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H^1- GALERKIN MIXED FINITE ELEMENT APPROXIMATION OF ONE CLASS OF HEAT TRANSPORT EQUATIONS

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Abstract. In this paper, we investigate a fully discrete H^1-Galerkin mixed finite element approximation of a class of heat transport equations. The Crank-Nicloson scheme is used for time discretization. Optimal error estimates in L^2-norm for the unknown function and its gradient function are obtained. A numerical example is presented to illustrate the theoretical findings.

Keywords: Heat transport equations; H^1-Galerkin mixed finite element method; Crank-Nicloson scheme; error estimate.

2000 AMS Subject Classification: 65N30

1. Introduction

The primary interest of this paper is to investigate a Crank-Nicloson H^1-Galerkin mixed finite element scheme for the following heat transport problems

$$\begin{cases} (a)\ \frac{1}{\delta}u_t + u_{tt} = au_{xx} + bu_{xxt} + f(x,t), & (x,t) \in I \times (0,T], \\ (b)\ u(0,t) = 0, u(L,t) = 0, & 0 \le t \le T, \\ (c)\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in I. \end{cases} \tag{1}$$

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where $I = [0, L]$, u denotes temperature, f is a heat source, δ, a and b are positive constants.

The above equations are widely used to describe the thermal behavior of thin films and other microstructures, see, for example, [1], [2], etc..

In recent years, a variety of numerical methods are proposed to resolve this problem, such as finite difference methods, finite element methods and mixed finite element methods. One can refer to [3], [5], [6], [7], [8], etc.. Recently, an H^1 Galerkin mixed finite element method was discussed for problem (1) in [15]. Comparing to standard mixed finite element methods the finite element spaces are free of the LBB stability condition in this formulation, which makes the choice of finite element spaces more flexible.

To improve the convergence order for time discretization a Crank-Nicloson H^1 mixed finite element scheme is proposed in this paper. An optimal a priori error estimates for the scalar unknown u and its flux q in L^2-norm are achieved. Moreover, a numerical example is presented to illustrate our theoretical analysis.

Throughout the paper, we use the standard notation $W^{m,q}(\Omega)$ for Sobolev space on Ω with a norm $\|\cdot\|_{m,q}$ and a semi-norm $|\cdot|_{m,q}$. For $q = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $\|\cdot\|_m = \|\cdot\|_{m,2}$ and for $m = 0$, we denote $\|\cdot\| = \|\cdot\|_0$. Moreover, the inner products in $L^2(\Omega)$ are indicated by $(\cdot,\cdot)$. Let X be a Banach space and $\varphi(t) : [0, T] \longmapsto X$, we set

$$\|\varphi\|^2_{L^2(X)} = \int_0^T \|\varphi(s)\|^2_X ds, \qquad \|\varphi\|_{L^\infty(X)} = ess \sup_{0\le t\le T} \|\varphi\|_X.$$

In addition, C denotes a generic constant independent of the spatial mesh parameter h and time discretization parameter τ, and ε denotes an arbitrarily small positive constant.

The outline of this article is organized as follows: In Section 2 a Crank-Nicloson H^1 mixed finite element scheme is described. Optimal a priori error bounds are derived for in Section 3. A numerical example is given to verify the theoretical results in Section 4.

2. The Fully discrete Scheme

2.1 Weak Formulation

For the H^1-Galerkin mixed finite element procedure, we split (1a) into a system of two equations. Let $p = au_x + bu_{xt}$, then (1a) can be rewritten as follows:

$$\begin{cases} (a)\ au_x + bu_{xt} = p, \\ (b)\ \frac{1}{\delta}u_t + u_{tt} - q_x = f(x,t). \end{cases} \tag{2}$$

To consider the H^1-Galerkin mixed finite element approximation scheme for (2a), (2b), we first derive the weak formulation.

Let $H_0^1 = \{v \in H^1(I), v(0) = v(L) = 0\}$. Multiplying (2a) by v_x, $v \in H_0^1$, and integrating on interval I we obtain

$$(au_x, v_x) + (bu_{xt}, v_x) = (q, v_x),\ v \in H_0^1. \tag{3}$$

Multiplying (2b) by w_x, $w \in H^1$, and integrating on interval I yields

$$(\frac{1}{\delta}u_t + u_{tt}, w_x) - (q_x, w_x) = (f, w_x),\ w \in H^1.$$

Since $u_t(0,t) = u_t(L,t) = 0$, $u_{tt}(0,t) = u_{tt}(L,t) = 0$, then integrating on interval I we derive that

$$(\frac{1}{\delta}u_{xt} + u_{xtt}, w) + (q_x, w_x) + (f, w_x) = 0,\ w \in H^1. \tag{4}$$

For $q = au_x + bu_{xt}$, then

$$u_{xt} = \frac{1}{b}q - \frac{a}{b}u_x, \tag{5}$$

$$u_{xtt} = \frac{1}{b}q_t + \frac{a^2}{b^2}u_x - \frac{a}{b^2}q. \tag{6}$$

Setting $\alpha = \frac{1}{b}, \beta = \frac{a^2}{b^2} - \frac{a}{b\delta}, \gamma = \frac{1}{b\delta} - \frac{a}{b^2}$, it is easy to see that $\alpha > 0$. Using (5) and (6), (4) can be rewritten as follows:

$$(\alpha q_t, w) + \gamma(q, w) + (q_x, w_x) + (\beta u_x, w) + (f, w_x) = 0,\ w \in H^1. \tag{7}$$

Therefore, the weak formulation of (2a), (2b) is to find $\{u, q\} : [0, T] \mapsto H_0^1 \times H^1$ such that

$$\begin{cases} (a)\ (au_x, v_x) + (bu_{xt}, v_x) = (q, v_x), \quad v \in H_0^1, \\ (b)\ (\alpha q_t, w) + \gamma(q, w) + (q_x, w_x) + (\beta u_x, w) + (f, w_x) = 0, \quad w \in H^1. \end{cases} \tag{8}$$

2.2 The Fully Discrete Scheme

In this Section, we briefly describe a fully discrete scheme for (8a),(8b). For the temporal discretization, we consider the Crank-Nicolson method, which is second-order in time.

Let V_h, W_h be finite dimensional subspaces of H_0^1 and H^1, respectively, with the following approximation properties:

$$\inf_{v_h \in V_h} \{\|v - v_h\|_{0,p} + h\|v - v_h\|_{1,p}\} \ \leq \ Ch^{k+1}\|v\|_{k+1,p}, \quad v \in H_0^1 \cap W^{k+1,p}(I),$$

and

$$\inf_{w_h \in W_h} \{\|w - w_h\|_{0,p} + h\|w - w_h\|_{1,p}\} \ \leq \ Ch^{r+1}\|w\|_{r+1,p}, \quad w \in W^{r+1,p}(I),$$

where $1 \leq p \leq \infty, k, r$ are integers.

Let $0 = t^0 < t^1 < \cdots < t^N = T$ be a given partition of the time interval $[0, T]$ with step length $\tau = \frac{T}{N}$, for some positive integer N. Define $t^n = n\tau$, $t^{n-\frac{1}{2}} = t^n - \frac{1}{2}\tau$, $\phi^n = \phi(t^n), \bar{\partial}_t\phi^n = (\phi^n - \phi^{n-1})/\tau$ for a smooth function ϕ. Let U^n and Q^n be the approximation of u and q at $t = t^n$ which are defined through the following explicit scheme.

$$\begin{cases} (a) \ \left(a\frac{U_x^n+U_x^{n-1}}{2}, v_{hx}\right) + \left(b\frac{U_x^n-U_x^{n-1}}{\tau}, v_{hx}\right) = \left(\frac{Q^n+Q^{n-1}}{2}, v_{hx}\right), v_h \in V_h, \\ (b) \ \left(\alpha\frac{Q^n-Q^{n-1}}{\tau}, w_h\right) + \left(\gamma\frac{Q^n+Q^{n-1}}{2}, w_h\right) + \left(\frac{Q_x^n+Q_x^{n-1}}{2}, w_{hx}\right) \\ \qquad\qquad + \left(\beta\frac{U_x^n+U_x^{n-1}}{2}, w_h\right) + (f^{n-\frac{1}{2}}, w_{hx}) = 0, \quad w_h \in W_h, \end{cases} \tag{9}$$

with U^0 and Q^0 to be defined latter.

3. Convergence Analysis

3.1 Preliminaries

We begin by reviewing some preliminary knowledge that will be used in the following convergence analysis. From [12] we define the Ritz-Volterra projection $\tilde{u}_h(t) \in V_h$, which satisfies:

$$(\int_0^t a(u(s) - \tilde{u}_h(s))_x ds + b(u(t) - \tilde{u}_h(t))_x, v_{hx}) = 0, v_h \in V_h.$$

It is easy to see that $\tilde{u}_h(t)$ is reduced to Ritz projection of $u(0)$ when $t=0$ and $\tilde{u}_h(t)$ also satisfies the following equation:

$$(a(u-\tilde{u}_h(t))_x + b(u_t - \tilde{u}_{ht}(t))_x, v_{hx}) = 0, v_h \in V_h. \tag{10}$$

Following [13], we define an elliptic projection $\tilde{q}_h \in W_h$, such that:

$$A(q-\tilde{q}_h, w_h) = 0, \qquad \forall w_h \in W_h, \tag{11}$$

where $A(u,v) = (u_x, v_x) + \lambda(u,v)$. Here λ is chosen appropriately so that A is H^1-coercive, i.e.,

$$A(v,v) \geq \alpha_0 \parallel v \parallel_1^2,$$

where α_0 is a positive constant. Moreover, it is easy to see that $A(\cdot,\cdot)$ is bounded.

Let $\eta = u - \tilde{u}_h$, $\rho = q - \tilde{q}_h$, then η and ρ satisfy the following estimates from [12] and [13]:

$$\parallel \eta(t) \parallel_j \leq Ch^{k+1-j} \parallel\mid u \mid\parallel_{k+1,0}, j=0,1, \tag{12}$$

$$\parallel \eta_t(t) \parallel_j \leq Ch^{k+1-j} \parallel\mid u \mid\parallel_{k+1,1}, j=0,1, \tag{13}$$

$$\parallel \eta_{tt}(t) \parallel_j \leq Ch^{k+1-j} \parallel\mid u \mid\parallel_{k+1,2}, j=0,1, \tag{14}$$

$$\parallel \eta_{ttt}(t) \parallel_j \leq Ch^{k+1-j} \parallel\mid u \mid\parallel_{k+1,3}, j=0,1, \tag{15}$$

and

$$\parallel \rho(t) \parallel_j + \parallel \rho_t(t) \parallel_j \leq Ch^{r+1-j}(\parallel q \parallel_{r+1} + \parallel q_t \parallel_{r+1}), j=0,1, \tag{16}$$

where

$$\parallel\mid u \mid\parallel_{k+1,m} = \sum_{i=0}^{m} \{ \parallel \frac{\partial^i u}{\partial t^i} \parallel_{k+1} + \int_0^t \parallel \frac{\partial^i u}{\partial t^i} \parallel_{k+1} ds \}.$$

3.2 Error Analysis

For the fully discrete error estimates, we split the errors into

$$u(t^n) - U^n = u(t^n) - \tilde{u}_h(t^n) + \tilde{u}_h(t^n) - U^n = \eta^n + \zeta^n,$$

$$q(t^n) - Q^n = q(t^n) - \tilde{q}_h(t^n) + \tilde{q}_h(t^n) - Q^n = \rho^n + \xi^n.$$

Since the estimates of η^n and ρ^n can be found out easily from (12) and (16) at $t=t^n$, it is enough to estimate ζ^n and ξ^n.

Setting $t = t^{n-\frac{1}{2}}$ in (8a), (8b) and combining (9a), (9b) with auxiliary projections we obtain the error equations in ζ^n and ξ^n

$$\begin{cases} (a)\ \left(a\frac{\zeta_x^n+\zeta_x^{n-1}}{2}, v_{hx}\right) + (b\bar{\partial}_t\zeta_x^n, v_{hx}) = a\left(\frac{\tilde{u}_{hx}^n+\tilde{u}_{hx}^{n-1}}{2} - \tilde{u}_{hx}^{n-\frac{1}{2}}, v_{hx}\right) \\ \quad +b\left(\bar{\partial}_t\tilde{u}_{hx}^n - \tilde{u}_{hxt}^{n-\frac{1}{2}}, v_{hx}\right) + \left(q^{n-\frac{1}{2}} - \frac{q^n+q^{n-1}}{2}, v_{hx}\right) \\ \quad + \left(\frac{\rho^n+\rho^{n-1}}{2}, v_{hx}\right) + \left(\frac{\xi^n+\xi^{n-1}}{2}, v_{hx}\right), \quad v_h \in V_h, \\ (b)\ (\alpha\bar{\partial}_t\xi^n, w_h) + A\left(\frac{\xi^n+\xi^{n-1}}{2}, w_h\right) = -(\alpha\bar{\partial}_t\rho^n, w_h) \\ \quad + \left((\lambda-\gamma)(\frac{\rho^n+\rho^{n-1}}{2} + \frac{\xi^n+\xi^{n-1}}{2}), w_h\right) + \beta\left(\frac{\eta^n+\eta^{n-1}}{2}, w_{hx}\right) \\ \quad +\beta\left(\frac{u_x^n+u_x^{n-1}}{2} - u_x^{n-\frac{1}{2}}, w_h\right) + \gamma\left(\frac{q^n+q^{n-1}}{2} - q^{n-\frac{1}{2}}, w_h\right) \\ \quad + \left(\frac{q_x^n+q_x^{n-1}}{2} - q_x^{n-\frac{1}{2}}, w_{hx}\right) + \alpha(\bar{\partial}_t q^n - q_t^{n-\frac{1}{2}}, w_h) - \beta\left(\frac{\zeta_x^n+\zeta_x^{n-1}}{2}, w_h\right),\ w_h \in W_h. \end{cases} \tag{17}$$

Theorem 3.1. *Assume that $U^0 = \tilde{u}_h(0), Q^0 = \tilde{q}_h(0)$ and $0 \le m \le N$. Then there exists a positive constant C independent of h and τ such that for sufficiently small τ*

$$\begin{aligned} & \| u^m - U^m \| + \| q^m - Q^m \| \\ \le\ & Ch^{\min\{k+1,\ r+1\}}(\| u \|_{L^\infty(H^{k+1})} + \| q \|_{L^\infty(H^{r+1})} + \| q_t \|_{L^\infty(H^{r+1})}) + \\ & C\tau^2(\| u \|_{L^2(H^1)} + \| u_t \|_{L^2(H^1)} + \| u_{tt} \|_{L^2(H^1)} + \| u_{ttt} \|_{L^2(H^1)} + \| q_{tt} \|_{L^2(H^1)} + \| q_{ttt} \|_{L^2(L^2)}). \end{aligned}$$

Proof. Choose $v_h = \frac{\zeta^n+\zeta^{n-1}}{2}$ in (17a) to obtain for $n = 0, 1, \cdots, N$

$$\begin{aligned} & \left(a\frac{\zeta_x^n+\zeta_x^{n-1}}{2}, \frac{\zeta_x^n+\zeta_x^{n-1}}{2}\right) + \left(b\bar{\partial}_t\zeta_x^n, \frac{\zeta_x^n+\zeta_x^{n-1}}{2}\right) \\ & = a\left(\frac{\tilde{u}_{hx}^n+\tilde{u}_{hx}^{n-1}}{2} - \tilde{u}_{hx}^{n-\frac{1}{2}}, \frac{\zeta_x^n+\zeta_x^{n-1}}{2}\right) + b\left(\bar{\partial}_t\tilde{u}_{hx}^n - \tilde{u}_{hxt}^{n-\frac{1}{2}}, \frac{\zeta_x^n+\zeta_x^{n-1}}{2}\right) \\ & + \left(q^{n-\frac{1}{2}} - \frac{q^n+q^{n-1}}{2}, \frac{\zeta_x^n+\zeta_x^{n-1}}{2}\right) + \left(\frac{\rho^n+\rho^{n-1}}{2}, \frac{\zeta_x^n+\zeta_x^{n-1}}{2}\right) + \left(\frac{\xi^n+\xi^{n-1}}{2}, \frac{\zeta_x^n+\zeta_x^{n-1}}{2}\right) \end{aligned}$$

Note that a and b are positive constants. Using Cauchy inequality we conclude that

$$\begin{aligned} & \tfrac{1}{2}b\tfrac{\|\zeta_x^n\|^2-\|\zeta_x^{n-1}\|^2}{\tau} \\ & \le C\Big(\| \tfrac{\tilde{u}_{hx}^n+\tilde{u}_{hx}^{n-1}}{2} - \tilde{u}_{hx}^{n-\frac{1}{2}} \|^2 + \| \bar{\partial}_t\tilde{u}_{hx}^n - \tilde{u}_{hxt}^{n-\frac{1}{2}} \|^2 \\ & + \| q^{n-\frac{1}{2}} - \tfrac{q^n+q^{n-1}}{2} \|^2 + \| \tfrac{\rho^n+\rho^{n-1}}{2} \|^2 + \| \tfrac{\xi^n+\xi^{n-1}}{2} \|^2 \Big). \end{aligned} \tag{18}$$

Multiplying (17) by 2τ and summing from $n=1$ to m, we can get the following estimate easily by choosing $U^0=\tilde{u}_h^0$.

$$\begin{aligned} b\parallel \zeta_x^m \parallel^2 \leq C\tau \sum_{n=1}^{m} \Big(&\parallel \tfrac{\tilde{u}_{hx}^n+\tilde{u}_{hx}^{n-1}}{2}-\tilde{u}_{hx}^{n-\frac{1}{2}} \parallel^2 + \parallel \bar{\partial}_t \tilde{u}_{hx}^n - \tilde{u}_{hxt}^{n-\frac{1}{2}} \parallel^2 \\ &+ \parallel q^{n-\frac{1}{2}} - \tfrac{q^n+q^{n-1}}{2} \parallel^2 + \parallel \tfrac{\rho^n+\rho^{n-1}}{2} \parallel^2 + \parallel \tfrac{\xi^n+\xi^{n-1}}{2} \parallel^2 \Big). \end{aligned} \tag{19}$$

By the Taylor formula with integral reminder we have that

$$\parallel \frac{\tilde{u}_{hx}^n+\tilde{u}_{hx}^{n-1}}{2}-\tilde{u}_{hx}^{n-\frac{1}{2}} \parallel^2 \leq C(\tau)^3 \int_{t^{n-1}}^{t^n} \parallel \tilde{u}_{hxtt} \parallel^2 ds.$$

Similarly, we have

$$\parallel \bar{\partial}_t \tilde{u}_{hx}^n - \tilde{u}_{hxt}^{n-\frac{1}{2}} \parallel^2 \leq C(\tau)^3 \int_{t^{n-1}}^{t^n} \parallel \tilde{u}_{hxttt} \parallel^2 ds.$$

and

$$\parallel q^{n-\frac{1}{2}} - \frac{q^n+q^{n-1}}{2} \parallel^2 \leq C(\tau)^3 \int_{t^{n-1}}^{t^n} \parallel q_{tt} \parallel^2 ds.$$

Thus

$$\begin{aligned} \parallel \zeta_x^m \parallel^2 \leq C(\tau)^4 \Big(&\textstyle\int_0^{t^m} \parallel \tilde{u}_{hxtt} \parallel^2 ds + \int_0^{t^m} \parallel \tilde{u}_{hxttt} \parallel^2 ds + \int_0^{t^m} \parallel q_{tt} \parallel^2 ds \Big) \\ &+ C\tau \sum_{n=1}^{m} \Big(\parallel \tfrac{\rho^n+\rho^{n-1}}{2} \parallel^2 + \parallel \tfrac{\xi^n+\xi^{n-1}}{2} \parallel^2 \Big). \end{aligned} \tag{20}$$

Setting $w_h=\frac{\xi^n+\xi^{n-1}}{2}$ in (17b) yields

$$\begin{aligned} &\left(\alpha\bar{\partial}_t\xi^n, \tfrac{\xi^n+\xi^{n-1}}{2}\right) + A\left(\tfrac{\xi^n+\xi^{n-1}}{2}, \tfrac{\xi^n+\xi^{n-1}}{2}\right) \\ &= \left(\alpha\bar{\partial}_t\rho^n, \tfrac{\xi^n+\xi^{n-1}}{2}\right) + \left((\lambda-\gamma)(\tfrac{\rho^n+\rho^{n-1}}{2}+\tfrac{\xi^n+\xi^{n-1}}{2}), \tfrac{\xi^n+\xi^{n-1}}{2}\right) \\ &+\beta\left(\tfrac{\eta^n+\eta^{n-1}}{2}, \tfrac{\xi_x^n+\xi_x^{n-1}}{2}\right) + \beta\left(\tfrac{u_x^n+u_x^{n-1}}{2}-u_x^{n-\frac{1}{2}}, \tfrac{\xi^n+\xi^{n-1}}{2}\right) \\ &+\gamma\left(\tfrac{q^n+q^{n-1}}{2}-q^{n-\frac{1}{2}}, \tfrac{\xi^n+\xi^{n-1}}{2}\right) + \left(\tfrac{q_x^n+q_x^{n-1}}{2}-q_x^{n-\frac{1}{2}}, \tfrac{\xi_x^n+\xi_x^{n-1}}{2}\right) \\ &+\alpha\left(\bar{\partial}_t q^n - q_t^{n-\frac{1}{2}}, \tfrac{\xi^n+\xi^{n-1}}{2}\right) - \beta\left(\tfrac{\zeta_x^n+\zeta_x^{n-1}}{2}, \tfrac{\xi^n+\xi^{n-1}}{2}\right) \end{aligned} \tag{21}$$

Using ε inequality and the coercive property of $A(\cdot,\cdot)$ we derive that

$$\begin{aligned} &\frac{\alpha}{2\tau}(\parallel \xi^n \parallel^2 - \parallel \xi^{n-1} \parallel^2) + (\alpha_0-\varepsilon) \parallel \frac{\xi^n+\xi^{n-1}}{2} \parallel_1^2 \\ &\leq C\Big(\parallel \bar{\partial}_t\rho^n \parallel^2 + \parallel \frac{\rho^n+\rho^{n-1}}{2} \parallel^2 + \parallel \frac{\xi^n+\xi^{n-1}}{2} \parallel^2 + \parallel \frac{\eta^n+\eta^{n-1}}{2} \parallel^2 \\ &+ \parallel \frac{u_x^n+u_x^{n-1}}{2} - u_x^{n-\frac{1}{2}} \parallel^2 + \parallel \frac{q^n+q^{n-1}}{2} - q^{n-\frac{1}{2}} \parallel^2 + \parallel \frac{q_x^n+q_x^{n-1}}{2} - q_x^{n-\frac{1}{2}} \parallel^2 \\ &+ \parallel \bar{\partial}_t q^n - q_t^{n-\frac{1}{2}} \parallel^2 + \parallel \frac{\zeta_x^n+\zeta_x^{n-1}}{2} \parallel^2 \Big). \end{aligned}$$

Since

$$\| \bar{\partial}_t \rho^n \|^2 \leq C\frac{1}{\tau}\int_{t^{n-1}}^{t^n} \| \rho_t \|^2 \, ds,$$

$$\| \zeta_x^J \|^2 \leq C(\tau)^4\Big(\int_0^{t^m} \| \tilde{u}_{hxtt} \|^2 \, ds + \int_0^{t^m} \| \tilde{u}_{hxttt} \|^2 \, ds + \int_0^{t^m} \| q_{tt} \|^2 \, ds\Big)$$
$$+C\tau\sum_{n=1}^{m}\Big(\| \frac{\rho^n+\rho^{n-1}}{2} \|^2 + \| \frac{\xi^n+\xi^{n-1}}{2} \|^2 \Big),$$

and

$$\| \frac{\xi^n+\xi^{n-1}}{2} \|^2 \leq \frac{\| \xi^n \|^2 + \| \xi^{n-1} \|^2}{2},$$

$$\| \frac{u_x^n+u_x^{n-1}}{2} - u_x^{n-\frac{1}{2}} \|^2 \leq C(\tau)^3\int_{t^{n-1}}^{t^n} \| u_{xtt} \|^2 \, ds,$$

$$\| \frac{q^n+q^{n-1}}{2} - q^{n-\frac{1}{2}} \|^2 \leq C(\tau)^3\int_{t^{n-1}}^{t^n} \| q_{tt} \|^2 \, ds,$$

$$\| \frac{q_x^n+q_x^{n-1}}{2} - q_x^{n-\frac{1}{2}} \|^2 \leq C(\tau)^3\int_{t^{n-1}}^{t^n} \| q_{xtt} \|^2 \, ds,$$

$$\| \bar{\partial}_t q^n - q_t^{n-\frac{1}{2}} \|^2 \leq C(\tau)^3\int_{t^{n-1}}^{t^n} \| q_{ttt} \|^2 \, ds,$$

then, multiplying by 2τ and summing from 1 to m we conclude that

$$\begin{aligned}
& (\alpha - C\tau) \| \xi^J \|^2 + 2(\alpha_0-\varepsilon)\tau\sum_{n=1}^{m} \| \frac{\xi^n+\xi^{n-1}}{2} \|_1^2 \\
\leq \;& C\int_0^{t^m} \| \rho_t \|^2 \, ds + C\tau\sum_{n=0}^{m}\Big(\| \rho^n \|^2 + \| \eta^n \|^2 \Big) \\
+ \;& C\tau\sum_{n=0}^{J-1} \| \xi^n \|^2 + C(\tau)^4\Big(\int_0^{t^m} \| q_{tt} \|^2 \, ds + \int_0^{t^m} \| q_{ttt} \|^2 \, ds + \int_0^{t^m} \| q_{xtt} \|^2 \, ds \\
+ \;& \int_0^{t^m} \| u_{xtt} \|^2 \, ds + \int_0^{t^m} \| \tilde{u}_{hxtt} \|^2 \, ds + \int_0^{t^m} \| \tilde{u}_{hxttt} \|^2 \, ds\Big).
\end{aligned}$$

Taking τ_1, let $0 < \tau \leq \tau_1$, such that $\alpha - C\tau > 0$, then by discrete Gronwall's lemma we obtain that

$$\begin{aligned}
& \| \xi^J \|^2 + \alpha_0\tau\sum_{n=1}^{m} \| \tfrac{\xi^n+\xi^{n-1}}{2} \|_1^2 \\
\leq \;& C\int_0^{t^m} \| \rho_t \|^2 \, ds + C\tau\sum_{n=0}^{m}\Big(\| \rho^n \|^2 + \| \eta^n \|^2 \Big) \\
+ \;& C(\tau)^4\Big(\int_0^{t^m} \| q_{tt} \|^2 \, ds + \int_0^{t^m} \| q_{ttt} \|^2 \, ds + \int_0^{t^m} \| q_{xtt} \|^2 \, ds \\
+ \;& \int_0^{t^m} \| u_{xtt} \|^2 \, ds + \int_0^{t^m} \| \tilde{u}_{hxtt} \|^2 \, ds + \int_0^{t^m} \| \tilde{u}_{hxttt} \|^2 \, ds\Big).
\end{aligned} \tag{22}$$

Note that

$$\| \tilde{u}_{hxtt} \| \leq \| \tilde{u}_{hxtt} - u_{xtt} \| + \| u_{xtt} \| \leq \| \eta_{tt} \|_1 + \| u_{tt} \|_1 .$$

Therefore

$$\begin{aligned} \| \xi^m \|^2 &\leq C(\| \eta \|^2_{L^\infty(L^2)} + \| \rho \|^2_{L^\infty(L^2)} + \| \rho_t \|^2_{L^2(L^2)}) \\ &+ C\tau^4(\| q_{tt} \|^2_{L^2(H^1)} + \| q_{ttt} \|^2_{L^2(L^2)} \\ &+ \tau^4(\| u_{ttt} \|^2_{L^2(H^1)} + \| u_{tt} \|^2_{L^2(H^1)} + \| u_t \|^2_{L^2(H^1)} + \| u \|^2_{L^2(H^1)}). \end{aligned} \tag{23}$$

Then, (20) and (23) imply that

$$\begin{aligned} \| \zeta^m_x \|^2 &\leq C(\| \eta \|^2_{L^\infty(L^2)} + \| \rho \|^2_{L^\infty(L^2)} + \| \rho_t \|^2_{L^2(L^2)}) \\ &+ C\tau^4(\| q_{tt} \|^2_{L^2(H^1)} + \| q_{ttt} \|^2_{L^2(L^2)} \\ &+ \tau^4(\| u_{ttt} \|^2_{L^2(H^1)} + \| u_{tt} \|^2_{L^2(H^1)} + \| u_t \|^2_{L^2(H^1)} + \| u \|^2_{L^2(H^1)}). \end{aligned} \tag{24}$$

Combining (23), (24) and the estimates of η^n, ρ^n, by the triangle inequality we can complete the proof.

Remark 3.2. In this paper, we only discuss the H^1-Galerkin mixed finite element schemes for the one-dimensional problem. In fact these schemes can be extended to several dimensional problem without introducing *rot* operator which was used in [4]. We use standard finite element space to approximate the unknown function u, while the gradient function q is approximated by the vector function space of the standard mixed finite element spaces(e.g., Raviart-Thomas spaces). The more details one can see [14].

4. Numerical Example

In this section a numerical example is given to verify the theorems presented in this paper.

Example 4.1. Let us consider the following initial and boundary problem:

$$\begin{cases} \frac{1}{\delta}u_t + u_{tt} = au_{xx} + bu_{xxt} + f(x,t), & (x,t) \in [0,1] \times (0,1], \\ u(0,t) = u(1,t) = 0, & t \in (0,1], \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in [0,1], t = 0, \end{cases} \tag{25}$$

where $a = \frac{1}{3}, b = \frac{1}{6}, \delta = \frac{6}{6+\pi^2}$, and the exact solution is chosen as $u(x,t) = e^{-t} sin(\pi x)$. This example is taken from [15]

We solve this problem by H^1-Galerkin mixed finite element method. Piecewise linear finite element spaces are used to approximate the unknown function u and its flux q, respectively.

The errors of $u - u_h$ and $q - q_h$ in L^2 norm for different time are shown in Table 4.1 and 4.2, respectively. The order of convergence for u and q in L^2 norm are displayed in Table 4.3. We observe that the rate of convergence is approximately equal to 2, which are in agreement with our theoretical results proposed in Section 3.

Table 4.1. The errors of $\| u - u_h \|$ at different time.

Time	t=0.2		t=0.4		t=0.8		t=1.0	
$h = \tau$	Error	Order	Error	Order	Error	Order	Error	Order
1/20	2.7559e-004	\	4.6745e-004	\	6.5543e-004	\	6.8058e-004	\
1/40	6.8872e-005	1.9131	1.1682e-004	1.9850	1.6383e-004	1.9809	1.7015e-004	2.0000
1/80	1.7216e-005	1.9913	2.9203e-005	1.9899	4.0957e-005	2.0082	4.2537e-005	1.9069

Table 4.2. The errors of $\| q - q_h \|$ at different time.

Time	t=0.2		t=0.4		t=0.8		t=1.0	
$h = \tau$	Error	Order	Error	Order	Error	Order	Error	Order
$\frac{1}{20}$	2.1945e-004	\	3.4766e-004	\	4.4515e-004	\	4.4834e-004	\
$\frac{1}{40}$	5.3504e-005	2.0362	8.4796e-005	2.0356	1.0865e-004	2.0346	1.0946e-004	2.0342
$\frac{1}{80}$	1.3210e-005	2.0180	2.0938e-005	2.0179	2.6833e-005	2.0176	2.7035e-005	2.0175

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