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ON THE LUCAS TRIANGLE AND ITS RELATIONSHIP WITH THE k-LUCAS NUMBERS

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Abstract. After defining the k-Lucas numbers of similar form to as the k-Fibonacci numbers are defined, a table with the polynomic expression of the first numbers of Lucas is indicated. The coefficients of these polynomials, properly placed, constitute a table that receives the name of Lucas triangle.

Later, we study some properties of this triangle and the sequences obtained from this one, either are by rows, or by diagonals or antidiagonals.

Finally we generate the classical Pascal trinagle from the k-Lucas triangle.

Keywords: k-Lucas numbers, The Pascal 2-triangle, The Lucas triangle.

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1. Introduction

In this section, we will remember the generation of the k-Lucas numbers and some of its properties.

In [5] we have created the k-Lucas numbers as an extension of the k-Fibonacci numbers [3, 4] and proved some of its many properties.

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Definition 1.1. *For any integer number $k \geq 1$, k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbf{N}}$ is defined recurrently by:*

$$F_{k,0} = 0,\ F_{k,1} = 1,\ \text{and } F_{k,n+1} = k\,F_{k,n} + F_{k,n-1} \text{ for } n \geq 1$$

Later we have defined the k-Lucas sequences in a similar form as it follows [5]:

Definition 1.2. *For any integer number $k \geq 1$, the k-Lucas sequence, say $\{L_{k,n}\}_{n\in\mathbf{N}}$, is defined recurrently by:*

$$L_{k,0} = 2,\ L_{k,1} = k,\ \text{and } L_{k,n+1} = k\,L_{k,n} + L_{k,n-1} \text{ for } n \geq 1 \tag{1}$$

Among its properties, we indicate the following:

(1) *Generation of the k-Lucas numbers*: $L_{k,n}^2 = (k^2+4)F_{k,n}^2 + 4(-1)^n$

(2) *Binet formula*: $L_{k,n} = \sigma_k^n + (-\sigma_k)^{-n}$ where σ_k is the positive characteristic root $\sigma_k = \dfrac{k+\sqrt{k^2+4}}{2}$

(3) *Relationship with the k-Fibonacci numbers*: $L_{k,n} = F_{k,n-1} + F_{k,n+1}$

(4) *Combinatorial formula*: $L_{k,n} = \dfrac{1}{2^{n-1}}\displaystyle\sum_{j=0}^{n}\binom{n}{2j}k^{n-2j}(k^2+4)^j$

2. Preliminaries

In this section, the Lucas triangle is presented and studied by mean of the k-Lucas sequences, and some elementary properties of this triangle are proven straightforwardly. As a particular case of the Lucas triangle, the classical Pascal triangle is obtained.

From relationship (1), we can writte the first k-Lucas numbers as follows:

TABLE 1. Polynomic expression of the first k-Lucas numbers

$$\begin{aligned}
L_{k,0} &= 2\\
L_{k,1} &= k\\
L_{k,2} &= k^2+2\\
L_{k,3} &= k^3+3k\\
L_{k,4} &= k^4+4k^2+2\\
L_{k,5} &= k^5+5k^3+5k
\end{aligned}$$

It is worthy to be noted that the coefficients arising in the previous list can be written in triangular position, in such a way that every side of the triangle is double. This triangle will be called *Lucas triangle* [2, 7]:

TABLE 2. The Lucas triangle

0					2					
1					1					
2				1		2				
3				1		3				
4			1		4		2			
5			1		5		5			
6		1		6		9		2		
7		1		7		14		7		
8	1		8		20		16		2	

Note that the numbers belonging to the same row of the Lucas triangle are the coefficients of $L_{k,n}$ as they are expressed in Table 1. The first of these numbers is 1. If we note by $L_{k,n}^{(j)}$ the j-th coefficient in the expression of $L_{k,n}$ as polynomial on k, for $j = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$, and $n \geq 3$, then the subsequent numbers into the Lucas triangle can be calculated from the previous rows as the following equation establishes:

$$L_{k,n}^{(j)} = L_{k,n-1}^{(j)} + L_{k,n-2}^{(j-1)} \text{ for } 0 \leq j \leq \lfloor \frac{n}{2} \rfloor \tag{2}$$

For $k = 1$, this equation is the Formula (1.8) given by N. Robbins in [7].

If we write the diagonal of the Lucas triangle as rows of a new triangle, then we obtain the classical Lucas triangle [2, 7]:

TABLE 3. The classical Lucas triangle

2							
1	2						
1	3	2					
1	4	5	2				
1	5	9	7	2			
1	6	14	16	9	2		
1	7	20	30	25	11	2	
1	8	27	50	55	36	13	2

3. Main results

Next, we will prove the second combinatorial formula for k-Lucas numbers, but previously, we need the following Lemma.

Lemma 3.1. *For $n \geq 2$ and $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$:*

$$L_{k,n}^{(j)} = \frac{n}{j}\binom{n-1-j}{j-1} \tag{3}$$

Proof. We will prove this emma by induction. For $n = 2$, it is $j = 1$ and, consequently, $L_{k,1}^{(1)} = \frac{2}{1}\binom{1}{1} = 2$.

Let us suppose this formula is true until $n-1$. Applying Formula (2) and taking into account $\binom{m}{r} = \frac{m-r+1}{r}\binom{m}{r-1}$ [6]:

$$\begin{aligned}
L_{k,n}^{(j)} &= \frac{n-1}{j}\binom{n-2-j}{j-1} + \frac{n-2}{j-1}\binom{n-2-j}{j-2} \\
&= \frac{n-1}{j} \cdot \frac{n-2-j-(j-1)+1}{j-1}\binom{n-2-j}{j-2} + \frac{n-2}{j-1}\binom{n-2-j}{j-2} \\
&= \frac{1}{j-1}\left(\frac{n-1}{j}(n-2i)+n-2\right)\binom{n-2-j}{j-2} \\
&= \frac{1}{j} \cdot \frac{n^2-(j+1)n}{j-1}\binom{n-2-1}{j-2} \\
&= \frac{1}{j} \cdot \frac{n(n-1-j)}{j-1}\frac{j-1}{n-1-j}\binom{n-1-j}{j-1} \\
&= \frac{n}{j}\binom{n-1-j}{j-1}
\end{aligned}$$

because $\binom{m}{r} = \frac{m}{r}\binom{m-1}{r-1} \to \binom{m-1}{r-1} = \frac{r}{m}\binom{m}{r}$

This formula coincides with Formula (1.1) given by N. Robbins in [7] and is also in Introduction of [2].

As a consequence of Formulas (1) and (3), the n-th k-Lucas number can be calculated by means of this second combinatorial formula

$$L_{k,n} = k^n + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{j}\binom{n-1-j}{j-1}k^{n-2j}$$

Below, the generating function for the k-Lucas sequences is given. As a result, the terms of the k-Lucas sequences are seen as the coefficients of the corresponding generating function [6, 9].

Theorem 3.2. *The generating function of the k-Lucas numbers is* $l_k(x) = \dfrac{2 - k\,x}{1 - k\,x - x^2}$

Proof. Let us suppose the Lucas numbers are the coefficients of a potential series centered at the origin, and let us consider the corresponding analytic function $l_k(x)$. The function defined in such a way is called the generating function of the k-Lucas numbers. So,

$$\begin{aligned} l_k(x) &= L_{k,0} + L_{k,1}x + L_{k,2}x^2 + \ldots + L_{k,n}x^n + \ldots \\ k\,x\,l_k(x) &= k\,L_{k,0}x + k\,L_{k,1}x^2 + k\,L_{k,2}x^3 + \ldots + k\,L_{k,n}x^{n+1} + \ldots \\ x^2 l_k(x) &= L_{k,0}x^2 + L_{k,1}x^3 + L_{k,2}x^4 + \ldots + L_{k,n}x^{n+2} + \ldots \end{aligned}$$

from where, since $L_{k,j} = k\,L_{k,j-1} + L_{k,j-2}$, $L_{k,0} = 2$, and $L_{k,1} = k$, it is obtained: $(1 - k\,x - x^2)l_k(x) = 2 - k\,x$. So, the generating function for the k-Lucas sequence, $\{L_{k,n}\}_{n=0}^{\infty}$, is

$$l_k(x) = \frac{2 - k\,x}{1 - k\,x - x^2} \tag{4}$$

Note that by doing the quotient of the generating function a power series, centered at the origin appears:
$l_k(x) = 2 + k\,x + (k^2+2)x^2 + k^3 + 3k)x^3 + (k^4 + 4k^2 + 2)x^4 + (k^5 + 5k^3 + 5k)x^5 + \cdots$, where the coefficients are precisely the k-Lucas numbers (see Table 1).

Moreover, if we substitute x for a small value in Equation (4), for example $x = 10^{-r}$, for $r \in N$, then the quotient of $l_k(x)$ if written as a decimal number has its integer part equal to 2, and its decimal part can be seen as r-uplas showing the first terms of the n-th anti-diagonal of the Lucas triangle. For example,

$$\begin{aligned} l_3(10^{-3}) &= \frac{2 - 3\cdot 0.001}{1 - 3\cdot 0.001 - 0.001^2} = 2.\underbrace{003}\underbrace{011}\underbrace{036}\underbrace{119}\underbrace{393}\ldots \\ &\longrightarrow \{2, 3, 11, 36, 119, 393, \ldots\} = \{L_{3,n}\} \end{aligned}$$

which are the first terms in the <u>third</u> anti-diagonal of Lucas triangle.

3.3. The Lucas triangle in rectangular form.

It will be called *diagonal* each line of numbers from left to right and from top to bottom. Numerating these diagonals from 0, for example, the 2-diagonal is: $\{1, 4, 9, 16, 25, 36, 49, \ldots\}$.

It will be called *anti-diagonal* to each of the different lines of couples of adjacent numbers on the Lucas triangle as shown in Table 2, from right to left and from first row to bottom. Starting by 0, the second anti-diagonal is $\{2-5, 9-14, 20-27, 35-44, 54-65, \ldots\}$.

The table corresponding to the Lucas triangle (Table 2), can be written in rectangular form, as follow: all the terms of the first row are 2 and all the terms of the first column (excepting the first) are 1. If $l_{i,j}$ is the term of the row r and of the column n, then $l_{r,n} = l_{r,n-1} + l_{r-1,n}$. As a consequence, the rows of this rectangle are the diagonals and the columns are the antidiagonals of the Lucas triangle, respectively. See Table(4).

TABLE 4. The Lucas triangle in rectangular form

$r\backslash n$	0.	1.	2.	3.	4.	5.	6.	7.	8.
0.	2	2	2	2	2	2	2	2	2
1.	1	3	5	7	9	11	13	15	17
2.	1	4	9	16	25	36	49	64	81
3.	1	5	14	30	55	91	140	204	285
4.	1	6	20	50	105	196	336	540	825
5.	1	7	27	77	182	378	714	1254	2079
6.	1	8	35	112	294	672	1386	2640	4719

In this Table, each element is $l_{r,n} = l_{r,n-1} + l_{r,n-1}$.

If in the Table 4 each row is moved two positions to the right with respect the preceding row, Table 5 is obtained [4].

TABLE 5. A new Lucas triangle

2	2	2	2	2	2	2	2	2	2	2	2	2
		1	3	5	7	9	11	13	15	17	19	21
				1	4	9	16	25	36	49	64	81
						1	5	14	30	55	91	140
									1	6	20	50
											1	7

Note that each column of Table 5 equalizes the corresponding diagonal of the Lucas triangle.

3.4. Properties of the diagonals of the Lucas triangle.

The r-th element of the n-th diagonal $\{1, j+2, \ldots\}$ verify the equation

$$d_{j,n} = \frac{n+2r}{r}\binom{n-1+r}{r-1} \tag{5}$$

and its proof is similar to Lemma 3.1.

The diagonal sequences are listed in The Online Encyclopedia of Integer Sequences, from now on OEIS [8].

Some remarks are bellow in order:

- Each diagonal sequence is the partial sums of the preceding sequence.
- The second diagonal sequence is the sequence of the odd numbers.

  Proof. It is enough to substitute $n = 1$ in Equation (3).
- The third diagonal sequence correspond to the square numbers and is the sum of two consecutive triangular numbers.

  Proof. Substitute $n = 2$ in Equation (3).
- The fourth diagonal sequence is the square of pyramidal numbers
- The fifth diagonal sequence is the 4-dimensional pyramidal numbers and so on.
- Each element of a diagonal sequence results in the sum of the same order element and all the previous elements in the preceding diagonal.

The n-th element of the r-th column of the classical Lucas triangle verifies the equation $l_{r,n} = \frac{n+2r}{r}\binom{n-1+r}{r-1}$ and its proof is similar to the used one in Lemma (**??**).

The first column sequences and the diagonal sequences are listed in OEIS.

3.5. Properties of the rows of the Lucas triangle.

By inspection of Table 2, may be observed some properties of the rows of the Lucas triangle:

- As we have previously obtained (formula 3), each element of the row n for $i > 1$ is $L_{k,n}^{(j)} = \frac{n}{j}\binom{n-1-j}{j-1}$ being $L_{k,n}^{(0)} = 1$.
- The sum of the elements of the n-th row equalizes the corresponding classical Lucas number obtaining the Lucas sequence $\{2, 1, 3, 4, 7, 11, 18, \ldots\}$ listed as A000032 in

OEIS. An element of this sequence is:
$\sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{j}\binom{n-1-j}{j-1} = L_n$, as it was obtained before.

- The sum of the elements of the $(n-1)$-th row with the terms of the $(n+1)$-th row gives the corresponding Lucas number. That is, $\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} L_{k,n-1}^{(j)} + \sum_{j=0}^{\lfloor \frac{n+2}{2} \rfloor} L_{k,n+1}^{(j)} = L_{k,n}$.

  This expression is another version of the well-know relation between Fibonacci and Lucas numbers: $L_{k,n} = F_{k,n+1} + F_{k,n-1}$.

3.6. The Pascal triangle from the Lucas triangle.

To derive the Pascal triangle of the Lucas triangle, we will write the Lucas triangle of the following form:

TABLE 6. The right Lucas triangle

	1.	2.	3.	4.	5.
1.	1				
2.	1	2			
3.	1	3			
4.	1	4	2		
5.	1	5	5		
6.	1	6	9	2	
7.	1	7	14	7	
8.	1	8	20	16	2
9.	1	9	27	30	9

Next we construct another table dividing each number by the order i of its row and multiplying consecutively by $r,\ r-1,\ r-2,\ \lfloor \frac{n+1}{2} \rfloor$ (until to finish the row). For instance, the 9-th new row is $1 \cdot \frac{9}{9},\ 9 \cdot \frac{8}{9},\ 27 \cdot \frac{7}{9},\ 30 \cdot \frac{6}{9},\ 9 \cdot \frac{5}{9}$. The new table is

	1.	2.	3.	4.	5.
1.	1				
2.	1	1			
3.	1	2			
4.	1	3	1		
5.	1	4	3		
6.	1	5	6	1	
7.	1	6	10	4	
8.	1	7	15	10	1
9.	1	8	21	20	5

Finaly, we can writte the diagonals of this triangle as rows and we obtain the classical Pascal triangle:

Table 7. The Pascal triangle

0.	1							
1.	1	1						
2.	1	2	1					
3.	1	3	3	1				
4.	1	4	6	4	1			
5.	1	5	10	10	5	1		
6.	1	6	15	20	15	6	1	
7.	1	7	21	35	21	21	7	1

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