributed to the theory, and Legendre (1786) laid down a method, not entirely satisfactory, for the discrimination of maxima and minima. Isaac Newton and Gottfried Leibnitz also gave some eurly attention to the Subject. To this discrimination Vincenzo Brunacci (1810), Carl Friedrich Gauss (1829), Simon Poisson (1831), Mikhali Ostrogradsky (1834),

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and Carl Jacobi (1837) have been among the contributors. An important general work is that of Sarrus (1842) which was condensed and improved by Cauchy (1844). Other valuable treatisee and memoirs have been written by Strauch(1849), Jellet(1850), Otto Hesse(1857), Alfred Clebsh(1858), and Carll(1885), but perhaps the most important work of the centry is that of Weierstrass. His celebrated course on the theory is epoch-making, and it may be asserted that he was the first to place it on a firm and unquestionable foundation.The 20th and the 23rd Hilbert problems published in 1900 encouraged further development. In the 20th centry David Hilbert, Emmy Noether, Leonida Tonelli, Henri Lebesgue and Jaques Hadamard among others made significant contributions. Marston Morse applied calculus of variations in what is now called Morse theory. Lev Pontryagin, Rockafellar and Clarcke developed new mathematical tools for optimal control theory, a generalisation of calculus of variations.

In this peaper we consider the functionals of the kind

$$J(u) = \int_{\Omega} f(x, u, \nabla u) dx$$

for a bounded domain $\Omega \subset I\!R^N$, and for function u in some weighted Olicz-Sobolev spaces $W^1L_M(\Omega, \rho)$ coresponding to in N-fonction M and to the weight fonction ρ . In the L^p case (when $M(t) = \frac{|t|^p}{p}$ and $\rho(x) = 1$) the surch of suffisent conditions to secure that those functionals attain an extrem value has a long history (see [3]).The most important problem is to verify the weak lower semicontinuity of those functionals with respect to the space involved. Usually this involves hypothesis that the integrand f is convex with respect to the gradient .

In 1992 R.Landes in [3] has studied the revese problem at a fixed level set and an many situations be has showed that if J is weakely lower semi-continous at one fixed (nonvoid) level set then this partical level set is an extrem value of J or the defining function f is convex in the gradient. The above statement for f as function of u (or of x and u) is not hard to prove (see[3]) but when $f = f(x, \nabla u)$ or $f = f(x, u, \nabla u)$ this is due to an approximation result in Sobolev-spaces .

In 2001 E.Azroul and A.Benkirane have studied the same work that R.Landes in the case of Orlicz-Sobolev spaces $W^1L_M(\Omega)$.

Since this approximation is important for possible application in calculus of variations , one of the main purpose in this paper is to extend the above approximation result to the setting of weighted Orlicz-Sobolev spaces $W^1L_M(\Omega,\rho)$. In the first part of this paper we study two simple case $f=f(u)$ and $f=f(x,u)$, in the second part we establich the same approximation in the more general settings of the weighted Orlicz-Sobolev spaces $W^1L_M(\Omega,\rho)$ and the third part of this article is devoted to the application of this approximation in the calculus of variations. However we prove when $f=f(x,\nabla u)$ that if J is weakly lower semi-continuous at one fixed level set H_μ in the space $W^1L_M(\Omega,\rho)$ then H_μ is an extreme value of J or the function f is convex with respect to the gradient .

2. Preliminaries

This section present, some definitions and well-known about N-functions , weighted Orlicz-Sobolev spaces (standard references are in [1], [5] and [8]).

A) The N- functions.

Let $M : I\!R^+ \to I\!R^+$ be an N- function , ie. M is continuous, convex, with $M(t)\succ 0$ for $t\succ 0$, $\quad M(t)/t \to 0$ as $t\to 0$ and $\frac{M(t)}{t}\to\infty$ as $t\to\infty$.

Equivalently. M admits the representation:

$$M(t)=\int_0^t m(\tau)d\tau.$$

Where $m : I\!R^+ \to I\!R^+$ is non-decreasing right continous , with $m(0)=0$, $m(t)\succ 0$ for $t\succ 0$ and $m(t)\to\infty$ as $t\to\infty$ The N-function $\overline{M}$ conjugate to M defined by

$$\overline{M}(t)=\int_0^t \overline{m}(\tau)d\tau.$$

Where $\overline{m} : I\!R^+ \to I\!R^+$ is given bay $\overline{m}(t)=\sup\{s : m(s)\le t\}$. Clearly $\overline{\overline{M}}=M$ and has young's inequality $st\le M(t)+\overline{M}(s)$ for all $s,t\ge 0$.

It is well known that we can assume that m and $\overline{m}$ are continuous and strictly increasing.

We will extend the N-functions into even function on all $I\!R$

The N-function M is said to satisfy the Δ_2-condition every where (resp. infinity) if there exist $k \succ 0$(resp. $t_0 \succ 0$) such that ,$M(2t) \leq kM(t)$ for all $t \geq 0$ (resp.$t \geq t_0$).

B) Orlicz-Sobolev space.

Let Ω be a open subset of $I\!R^N$, and let M be an N-fonction .

The Orlicz classe $K_M(\Omega)$ (resp the Orlicz spaces $L_M(\Omega)$ is the set of all (equivalence classes modulo equality a.e.in Ω of) real-valued measurable functions u defined in Ω and satisfying $\int_\Omega M(u(x))dx \prec \infty$ (resp $\int_\Omega M(\frac{|u(x)|}{\lambda})dx \prec \infty$ for some $\lambda \succ 0$).

$L_M(\Omega)$ is a Banach space under the norm:

$$\|u\|_{M,\Omega} = \inf\left\{\lambda \succ 0 : \int_\Omega M(\frac{|u(x)|}{\lambda})dx \leq 1\right\}. \tag{2.1}$$

The closure in $L_M(\Omega)$ of the set of bounded measurable function with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$ (we have usual $E_M(\Omega) \subset K_M(\Omega) \subset L_M(\Omega)$).

The equality $L_M(\Omega) = E_M(\Omega)$ hold if and only if M satisfies the Δ_2-condition , for all t or for t large according to whether Ω has a infinite measure or note .

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$ where $u \in L_M(\Omega)$ and $v \in L_{\overline{M}}(\Omega)$ and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\|_{\overline{M},\Omega}$.

The space $L_M(\Omega)$ is reflexive if and only if M an $\overline{M}$ satisfy the Δ_2-condition for all t or for t large,according to wheither Ω be infinite measure or note.

We return now to the Orlicz-Sobolev spaces $W^1L_M(\Omega)$ (resp $W^1E_M(\Omega)$) is the space of all function u such that u and its distibutional derivatives up to order 1 lie in $\in L_M(\Omega)$ (resp $\in E_M(\Omega)$) .

It's Banach space under the norm :

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_{M,\Omega}. \tag{2.2}$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of $\prod L_M$ we have the weak topology $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ (resp $W_0^1L_M(\Omega)$ is defined by the closure of $D(\Omega)$ in $W^1E_M(\Omega)$ (resp $W^1L_M(\Omega)$ for the norm 2.2 (resp for the topology $\sigma(\prod L_M, \prod E_{\overline{M}})$.

Definition 2.1 The sequence u_n converges to u in $L_M(\Omega)$ for the modular convergence (denoted by $u_n \to u$ (mod) $L_M(\Omega)$) if $\int_\Omega M(\frac{|u_n - u|}{\lambda})dx \to 0$ as $n \to \infty$ for some $\lambda \succ 0$.

C) Weighted Orlicz- Sobolev space.

Let Ω be a domain in $I\!R^N$, and let M be an N-fonction and $\rho(x)$ be a weight function on Ω, ie .measurable positive a.e on Ω.

The weighted Orlicz classe $K_M(\Omega, \rho)$ (resp the weighted Orlicz space $L_M(\Omega, \rho)$) is the set of all (equivalence classes modulo equality a.e.in Ω) of real-valued mesurable functions u defined in Ω and satisfying $m_\rho(u, M) = \int_\Omega M(|u(x)|)\rho(x)dx \prec \infty$ (resp $m_\rho(\frac{u}{\lambda}, M) = \int_\Omega M(\frac{|u(x)|}{\lambda})\rho(x)dx \prec \infty$ for some $\lambda \succ 0$).

$L_M(\Omega, \rho)$ is a Banach space under the norm:

$$\|u\|_{M,\rho} = \inf\left\{\lambda \succ 0; m(\frac{u}{\lambda}, M) \leq 1\right\}. \tag{2.3}$$

The closure in $L_M(\Omega, \rho)$ of the set of bounded measurable function with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega, \rho)$ (we have usual $E_M(\Omega, \rho) \subset K_M(\Omega, \rho) \subset L_M(\Omega, \rho)$).

The equality $L_M(\Omega, \rho) = E_M(\Omega, \rho)$ hold if and only if M satisfies the Δ_2-condition , for all t or for t large according to whether Ω has a infinite measure or note .

The dual of $E_M(\Omega, \rho)$ can be identified with $L_{\overline{M}}(\Omega, \rho)$ by means of the pairing $\int_\Omega u(x)v(x)\rho(x)dx$ where $u \in L_M(\Omega, \rho)$ and $v \in L_{\overline{M}}(\Omega, \rho)$ and the dual norm on $L_{\overline{M}}(\Omega, \rho)$ is equivalent to $\|\|_{\overline{M},\Omega}$.

The space $L_M(\Omega, \rho)$ is reflexive if and only if M an $\overline{M}$ satisfy the Δ_2-condition for all t or for t large,according to wheither Ω be infinite measure or note.

We return now to the weighted Orlicz-Sobolev spaces $W^1L_M(\Omega, \rho)$ (resp $W^1E_M(\Omega, \rho)$) is the space of all function u such that $u \in L_M(\Omega)$ (resp $u \in E_M(\Omega)$) and its distibutional derivatives up to order 1 lie in $L_M(\Omega, \rho)$ (resp $E_M(\Omega, \rho)$).

It's Banach space under the norm :

$$\|u\|_{1,M,\rho} = \|u\|_M + \|\nabla u\|_{M,\rho}. \tag{2.4}$$

(where $\|u\|_M = \|u\|_{M,\Omega}$). Thus $W^1L_M(\Omega, \rho)$ and $W^1E_M(\Omega, \rho)$) can be identified with subspaces of $\prod L_{M,\rho} = L_M \times \prod L_M(\Omega, \rho)$ we have the weak topology $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$

and $\sigma(\prod L_{M,\rho}, \prod L_{\overline{M},\rho})$.

The space $W_0^1 E_M(\Omega, \rho)$ (resp $W_0^1 L_M(\Omega, \rho)$) is defined by the closure of $D(\Omega)$ in $W^1 E_M(\Omega, \rho)$) (resp $W^1 L_M(\Omega, \rho)$) for the norm (2.4) (resp for the topology $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$.

Definition 2.2. The sequence u_n converges to u in $L_M(\Omega, \rho)$ for the modular convergence (denoted by $u_n \to u$ (mod) $L_M(\Omega, \rho)$) if $\int_\Omega M(\frac{|u_n - u|}{\lambda})\rho(x)dx \to 0$ as $n \to \infty$ for some $\lambda \succ 0$.

Definition 2.3. The sequence u_n converges to u in $W^1 L_M(\Omega, \rho)$ for the modular convergence (denoted by $u_n \to u$ (mod) $W^1 L_M(\Omega, \rho)$) if for some $\lambda \succ 0$ $\int_\Omega M(\frac{|u_n - u|}{\lambda})dx \to 0$ as $n \to \infty$ and $\int_\Omega M(\frac{|D^\alpha(u_n - u)|}{\lambda})\rho(x)dx \to 0$ as $n \to \infty$ for $|\alpha| = 1$.

Lemma 2.4(see [2,lemma 10-1]). Let M be an N-function . If $u_n \in L_M(\Omega)$ converges a.e. to u and u_n bounded in $L_M(\Omega)$, then $u \in L_M(\Omega)$ and $u_n \to u$ for the topology $\sigma(L_M(\Omega), E_{\overline{M}}(\Omega))$.

Lemma 2.5 (see[2, lemma 10-2]). If the sequence $u_n \in L_M(\Omega, \rho)$ converges to u a.e. and bounded in $L_M(\Omega, \rho)$, then $u \in L_M(\Omega, \rho)$ and $u_n \to u$ for the topology $\sigma(L_M(\Omega, \rho), E_{\overline{M}}(\Omega, \rho))$.

D) Compactness results.

Let Ω an open bounded locally-border lipscitzien in $I\!R^N$, ρ the weight function , and the N-function M such that the assumptions **(H)** are satisfied,

(H):There is a real $s \succ 0$ such that:

(H_1) : $(M(t))^{\frac{s}{s+1}}$ be N-function and that $\rho^{-s} \in L^1(\Omega)$.

(H_2) : $\displaystyle\int_1^\infty \frac{t}{M(t)^{1+\frac{s}{N(s+1)}}} dM(t) = \infty$.

(H_3) : $\displaystyle lim_{t\to\infty} \frac{1}{M^{-1}(t)} \int_0^{t^{\frac{s+1}{s}}} \frac{M^{-1}(u)}{u^{1+\frac{s}{N(s+1)}}} du = 0$.

Remark 2.6. In the particlar case where $M(t) = \frac{t^p}{p}$ $(1 \prec p \prec \infty)$, the first part of (H_1) is satisfied if $s \succ \frac{1}{p-1}$.

Theorem 2.7(see [2, theorem 9-5]). Let Ω an open bounded locally-border lipscitzien in $I\!R^N$ and M an N -function .

Suppose that assumptions **(H)** are satisfied.So we have the following compact injection: $W^1L_M(\Omega,\rho) \hookrightarrow\hookrightarrow E_M$.

3.Functional depending on u or x and u.

On a bonded domain $\Omega \subset I\!R^N$, we consider the functional of kind defined by (3.1) or (3.2).

$$J(u) = \int_\Omega f(u)dx. \tag{3.1}$$

$$J(u) = \int_\Omega f(x,u)dx. \tag{3.2}$$

Let's note that the obtained results whithout any restruction on the N-function M. Where $J : L_M(\Omega,\rho) \to \mathbb{R}$, $\rho \in L^1(\Omega)$, $f : I\!R \to I\!R$ or $f : \Omega \times I\!R^N \to \mathbb{R}$ and f is the Carathéodory-function .

For each μ, we write H_μ for the level set of the functional J ie. $H_\mu = \{u \in L_M(\Omega,\rho)/J(u) = \mu\}$ $\overline{H}^w_\mu$ for the closure of H_μ in $W^1L_M(\Omega,\rho)$ for the weakly topology $\sigma(L_M(\Omega,\rho), E_{\overline{M}}(\Omega,\rho))$

Definition 3.1. A functional $J : L_M(\Omega,\rho) \to I\!R^N$ is weakly lower semicontinuous at a level set H_μ , if $J(u) \le \mu$ for all $u \in \overline{H}^w_\mu$.

Remark 3.2. Note that this definition does not imply that $J_{/\overline{H}^w_\mu}$ is weakly lower semicontinuous .

Theorem 3.3. Let $f : I\!R \to I\!R$ be a real valued such that $J(u) = \displaystyle\int_\Omega f(u)dx$ is defined for all $u \in L_M(\Omega,\rho)$ and $\rho \in L^1(\Omega)$.

Suppose that for some fixed $\mu \in I\!R$ the level set H_μ is nonvoid and J is weakly lower semicontinous at H_μ, then we have : Either the value μ is an extreme value of the functional J or f is convex.

Proof of theorem 3.3

If μ is not an value extreme of J then there are α_1 and α_2 such that $|\Omega|\, f(\alpha_1) \prec \mu \prec |\Omega|\, f(\alpha_2)$, where $|\Omega| = \displaystyle\int_\Omega dx$. Suppose that $\xi, \xi^* \in I\!R$ and $\lambda \in [0,1]$ are given. We fix the numbers $\alpha = \min\left\{\frac{\mu}{|\Omega|} - f(\alpha_1); f(\alpha_2) - \frac{\mu}{|\Omega|}\right\}$ and $\beta = \min\{|f(\xi)|\,; |f(\xi^*)|\}$.

We define the sequence:

$$v_n(x) = \xi g_n(x) + \xi^*(1 - g_n(x)). \tag{3.3}$$

where $g_n(x) = g_\lambda(nx_1), \lambda \in [0,1]$,and $x = (x_1, ..., x_N)$.

$$g_\lambda(t) = \begin{cases} 1 & \text{if} \quad 0 \prec t \prec \lambda \\ 0 & \text{if} \quad \lambda \prec t \prec 1 \end{cases} \tag{3.4}$$

We recall the fact that (see [3]) $g_n(x) \rightharpoonup^* \lambda$ in $L^\infty(\Omega)$ weak star and $(1 - g_n(x)) \rightharpoonup^* (1 - \lambda)$ in $L^\infty(\Omega)$ weak star. We fix a ball $B \subset \Omega$ small enough such that $\mu - \frac{\mu}{|\Omega|} |\Omega \backslash B| \prec \alpha |\Omega \backslash B| - \beta |B|$. For $r \in I\!R_+^*$, we write $\Omega_r = \{x \in \Omega \backslash B : x_1 \prec r\}$, as $f(\alpha_1) [|B| + |\Omega \backslash (B \cup \Omega_r)| + |\Omega_r|] \prec \mu \prec f(\alpha_2) [|B| + |\Omega \backslash (B \cup \Omega_r)| + |\Omega_r|]$. Due our definitions, there are numbers t_n such that $\mu = f(\alpha_1) |\Omega_{t_n}| + f(\alpha_2) |\Omega \backslash (B \cup \Omega_{t_n}| + \int_B f(v_n) dx$. Thus we define the sequence $h_n(x)$ by

$$h_n(x) = \begin{cases} v_n(x) & \text{if} \quad x \in B \\ \alpha_1 & \text{if} \quad x \in \Omega_{t_n} \\ \alpha_2 & \text{if} \quad x \in \Omega \backslash (B \cup \Omega_{t_n}) \end{cases} \tag{3.5}$$

It's clear that $J(h_n) = \mu$. Remain to show that $h_n \in L_M(\Omega, \rho)$

In fact:

$$\int_\Omega M(|h_n(x)|)\rho(x)dx = \int_{\Omega_{t_n}} M(|\alpha_1|)\rho(x)dx + \int_{\Omega/(\Omega_{t_n} \cup B)} M(|\alpha_2|)\rho(x)dx + \int_B M(|\xi g_n(x) + \xi^*(1 - g_n(x))|)\rho(x)dx,$$

since $\rho(x) \in L_1(\Omega)$, then the first and the second term to right of equality are finished. It suffices to show that the third term is finished.

Since M is convex , then

$$\begin{aligned} \int_B M(|\xi g_n(x) + \xi^*(1 - g_n(x))|)\rho(x)dx &\leq \int_B M(|\xi| g_n(x) + |\xi^*| (1 - g_n(x)))\rho(x)dx \\ &\leq \int_B M(|\xi|)g_n(x)\rho(x)dx + \int_B M(|\xi^*|)(1 - g_n(x))\rho(x)dx \\ &\leq \int_{B_1} M(|\xi|)\rho(x)dx + \int_{B_2} M(|\xi^*|)\rho(x)dx \\ &\leq M(|\xi|) \int_\Omega \rho(x)dx + M(|\xi^*|) \int_\Omega \rho(x)dx \\ &\leq k, \end{aligned}$$

where $B_1 = B \bigcap \left\{x \in B / 0 \prec x_1 \prec \frac{\lambda}{n}\right\}$ and $B_2 = B \bigcap \left\{x \in B / \frac{\lambda}{n} \prec x_1 \prec \frac{1}{n}\right\}$.

Then $\int_\Omega M(|h_n|)\rho(x)dx \leq k'$, which imply that $h_n \in L_M(\Omega, \rho)$ for all n, and $(h_n)_n$ is bounded in $L_M(\Omega, \rho)$.

In fact:

First case) if $k^{'} \leq 1$, then $\|h_n\|_{M,\rho} \leq 1$ for all n

Second case) if $k^{'} \succ 1$, since M is convex, then $\int_{\Omega} M(\frac{|h_n|}{k^{'}})\rho(x)dx \leq \int_{\Omega} \frac{1}{k^{'}} M(|h_n|)\rho(x)dx \leq 1$.

Therefore $\|h_n\|_{M,\rho} \leq k^{'}$ for all n .

In the two case one has $\|h_n\|_{M,\rho} \leq c$ with $c = \max(k^{'}, 1)$ for all n.

Choosing now a convergent subsequence (t_{n_k}) of (t_n) with limit t_0, as $k \to \infty$, we note by h_{n_k} the corresponding subsequence , on the other hand $h_{n_k} \to h$ a.e.x, where $h(x)$ defined by

$$h(x) = \begin{cases} \lambda\xi + (1-\lambda)\xi^{*} & \text{if} \quad x \in B, \\ \alpha_1 & \text{if} \quad x \in \Omega_{t_0} \\ \alpha_2 & \text{if} \quad x \in \Omega \backslash (B \cup \Omega_{t_0}) \end{cases} \tag{3.6}$$

Lemma 2.5 , imply that $h_{n_k} \to h$ for the topology $\sigma(L_M(\Omega,\rho), E_{\overline{M}}(\Omega,\rho))$

As $lim_{k\to\infty} \int_{\Omega\backslash B} f(h_{n_k})dx = lim_{k\to\infty} \int_{\Omega\backslash B} f(h)dx.$

J is weakly lower semicontinous at N_{μ} then

$$\begin{aligned} \int_B f(h)dx &= |B|\, f(\lambda\xi + (1-\lambda)\xi^{*}) \\ &\leq lim_{k\to\infty} \int_B f(h_{n_k})dx \\ &\leq f(\xi) lim_{k\to\infty} \int_B g_{n_k}(x)dx + f(\xi^{*}) lim_{k\to\infty} \int_B (1 - g_{n_k}(x))dx \\ &\leq |B|\, (\lambda f(\xi) + (1-\lambda) f(\xi^{*})). \end{aligned}$$

This proves theorem 3.3.

Definition 3.4 A function $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is called the Carathéodory-function if

i) $f(.,\xi) : \Omega \to \mathbb{R}$ is mesurable for all $\xi \in \mathbb{R}^N$.

ii)$f(x,.) : \mathbb{R}^N \to \mathbb{R}$ is continuous for almost all $x \in \Omega$

Theorem 3.5 Suppose that $J(u) = \int_{\Omega} f(x,u)dx$ where $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory-function and $f(x,\xi)$ is integrable for all ξ.

Suppose that for some fixed $\mu \in \mathbb{R}$ the level set H_{μ} is nonvoid and J is weakly lower semicontinous at H_μ, then we have the alternative : Either μ is an extreme value of the functional J or for almost all $x \in \Omega$ the function $f(x, .)$ is convex.

Proof of theorem 3.5

Let us assume that $\mu \in I\!R$ is not an extreme value of J , then there are two functions u_1 and u_2 in $L_M(\Omega, \rho)$ such that

$$J(u_1) \prec \mu \prec J(u_2).$$

Next let $E \subset \Omega$ be the set defined by

$E = \{x \in \Omega$:x is the Lebesgue point of $f(x, \xi)$ for all $\xi \in I\!Q^N\}$. Certainly we have $|\Omega \backslash E| = 0$ and for every $x_0 \in E$ there is a ball $B(x_0, r_0)$ (see[3]) with the following property:

Let $B(x_0, r)$ be any ball with radius $r \leq r_0$ and we define the functions $\tilde{u}_i$ by

$$\tilde{u}_i(x) = \begin{cases} u_i(x) & \text{if} \quad x \in \Omega \backslash B(x_0, r) \\ \quad 0 & \text{if} \quad x \in B(x_0, r) \end{cases} \tag{3.7}$$

for $i = 1, 2$, then we have

$$J(\tilde{u}_1) \prec \mu \prec J(\tilde{u}_2) \qquad \text{(see[3])}.$$

Now we fix x_0 and choose some $\xi, \xi^* \in I\!Q^N$ and $\lambda \in [0, 1]$. Then for $\alpha = \max\{|J(\tilde{u_1}) - \mu|, |J(\tilde{u_2}) - \mu|\}$ there is a ball $B(x_0, r_1)$ such that $\int_{B(x_O, r)} |f(x, 0)|\, dx$, $\int_{B(x_O, r)} |f(x, \xi)|\, dx$ and $\int_{B(x_O, r)} |f(x, \xi^*)|\, dx$ are less than $\frac{\alpha}{4}$ each for $r \leq r_1$.

For $r \leq \min(r_0, r_1)$, we define the functions $w_{t,n}$ bay

$$w_{t,n}(x) = \begin{cases} \widetilde{u}_1(x) \quad \text{if} \quad x \in \Omega \backslash B(x_0, r) \quad and \quad x_1 \leq t \\ v_n(x) \qquad\qquad \text{if} \quad x \in B(x_0, r) \\ \widetilde{u}_2(x) \quad \text{if} \quad x \in \Omega \backslash B(x_0, r) \quad and \quad x_1 \succ t \end{cases} \tag{3.8}$$

Where v_n is defined as in(3.3). Since $\int_{B(x_0, r)} |f(x, v_n)|\, dx \leq \dfrac{\alpha}{2}$, we have

$J(w_{+\infty}, n) \prec \mu \prec J(w_{-\infty}, n)$, and hence there is t_n such that $J(w_{t_n, n} = w_n) = \mu$.

Let t_0 be an accumulation point of the sequence t_n and choose a subsequence also denoted by t_n such that $t_n \to t_0$, then $w_n \to w$ weakly for the topology $\sigma(L_M(\Omega,\rho), E_{\overline{M}}(\Omega,\rho))$ what one goes the shown later in proposition 3.6. Where

$$w(x) = \begin{cases} \widetilde{u}_1(x) & \text{if} \quad x \in \Omega \backslash B(x_0,r) \quad and \quad x_1 \leq t_0 \\ \lambda\xi + (1-\lambda)\xi^* & \text{if} \quad x \in B(x_0,r) \\ \widetilde{u}_2(x) & \text{if} \quad x \in \Omega \backslash B(x_0,r) \quad and \quad x_1 \succ t_0 \end{cases} \tag{3.9}$$

Because of

$$\int_{\Omega \backslash B(x_0,r)} f(x,w_n)dx = \int_{\Omega \backslash B(x_0,r)} f(x,w)dx,$$

we obtain

$$\int_{B(x_0,r)} f(x,\lambda\xi + (1-\lambda)\xi^*)dx \leq \lambda \int_{B(x_0,r)} f(x,\xi) + (1-\lambda) \int_{B(x_0,r)} f(x,\xi^*)dx.$$

This inequality holds for all $r \leq \min(r_0, r_1)$ and hence

$$f(x_0,\lambda\xi + (1-\lambda)\xi^*) \leq \lambda f(x_0,\xi) + (1-\lambda)f(x_0,\xi^*) \ ,$$

yielding the theorem because of the continuity of f with respect to ξ.

Proposition 3.6 $w_n \in L_M(\Omega,\rho)$ for all $n \in \mathbb{N}$, and $w_n \to w$ for the topology $\sigma(L_M(\Omega,\rho), E_{\overline{M}}(\Omega,\rho))$.

Proof of proposition 3.6(see appendix)

Corollary 3.7 If an addition H_μ is weakly closed , then either μ is an extreme value of J or $f(x,.)$ is affine for almost all $x \in \Omega$.

4. Approximation result.

Theorem 4.1 Let Ω be a bounded domain in $\mathbb{R}^N$, let M be an N-function, and let ρ be an weight function such that $\rho \in L^1(\Omega)$. If $u \in W^1L_M(\Omega,\rho)$, for a.e. $x_0 \in \Omega$, there is $u_\alpha \in W^1L_M(\Omega,\rho)$, such that:

i) $u_\alpha \to u(\text{mod})$ in $W^1L_M(\Omega,\rho)$ as $\alpha \to 0$,

ii) $u_\alpha \equiv c(x_0,\alpha)$ in $B(x_0,\alpha)$.

Proof of theorem 4.1. Let Ψ_α be a C_0^∞ cut-off function with support in $B(0,2\alpha)$ such that $\Psi_\alpha \equiv 1$ in $B(0,\alpha)$ and $|\nabla\Psi_\alpha| \leq \frac{2}{\alpha}$.
Let x_0 be a Lebesgue point of the function u in Ω , hence we can take $c(x_0,\alpha) = u(x_0)$.
We define in Ω the function u_α by

$$u_\alpha(x) = u(x)(1-\Psi_\alpha(x-x_0)) + u(x_0)\Psi_\alpha(x-x_0). \tag{4.1}$$

First it's clear that $u_\alpha \in W^1L_M(\Omega,\rho)$
In fact since $u \in W^1L_M(\Omega,\rho)$, then there are the numbers $\lambda_i \succ 0$, $0 \leq i \leq N$, such that

$$\int_\Omega M(\frac{|u(x)|}{\lambda_0})dx \prec \infty$$

and

$$\int_\Omega M(\frac{1}{\lambda_i}\left|\frac{\partial u(x)}{\partial x_i}\right|)\rho(x)dx \prec \infty \text{ for } 1 \leq i \leq N$$

Let $\lambda \succ 0$, since M is a convex function, then

$$\begin{aligned}\int_\Omega M(\frac{|u_\alpha(x)|}{\lambda})dx &\leq \frac{1}{2}\int_\Omega M(\frac{2}{\lambda}|u(x)(1-\Psi_\alpha(x-x_0))|)dx + \frac{1}{2}\int_\Omega M(\frac{2}{\lambda}|u(x_0)\Psi_\alpha(x-x_0)|)dx \\ &\leq \frac{1}{2}\int_\Omega M(2k_1\frac{|u(x)|}{\lambda})dx + \frac{1}{2}M(\frac{2k'}{\lambda}|u(x_0)|)\int_{B(0,2\alpha)}dx \\ &\prec \infty,\end{aligned}$$

where $k_1 = \sup_{B(0,2\alpha)}|1-\Psi_\alpha(x-x_0)|$, $\lambda = 2k_1\lambda_0$ and $k' = \sup_{B(0,2\alpha)}|\Psi_\alpha(x-x_0)|$.
Remains to show that

$$\tfrac{\partial u_\alpha}{\partial x_i} \in L_M(\Omega,\rho), 1 \leq i \leq N.$$

By a simple calculation we find that

$$\tfrac{\partial u_\alpha}{\partial x_i} = \tfrac{\partial u(x)}{\partial x_i}(1-\Psi_\alpha(x-x_0)) + (u(x_0)-u(x))\tfrac{\partial\Psi_\alpha(x-x_0)}{\partial x_i}.$$

Then

$$\begin{aligned}\int_\Omega M(\frac{1}{\lambda}\left|\frac{\partial u_\alpha}{\partial x_i}\right|)\rho(x)dx &\leq \frac{1}{2}\int_\Omega M(\frac{2}{\lambda}\left|(1-\Psi_\alpha(x-x_0)\frac{\partial u(x)}{\partial x_i})\right|)\rho(x)dx \\ &+\frac{1}{2}\int_\Omega M(\frac{2}{\lambda}\left|(u(x)-u(x_0))\frac{\partial \Psi_\alpha(x-x_0}{\partial x_i})\right|)\rho(x)dx \\ &\leq \frac{1}{2}\int_\Omega M(\frac{2}{\lambda}\left|k_1\frac{\partial u(x)}{\partial x_i})\right|)\rho(x)dx \\ &+\frac{1}{2}\int_\Omega M(\frac{2}{\lambda}\left|(u(x)-u(x_0))\frac{\partial \Psi_\alpha(x-x_0}{\partial x_i}\right|)\rho(x)dx.\end{aligned}$$

Since $u \in W^1L_M(\Omega,\rho)$, then the first term on the right side of the inequality is finite. In addition we will show in lemma 4.2, that

$$I'_\alpha = \int_{\Omega_{2\alpha_0}}\int_{B(y,2\alpha)} M(\frac{\lambda\,|u(x)-u(y)|}{\alpha})\rho(x)dx\;dy \prec \infty,$$

then

$$\int_{B(y,2\alpha)} M(\frac{\lambda\,|u(x)-u(y)|}{\alpha})\rho(x)dx \prec \infty \quad \text{a.e},$$

which implies that the second term is finite. Thus $u_\alpha \in W^1L_M(\Omega,\rho)$.

It is clear by using the Lebesgue theorem that

$$u_\alpha \to u(mod)L_M(\Omega) \quad \text{as} \quad \alpha \to 0. \tag{4.2}$$

therefore, it remains to show that

$$\frac{\partial u_{\alpha_k}}{\partial x_i} \to \frac{\partial u}{\partial x_i}(mod)L_M(\Omega,\rho), 1 \leq i \leq N, \tag{4.3}$$

for the sequence α_k with $\alpha_k \to 0$ as $k \to \infty$.

By a simple calculation we find that : $\frac{\partial(u-u_\alpha)(x)}{\partial x_i} = \frac{\partial u(x)}{\partial x_i}\Psi_\alpha(x-x_0)+\frac{\partial \Psi_\alpha(x-x_0)}{\partial x_i}(u(x)-u(x_0))$ and the convexity of the N-function M we can write

$$\begin{aligned}\int_\Omega M(\lambda\left|\frac{\partial(u-u_\alpha)(x)}{\partial x_i}\right|)\rho(x)dx &\leq \frac{1}{2}\int_\Omega M(2\lambda\left|\frac{\partial u(x)}{\partial x_i}\Psi_\alpha(x-x_0)\right|)\rho(x)dx \\ &+\frac{1}{2}\int_\Omega M(2\lambda\left|(u(x)-u(x_0))\frac{\partial \Psi_\alpha(x-x_0)}{\partial x_i}\right|\rho(x)dx.\end{aligned}$$

By virtue of Lebesgue theorem, the first term in the expression right of the above inequality converges to zero as $\alpha \to 0$, so it remains to show that :

$$\int_\Omega M(2\lambda \left|(u(x)-u(x_0))\frac{\partial\Psi_\alpha(x-x_0)}{\partial x_i}\right|)\rho(x)dx \to 0 \quad \text{as} \quad \alpha \to 0 \tag{4.4}$$

for this we use the following lemma .

Lemma 4.2 For almost all $x_0 \in \Omega$, there exists a sequence $\alpha_k \succ 0$ with $\alpha_k \to 0$ as $k \to \infty$ such that $\int_{B(x_0,2\alpha)} M(\frac{\lambda|u(x)-u(x_0)|}{\alpha_k})\rho(x)dx \to 0$ as $k \to \infty$ for some $\lambda \succ 0$. Using the above lemma we conclude directly, which completes the proof of theorem 4.1.

Proof of lemma 4.2(see appendix)

Remark 4.3.

1) In the particular case when $\rho(x) = 1$, we obtain the statement of [2;lemma 2].

2) In the particular case when $\rho(x) = 1$, and $M(t) = \frac{|t|^p}{p}$,$1 \leq p \prec \infty$ we obtain the statement of [3;lemma 2-1] .

5. Functional depending on x and ∇u.

Let Ω be a bounded domain in $I\!R^N$, let M be an N-function, and let ρ be an weight function such that $\rho \in L^1(\Omega)$. We consider the functional of kind

$$J = \int_\Omega f(x, \nabla u)dx. \tag{5.1}$$

Where $J : W^1L_M(\Omega,\rho) \to I\!R$ is continuous and $f : \Omega \times I\!R^N \to I\!R$ is a Carathéodory function satisfying

$$|f(x,\xi)| \leq T(x)G(|\xi|). \tag{5.2}$$

for some nondecreasing function $G : I\!R \to I\!R$ and some $T(x) \in L^1(\Omega)$.

For each μ, we write H_μ for the level set of the functional J, ie. $H_\mu = \{u \in W^1L_M(\Omega,\rho) : J(u) = \mu\}$. And for $\overline{H}^w_\mu$ for the closure of H_μ in $W^1L_M(\Omega,\rho)$ for the weak topology $\sigma(\prod L_M(\Omega,\rho), \prod E_{\overline{M}}(\Omega,\rho))$.

Defintion 5.1 A functional $J : W^1L_M(\Omega,\rho) \to I\!R$ is called weakly lower semicontinuous at a level set H_μ . If $J(u) \leq \mu$ for all $u \in \overline{H}^w_\mu$.

Remark 5.2 Note that this definition does not imply that $J_{/\overline{H}_{\mu}^{w}}$ is weakly lower semicontinuous .

Theorem 5.3 Let Ω be a bounded domain in $I\!\!R^N$, and let ρ the weight function such that $\rho \in L^1(\Omega)$.
Let $J : W^1L_M(\Omega, \rho) \to I\!\!R$ be a continuous functional defined as (5.1) , with the Carathéodory function $f : \Omega \times I\!\!R^N \to I\!\!R$ satisfying (5.2).
If J is weakly lower semicontinous at nonvoid level set H_μ, then we have the alternative:
Either μ is an extreme value of J or for almost all $x \in \Omega$ the function $f(x, .)$ is convex .

Proof of theorem 5.3 Let assume that the level set μ is not an extreme value of J , then we shall show that

$$f(x, \lambda\xi + (1-\lambda)\xi^*) \leq \lambda f(x, \xi) + (1-\lambda) f(x, \xi^*)$$

for all $\lambda \in [0, 1]$, for all $\xi, \xi^* \in I\!\!R^N$ and for $a.e.x \in \Omega$.
We can assume that $\mu = 0$ and that in $W^1L_M(\Omega, \rho)$ there are two functions $\hat{a}_1$ and $\hat{a}_2$ such that $J(\hat{a}_1) \prec -\epsilon_0$ and $J(\hat{a}_2) \succ \epsilon_0$ for some $\epsilon_0 \succ 0$.
Let x_0 be a Lebesgue point of $f(x, \xi)$ for all $\xi \in I\!\!Q^N$. We can assume that $x_0 = 0$.
Using the continuity of the fonctional J and (theorem(4.1), there is a ball $B(0, R_0) \subset \Omega$ and there are $\bar{b}, \bar{b_1}$ and $\bar{b_2}$ (see[3]) such that

$$\nabla\bar{b} = \nabla\bar{b_1} = \nabla\bar{b_2} = 0 \quad \text{on} \quad B(0, R_0). \tag{5.3}$$

$$J(\bar{b_1}) \prec \frac{7}{8}\epsilon_0, \quad J(\bar{b_2}) \succ \frac{7}{8}\epsilon_0 \quad \text{and} \quad \left|J(\bar{b})\right| \prec \frac{1}{8}\epsilon_0. \tag{5.4}$$

Furthermore for all function $\bar{a}$ satisfying $|J(\bar{a})| \prec \frac{7}{8}\epsilon_0$ there is $t_i \in [0, 1]$ with $i = i(\bar{a}) \in \{1, 2\}$ such that the function $\bar{c} = \bar{a} + t_i(\bar{b_i} - \bar{a})$ lies in the level set N_0 , i.e. $J(\bar{c}) = 0$.
Let us now fix $\lambda \in [0, 1] \bigcap I\!\!Q$ and $\xi, \xi^* \in I\!\!Q^N$. We define the sequence of functions

$$\hat{c}_n(x) =< \xi^*, x > + \int_0^{<\xi-\xi^*, x>} g_\lambda(nt)dt,$$

where $<, >$ denotes the usual inner product in $I\!\!R^N$ and

$$g_\lambda(x) = \begin{cases} 1 & \text{if} \quad 0 \prec t \prec \lambda \\ 0 & \text{if} \quad \lambda \prec t \prec 1 \end{cases}$$

We recall the fact that (see[3])

$$g_n(x) \rightharpoonup^* \lambda \qquad \text{in} \qquad L^\infty(\Omega)$$

and

$$(1 - g_n(x)) \rightharpoonup^* (1 - \lambda) \qquad \text{in} \qquad L^\infty(\Omega).$$

It's clear that

$$\nabla \hat{c}_n(x) = \xi^* + (\xi - \xi^*) g_\lambda(n < \xi - \xi^*, x >),$$

from the boundedness on $W^1 L_M(\Omega, \rho)$ will be shown later in proposition 5.5, and convergence almost everywhere $\hat{c}_n(x) \to \hat{c}_0(x)$ we have convergence

$$\hat{c}_n \to \hat{c}_0 \qquad \text{in} \qquad W^1 L_M(\Omega, \rho) \quad \text{for} \qquad \sigma(\prod L_M(\Omega, \rho), \prod E_{\bar{M}}(\Omega, \rho)),$$

where

$$\hat{c}_0(x) =< \lambda \xi + (1 - \lambda) \xi^*, x >$$

Let $\psi : I\!R \to I\!R$ be a C^∞-function with support in the interval $(-1, 1)$ and $\psi(t) = 1$ for all $|t| \prec \frac{1}{2}$. Defining $\bar{c}_R(x)$ by $\bar{c}_R(x) = \psi(\frac{|x|}{R}) \hat{c}_0(x)$ for all $R \succ 0$, we calculate

$$\nabla \bar{c}_R(x) = \psi'(\tfrac{|x|}{R}) \tfrac{|x|}{R} \hat{c}_0(x) + \psi(\tfrac{|x|}{R}) \nabla \hat{c}_0(x)$$

Moreover, the function $\bar{c}_R(x) = \psi(\frac{|x|}{R}) \hat{c}_0(x)$ satisfying the properties (see[3:proposition 3.1]):

$$|\nabla \bar{c}_R(x)| \leq c \quad \text{in} \quad \Omega. \tag{5.5}$$

$$\int_{B(0,R)} f(x, \nabla \bar{c}_R(x)) dx \to 0 \quad \text{as} \quad R \to 0. \tag{5.6}$$

Note that (5.2) is used for to prove (5.6).

Next we consider the sequence $\hat{c}_n(x)$ in a ball $B(0, r)$, say. We shall show that is possible alter each element of the sequence $\hat{c}_n(x)$ in such a manner that it coincides with limit $\hat{c}_0(x)$ in the bundary.

The following lemma is generlization of [3:proposition 3.2] in weigthed Olicz-Sobolev spaces.

Lemma 5.4 There is a sequence $a_n(x)$ in $W^1L_M(\Omega,\rho)$ such that:

i) $a_n(x)=\hat{c}_0(x)=<\lambda\xi+(1-\lambda)\xi^*,x>$ in $\partial B(0,r)$

ii) $a_n-\hat{c}_n\to 0$ (mod) in $W^1L_M(\Omega,\rho)$ as $n\to\infty$

iii) $a_n\to\hat{c}_0$ in $W^1L_M(\Omega,\rho)$ for $\sigma(\prod L_M(\Omega,\rho),\prod E_{\overline{M}}(\Omega,\rho))$

iv) $\|\nabla a_n\|_\infty+\|\nabla\hat{c}_n\|_\infty\le c$

v) $\left|\int_{B(0,r)}f(x,\nabla\hat{c}_n)dx-\int_{B(0,r)}f(x,\nabla a_n)dx\right|\to 0$ as $n\to\infty$

vi) $\int_{B(0,r)}f(x,\nabla a_n)dx\to 0$ as $r\to 0$ uniformly in n.

Proof of lemma 5.4(see appendix)

Now, we are in a position to complete the proof of theorem 5.3. For $R\le R_0$ and $r=\frac{R}{2}$,we define the sequence:

$$\widehat{b}_n(x)=\begin{cases}\overline{b}(x) & \text{if}\quad x\in\Omega\backslash B(0,R),\\ \overline{b}(x)+\overline{c}_R(x) & \text{if}\quad x\in B(0,R)\backslash B(0,r),\\ \overline{b}(x)+a_n(x) & \text{if}\quad x\in B(0,r);\end{cases}$$

which converges in $W^1L_M(\Omega,\rho)$ for the weak topology $\sigma(\prod L_M(\Omega,\rho),\prod E_{\overline{M}}(\Omega,\rho))$ to

$$\widehat{b}_0(x)=\begin{cases}\overline{b}(x) & \text{for}\quad x\in\Omega\backslash B(0,R),\\ \overline{b}(x)+\overline{c}_{_R}(x) & \text{for}\quad x\in B(0,R).\end{cases}$$

We accont of (5.5) and (5.6) and lemma 5.4, (as in [3] and [2]). We have for $R\succ 0$ small enough $\left|J(\overline{b}_n)\right|\prec\frac{7}{8}\epsilon_0$ for all n . Hence for any n , we find numbers $t_n\in[0,1]$ and $i_n\in\{1,2\}$, such that for $b_n=\widehat{b}_n+t_n(\overline{b}_{i_n}-\widehat{b}_n)$ we have $J(b_n)=0$.

Now choosing a subsequence t_n such that $t_n\to t_0$ and $i_n=i$; $i\in\{1,2\}$, we have

$$b_n\to b_0\quad\text{in}\quad W^1L_M(\Omega,\rho)\quad\text{for}\quad\sigma(\textstyle\prod L_M(\Omega,\rho),\prod E_{\overline{M}}(\Omega,\rho)).$$

Because, of the continuity of J with strong topology of $W^1L_M(\Omega,\rho)$, we have

$$lim_{n\to\infty}J(\overline{b}+t_n(\overline{b}_{i_n}-\overline{b}))=J(\overline{b}+t_0(\overline{b}_i-\overline{b})),$$

and by constuction

$$f(x,\nabla(\overline{b}+t_n(\overline{b}_i-\overline{b}))=f(x,0)\text{ in }B(0,R)$$

because

$$\nabla\overline{b}=\nabla\overline{b}_1=\nabla\overline{b}_2=0\text{ in }B(0,R)$$

Yielding,

$$lim_{n\to\infty}\int_{B(0,R)} f(x,\nabla b_n(x))dx \geq \int_{B(0,R)} f(x,\nabla b_0(x))dx.$$

Since $b_n = b_0$ in $B(0,R)\backslash B(0,r)$, $r=\frac{R}{2}$, we finally get

$$\begin{aligned}\int_{B(0,r)} f(x,\lambda\xi+(1-\lambda)\xi^*)dx &= \int_{B(0,r)} f(x,\nabla b_0(x))dx\\ &\leq lim_{n\to\infty}\int_{B(0,r)} f(x,\nabla b_n)(x))dx\\ &= lim_{n\to\infty}\int_{B(0,R)} f(x,\nabla a_n(x))dx\\ &= \lambda\int_{B(0,r)} f(x,\xi)dx+(1-\lambda)\int_{B(0,r)} f(x,\xi^*)dx.\end{aligned}$$

Since the above inequality can be obtained for all $B(0,r)$ with radius $r\prec\frac{R}{2}$, we conclude that $f(x_0,\lambda\xi+(1-\lambda)\xi^*)\leq\lambda f(x_0,\xi)+(1-\lambda)f(x_0,\xi^*)$ for all $\lambda\in[0,1]\cap\mathbb{Q}$ and all $\xi,\xi^*\in Q^N$. It then follows by the continuity of $f(x,\xi)$ with respect to ξ, that the above inequality holds for all $\lambda\in[0,1]$ and all $\xi,\xi^*\in\mathbb{R}^N$.

Proposition 5.5 The sequence of function $\hat{c}_n$ defined by $\hat{c}_n(x)=<\xi^*,x>+\int_0^{<\xi-\xi^*,x>} g_\lambda(nt)dt$ satisfying the following propreties:

i) $\hat{c}_n(x)\to\hat{c}_0(x)$ for almost all $x\in\Omega$ where $\hat{c}_0(x)=<\lambda\xi+(1-\lambda)\xi^*,x>$

ii) $\hat{c}_n$ is bounded in $W^1L_M(\Omega,\rho)$

Proof of proposition 5.5 (see appendix)

Corollary 5.6. Under the same assumptions as in theorem suppose that there is a nonvoid weakly closed level set H_μ. If μ is not an extreme value of J, then the function $f(x,\nabla u(x))$ is affine in the gradient.

Remark 5.7.

1) In the particular case when $\rho(x)=1$, we obtain the statement of [2;theorem 6].

2)In the particular case when $\rho(x)=1$, and $M(t)=\frac{|t|^p}{p}$,$1\leq p\prec\infty$, we obtain the statement of [3;theorem 3-1].

6. Appendix

Proof of proposition 3.6 For all $n \succ 0$ and $\gamma \succ 0$, we have
$$\int_{\Omega} M(\frac{|w_n|}{\gamma})\rho(x)dx = \int_{B(x_0,r)} M(\frac{|v_n|}{\gamma})\rho(x) + \int_{\Omega_1} M(\frac{|u_1|}{\gamma})\rho(x) + \int_{\Omega_2} M(\frac{|u_2|}{\gamma})\rho(x)dx,$$
where $\Omega_1 = \Omega\backslash B(x_o,r) \cap \{x, x_1 \leq t\}$ and $\Omega_2 = \Omega\backslash B(x_o,r) \cap \{x, x_1 \succ t\}$.
Since $u_i \in L_M(\Omega,\rho)$ $i = 1,2$, then there are γ_i. $i = 1,2$, such that $\int_{\Omega_i} M(\frac{|u_i|}{\gamma_i})\rho(x)dx \prec \infty$.
Let $\gamma = \max\{\gamma_1, \gamma_2, 1\}$.

Since
$$\int_{B(x_0,r)} M(|v_n|)\rho(x)dx \leq (M(|\xi| + M(|\xi^*|)\,|\Omega|\,.$$
Then $\int_{\Omega} M(\frac{|w_n|}{\gamma})\rho(x)dx \leq k$ with k is a constant positive , then $w_n \in L_M(\Omega,\rho))$.
In fact.

If $k \leq 1$, then $\|w_n\|_{1,M,\rho} \leq \gamma$.
If $k \succ 1$ $\int_{\Omega} M(\frac{|w_n|}{k\gamma})\rho(x)dx \leq \frac{1}{k}\int_{\Omega} M(\frac{|w_n|}{\gamma})\rho(x)dx \leq 1$.
Then $\|w_n\|_{1,M,\rho} \leq \gamma'$, where $\gamma' = \max\{k\gamma, \gamma\}$. Then w_n is bounded in $L_M(\Omega,\rho)$.
Lemma 2.5 imply that $w_n \to w$ for the topology $\sigma(L_M(\Omega,\rho), E_{\overline{M}}(\Omega,\rho))$.

Proof of lemma 4.2. Let $x_0 \in \Omega$. For each $t \succ 0$, we define the set
$\Omega_t = \{x \in \Omega; dist(x,\partial\Omega) \succ t\}$.
Let $\alpha_0 \succ 0$. For $\alpha \prec \alpha_0$, we consider the function $\phi_\alpha : \Omega_{2\alpha_0} \to I\!R$ defined by
$$\phi_\alpha(y) = \int_{B(y,2\alpha)} M(\frac{\lambda\,|u(x) - u(y)|}{\alpha})\rho(x)dx. \tag{6.1}$$
Since $\phi_\alpha(y) = \int_{\Omega} M(\frac{\lambda\,|u(x) - u(y)|}{\alpha})\rho(x)\chi_{B(y,2\alpha)}dx$, then the function $\phi_\alpha : \Omega_{2\alpha_0} \to I\!R$ is measurable ; χ_E, as usual denotes the charateristic function of the set E.
For all $\alpha_0 \succ 0$, we shall show that :
$$|\phi_\alpha(y)| \to 0 \quad \text{in} \quad L^1(\Omega_{2\alpha_0}) \quad \text{as} \quad \alpha \to 0, \quad \alpha \prec \alpha_0 \qquad (6.2).$$
This obviously implies the statement of lemma 4.2, (because if (6.2) is satisfied , then there is a subsequence α_k converges at 0 as $k \to \infty$ and such that $\phi_{\alpha_k}(y) \to 0$ a.e. y in

$\Omega_{2\alpha_0}$).

Since α_0 is arbitrary , then the previous convergence is true a.e. x_0 in Ω.

To verify (6.2), we denotes by $u_\delta = u * \varphi_\delta$ the mollification of u, where $\varphi_\delta \in D(I\!R^N)$, $\varphi_\delta = 1$ for $|x| \geq \delta$, $\varphi_\delta \geq 0$ and $\int_{I\!R^N} \varphi_\delta(x)dx = 1$. Hence, φ_δ is well defined in $\Omega_{2\alpha_0}$ for $\delta \prec \alpha_0$ and we have

$$\begin{aligned}
\int_{\Omega_{2\alpha_0}} |\phi_\alpha(y)|\, dy &= \int_{\Omega_{2\alpha_0}} \int_{B(y,2\alpha)} M(\frac{\lambda\, |u(x)-u(x_0)|}{\alpha})\rho(x)dx\, dy \\
&\leq lim_{\delta\to 0} \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} M(\frac{\lambda\, |u_\delta(y-x)-u_\delta(y)|}{\alpha})\rho(x)dx\, dy
\end{aligned}$$

Since u_δ is continuously differentiable , we may estimate

$$I_\alpha = \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} M(\frac{\lambda\, |u_\delta(y-x)-u_\delta(y)|}{\alpha})\rho(x)dx\, dy$$

In fact, we have

$$\begin{aligned}
I_\alpha &\leq \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} M(\frac{\lambda \int_0^1 |\nabla u_\delta(y-tx)|\, |x|\, dt}{\alpha})\rho(x)dx\, dy \\
&\leq \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} M(2\lambda \int_0^1 |\nabla u_\delta(y-tx)|\, dt)\rho(x)dx\, dy
\end{aligned}$$

Then , it follows by Jensen's inequality that

$$\begin{aligned}
I_\alpha &\leq \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} \int_0^1 M(2\lambda\, |\nabla u_\delta(y-tx)|)\rho(x)dt\, dx\, dy \\
&\overset{(*)}{=} \int_0^1 \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} M(2\lambda \left| \int_{B(0,\delta)} \nabla u(y-tx-z)\varphi_\delta(z)dz \right|)\rho(x)dt\, dx\, dy \\
&\leq k_2 \int_0^1 \int_{\Omega_{2\alpha_0}} \int_{B(0,2\alpha)} \int_{B(0,\delta)} M(k_1\lambda\, |\nabla u(y-tx-z)|)\rho(x)dt\, dx\, dy\, dz \\
&= k_2 \int_0^1 \int_{B(0,2\alpha)} \int_{B(0,\delta)} (\int_{\Omega_{2\alpha_0}} M(k_1\lambda\, |\nabla u(y-tx-z)|)dy)\rho(x)dt\, dx\, dz \\
&\leq k_3 \int_0^1 \int_{B(0,2\alpha)} \int_{B(0,\delta)} \|M(k_1\lambda\, |\nabla u|)\|_1\, \rho(x)dt\, dx\, dz \\
&\leq k_3 \|M(k_1\lambda\, |\nabla u|)\|_1 \int_{B(0,2\alpha)} \int_{B(0,\delta)} dz\rho(x)dx \\
&\leq k_3(\frac{\sigma_N}{N})\delta^N \|M(k_1\lambda\, |\nabla u|)\|_1 \int_{B(0,2\alpha)} \rho(x)dx \\
&\leq k_4(\frac{\sigma_N}{N})\alpha^N \|M(k_1\lambda\, |\nabla u|)\|_1 \quad \text{(because} \quad \alpha \succ \delta)
\end{aligned}$$

for some positive constants k_1, k_2, k_3, and k_4 , (σ_N denotes the mesure of the unit sphere in $I\!R^N$). So we obtain $I_\alpha \to 0$ as $\alpha \to 0$.

Then it follows for $\alpha_0 \succ 0$ that $\int_{\Omega_{2\alpha_0}} |\phi_\alpha(y)|\, dy \to 0$ as $\alpha \to 0$ $\alpha \prec \alpha_0$,wich allows to conclude for almost every $x_0 \in \Omega$, we have $\phi_{\alpha_k}(x_0) \to 0$as $k \to \infty$. To justify (*), we recall that in $\Omega_{2\alpha_0}$ the differentiation and the mollification commutent for $\delta \prec \alpha \prec \alpha_0$, which proves the statement of lemma 4.2.

Proof of lemma 5.1.

Let $\tilde{\omega}_\delta$ be a C^∞ -function with support in $[-1,1]$ such that $\tilde{\omega}_\delta = 1$ for all $|t| \prec 1-\delta$ and $\left|\tilde{\omega}_\delta'\right| \prec \frac{2}{\delta}$ for all t.

Defining the function $\omega_\delta(x) = \tilde{\omega}_\delta(\frac{|x|}{r})$ and $a_{n,\delta}(x) = \hat{c}_0(x) + \omega_\delta(x)(\hat{c}_n(x) - \hat{c}_0(x))$

we have the following inequality

$$|\nabla(\hat{c}_n(x) - \hat{c}_0(x))|\,(1-\omega_\delta(x)) \le c^{'} r(|\xi^*| + |\xi|)(1-\omega_\delta(x)). \tag{6.3}$$

$$|\nabla\omega_\delta(x)|\,|\hat{c}_n(x) - \hat{c}_0(x)| \le O(n^{-1})\frac{1}{\delta}\chi_{supp(\nabla\omega_\delta)} \tag{6.4}$$

$$\int_\Omega M(|a_{n,\delta} - \hat{c}_n|)dx + \int_\Omega M(|\nabla(a_{n,\delta} - \hat{c}_n)|)\rho(x)dx \le O(\delta) + c\int_{B(0,r)} M(|\nabla(\hat{c}_n(x) - \hat{c}_0(x))(1-\omega_\delta(x))|)\rho(x)dx \tag{6.5}$$

for some positive constants c and $c^{'}$.

For (6.3) and (6.4) see the proof of [3 ,proposition 3.2].

Assume now that(6.5) is true, thus we get

$$\omega_\delta(x) = \begin{cases} 0 & \text{in} \quad \Omega\backslash\overline{B}(0,r) \\ 1 & \text{in} \quad B(0,(1-\delta)r) \\ \tilde{\omega_\delta}(\frac{|x|}{r}) & \text{in} \quad \overline{B}(0,r)\backslash B(0,(1-\delta)r) \end{cases}$$

which implies that

$$a_{n,\delta}(x) - \hat{c}_n(x) = \begin{cases} \hat{c}_0(x) - \hat{c}_n(x) & \text{in} \quad \Omega\backslash\overline{B}(0,r) \\ 0 & \text{in} \quad B(0,(1-\delta)r) \\ (1-\tilde{\omega_\delta}(\frac{|x|}{r}))(\hat{c}_0(x) - \hat{c}_n(x)) & \text{in} \quad \overline{B}(0,r)\backslash B(0,(1-\delta)r) \end{cases}$$

and

$$\nabla(a_{n,\delta}(x)-\hat{c}_n(x)) = \begin{cases} \nabla(\hat{c_0}(x)-\hat{c}_n(x)) & \text{in} \quad \Omega\backslash\overline{B}(0,r) \\ 0 & \text{in} \quad B(0,(1-\delta)r) \\ \nabla(\tilde{\omega_\delta}(\frac{|x|}{r}))(\hat{c}_n(x)-\hat{c}_0(x))+ & \\ (1-\tilde{\omega_\delta}(\frac{|x|}{r}))\nabla((\hat{c}_0(x)-\hat{c}_n(x))) & \text{in} \quad \overline{B}(0,r)\backslash B(0,(1-\delta)r) \end{cases}$$

Hence, we have the estimate

$$\int_\Omega M(|a_{n,\delta}-\hat{c}_n|)dx+\int_\Omega M(|\nabla(a_{n,\delta}-\hat{c}_n)|)\rho(x)dx$$
$$\leq O(\delta)+c\int_{B(0,r)\backslash B(0,(1-\delta)r)} M(|\nabla(\hat{c}_n(x)-\hat{c}_0(x))(1-\omega_\delta(x))|)\rho(x)dx$$
$$\leq O(\delta)+cM(c_1O(n^{-1})\frac{1}{\delta})\int_{B(0,r)\backslash B(0,(1-\delta)r)}\rho(x)dx$$
$$\leq O(\delta)+cc_2M(c_1O(n^{-1})\frac{1}{\delta})$$

with $c_2=\int_\Omega \rho(x)dx$ (because $\rho\in L^1(\Omega)$)

Selecting numbers δ_n such that $O(n^{-1})\frac{1}{\delta_n}=1$, this implies that $O(\delta_n)=O(n^{-1})$ and $\delta_n\to 0$ as $n\to\infty$.

then, we conclude that

$$\int_\Omega M(|a_{n,\delta}-\hat{c}_n|)dx+\int_\Omega M(|\nabla(a_{n,\delta}-\hat{c}_n)|)\rho(x)dx \leq O(n^{-1})+cc_2M(c_1O(n^{-1})\frac{1}{\delta})$$

wich convege to 0 as $n\to\infty$.

We define the functions $a_n=a_{n,\delta}$ and we have $a_{n,\delta}-\hat{c}_n\to 0$ (mod)$W^1L_M(\Omega,\rho)$ as $n\to 0$.

Which gives (ii) in lemma 5.1 and $a_n-\hat{c}_0=(a_n-\hat{c}_n)+(\hat{c}_n-\hat{c}_0)\to 0$ in $W^1L_M(\Omega,\rho)$ for $\sigma(\prod L_M(\Omega,\rho),\prod E_{\overline{M}}(\Omega,\rho))$ (because $(\hat{c}_n-\hat{c}_0)\to 0$ in $W^1L_M(\Omega,\rho)$ for $\sigma(\prod L_M(\Omega,\rho),\prod E_{\overline{M}}(\Omega,\rho))$.

The properties i),iv) and vi) are satisfied by the definition of a_n . Now, we return to show the inequality (6.5). In fact we can write

$$\int_\Omega M(|a_{n,\delta}-\hat{c}_n|)dx+\int_\Omega M(|\nabla(a_{n,\delta}-\hat{c}_n)|)\rho(x)dx=\int_{\overline{B}(0,r)} M(|\hat{c}_n-\hat{c}_0|\,(1-\omega_\delta))dx+$$
$$\int_{\Omega\backslash\overline{B}(0,r)} M(|\hat{c}_n-\hat{c}_0|)dx+\int_{\Omega\backslash\overline{B}(0,r)} M(|\nabla(\hat{c}_n-\hat{c}_0)|)\rho(x)dx+\int_{\overline{B}(0,r)} M(|\nabla(\hat{c}_n-\hat{c}_0)(1-\omega_\delta)|)\rho(x)dx.$$

Since

$$(1-\omega_\delta(x))\to 0 \text{ a.e in } \overline{B}(0,r)$$

and

$$\int_{\Omega\setminus\overline{B}(0,r)} M(|\hat{c}_n-\hat{c}_0|)dx+\int_{\Omega\setminus\overline{B}(0,r)} M(|\nabla(\hat{c}_n-\hat{c}_0)|)\rho(x)dx\to 0 \text{ as } n\to\infty\ ,$$

then we conclude that

$$\int_\Omega M(|a_{n,\delta}-\hat{c}_n|)dx+\int_\Omega M(|\nabla(a_{n,\delta}-\hat{c}_n)|)\rho(x)dx\le O(\delta)+c\int_{B(0,r)} M(|\nabla(\hat{c}_n(x)-\hat{c}_0(x))(1-\omega_\delta(x))|)\rho(x)d$$

Wich implies the inequality (6.5).

Proof of proposition 5.1 :

It is obvious to show i) (see[3])

Now show ii) Since M is convex and Ω bounded then

$$\begin{aligned}\int_\Omega M(|\hat{c}_n(x)|)dx &= \int_\Omega M(\left|<\xi^*,x>+\int_0^{<\xi-\xi^*,x>} g_\lambda(nt)dt\right|)\\ &\le \frac{1}{2}\int_\Omega M(2\,|<\xi^*,x>|)dx+\frac{1}{2}\int_\Omega M(2\left|\int_0^{<\xi-\xi^*,x>} g_\lambda(nt)dt\right|)dx\\ &\le K+\frac{1}{2}\int_\Omega M(2\,|<\xi-\xi^*,x>|)dx\\ &\le k'\end{aligned}$$

If $k'\le 1$, then $\|\hat{c}_n\|_M=\|\hat{c}_n\|_{M,\rho}\le 1$.
If $k'\succ 1$, since $\frac{1}{k'}\int_\Omega M(|\hat{c}_n(x)|)dx\le\int_\Omega M(\frac{1}{k'}\,|\hat{c}_n(x)|)dx$,
then $\|\hat{c}_n\|_M\le k_1$ (where k , k' and k_1 are positive constants).

$$\begin{aligned}\int_\Omega M(|\nabla\hat{c}_n|)\rho(x)dx &= \int_\Omega M(|\xi^*+(\xi-\xi^*)g_\lambda(n<\xi-\xi^*,x>)|)\rho(x)dx\\ &\le \tfrac{1}{2}\int_\Omega M(2\,|\xi^*|)\rho(x)dx+\frac{1}{2}\int_\Omega M(2\,|(\xi-\xi^*)g_\lambda(n<\xi-\xi^*,x>)|)\rho(x)dx\\ &\le k_2\end{aligned}$$

the study of the cases $k_2\le 1$ and $k_2\succ 1$ give $\|\nabla\hat{c}_n\|_{M,\rho}\le k_3$.

Thus $\|\hat{c}_n\|_{1,M,\rho}=\|\hat{c}_n\|_M+\|\nabla\hat{c}_n\|_{M,\rho}\le k_4$,

where k_2 , k_3 and k_4 are positive constants, then $\hat{c}_n$ is bounded in $W^1L_M(\Omega,\rho)$.

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