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ON SOME FUNDAMENTAL RESULTS ON LINEAR ALGEBRA

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Abstract. Jordan Exchange is a method for solving a given system of m linear equations in n unknowns by exchanging the roles of the dependent and independent variables. Using this method, we prove in this paper, some new and different proofs for several fundamental results of Linear algebra.

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1. Introduction

Linear Algebra is an essential part of the mathematical tool required in the modern study of many areas in the behavioral, natural, physical and social sciences, in engineering, in business, in computer science, and of course in pure and applied mathematics. We are very well aware of the importance of system of linear equations that arise in the application of mathematics to diverse areas and how matrices and their inverse are closely related to the problem of solving systems of linear equations. In this paper we develop a new mathematical technique for proving some of the fundamental well-known results in linear algebra.

Throughout this paper we adopt the following notation.

R^n : Real n-dimensional space

x, y : Column vectors

x_i : The i th component of the vector x

x^t : The transpose of the vector x

A, B : Matrices

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A_i : The i th row of the matrix A

A_j : The j th column of the matrix A

a_{ij} : Element in the i-th row and j-th column of the matrix A

A^{-1} : The inverse of the matrix A

A^t : The transpose of the matrix A

$\square$: End of proof

Ordinary Jordan Exchange

Consider the system $y_i = a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n$, $i = 1, ..., m$ of m linear equations in n unknowns $x_1, ..., x_n$. This system can be represented in the form of a compact table:

	$x_1 ... x_r ... x_n$
$y_1 =$	$a_{11} ... a_{1s} ... a_{1n}$
.	.
.	.
.	.
$y_r =$	$a_{r1} ... a_{rs} ... a_{rn}$
.	.
.	.
.	.
$y_m =$	$a_{m1} ... a_{ms} ... a_{mn}$

The original system can be read from this table by setting y_i equal to the sum of the products of the entries in the i-th row with the corresponding variables on the top of the table. By convention, a table is said to represent the point in R^n obtained by setting the independent variables on the top of that table to zero. Thus, the initial table represents the origin in R^n.

We shall use the expression ordinary Jordan exchange with pivot row r, pivot column s, and pivot $a_{rs} \neq 0$ to mean the operation of switching the roles of the dependent (slack) variable y_r and the independent variable x_s. This is done by solving the equation $y_r = a_{r1}x_1 + ... + a_{rs}x_s + ... + a_{rn}x_n$ for x_s, substituting for x_s in the remaining equations, and writing the system in a new table of the form

	$x_1 ... y_r ... x_n$
$y_1 =$	$a_{11} ... a_{1s} ... a_{1n}$
.	.
.	.
.	.
$x_s =$	$a'_{r1} ... a'_{rs} ... a'_{rn}$
.	.
.	.
.	.
$y_m =$	$a'_{m1} ... a'_{ms} ... a'_{mn}$

Rules for Jordan exchange

To derive general rules for Jordan exchange, we find the relationship between the entries of the current table and the entries of the previous one. We have

$$y_r = \sum_{j=1}^{n} a_{rj} x_j = a_{rs} x_s + \sum_{j=1}^{n} a_{rj} x_j$$

$j \neq s$ since $a_{rs} \neq 0$. Solving for x_s, we get $x_s = \frac{1}{a_{rs}} y_r + \sum_{j=1, j \neq s}^{n} -\frac{-a_{rj} x_j}{a_{rs}}$. Substituting for x_s in the remaining equations, we get for $i \neq r$ $y_i = a_{is} x_s + \sum_{j=1}^{n} a_{ij} x_j = \frac{a_{is}}{a_{rs}} y_r + \sum_{j=1, j \neq s}^{n} (a_{ij} - \frac{a_{rj} a_{is}}{a_{rs}} x_j)$.

Therefore by comparing the last two equations with the second table, we find that the general rules for Jordan exchange with pivot $a_{rs} \neq 0$ are:

$a'_{rs} = \frac{1}{a_{rs}}$ new pivot; $a'_{rj} = \frac{-a_{rj}}{a_{rs}}, j \neq s$ new pivot row;

$a'_{is} = \frac{a_{is}}{a_{rs}}, i \neq r$, new pivot column;

$a'_{ij} = a_{ij} - \frac{a_{rj} a_{is}}{a_{rs}}, i \neq r, j \neq s$, new off-pivot rows and columns.

In Section 2 we give either alternative proofs or new proofs using Jordan exchange technique for certain well known theorems in linear algebra. Also we include an appendix to illustrate these techniques with some numerical examples.

2. Application of Jordan exchange to Linear Algebra

Theorem 2.1. If all the equations of the linear system $y_i = A_i x$, $x \in R^n$, $i = 1, ..., m$ are linearly independent, then $m \leq n$.

Proof. Suppose that all the equations are linearly independent and that $m > n$. Then by exchanging y's and x's until it is no longer possible, we reach one of the following cases:

Case 1: All the x's can be shifted to the (leftmost column) side of the table. Rearranging and relabeling if necessary, the final table will be of the form

	y_I
$x =$	B
$y_{II} =$	C

Where $y_I = (y_1, y_2, ..., y_n)^t$, $y_{II} = (y_{n+1}, ..., y_m)^t$. Reading this table we get $y_{II} = Cy_I$ which means y_{II} is linearly dependent on y_I, a contradiction. Hence $m \leq n$.

Case 2. We are blocked by zeros (no further exchange are possible) while some x's are still on the top of the table.

Let k be the number of x's that have been exchanged to the side of the table. Rearranging and relabeling if necessary, we get a table of the form

	y_I	y_{II}
$x_I =$	B_I	B_{II}
$y_{II} =$	C	0

Where $y_I = (y_1, y_2, ..., y_k)^t$ and $y_{II} = (y_{k+1}, ..., y_m)^t$. Reading this table we get $y_{II} = Cy_I + 0x_{II} = Cy_I$ which means that y_{II} is linearly dependent on y_I, a contradiction. Hence $m \leq n$. □

Theorem 2.2. Given the system $y_i = A_i x . x \in R^n, i = 1, ..., m$ then all the dependent variables can be converted into independent variables if and only if the system of equations is linearly independent.

Proof. Suppose that the system is linearly dependent. Let $y_I = (y_1, ..., y_k), k < m$, represents the independent equations of the system. If all the y's can be converted into independent variables (exchanged to the top of the table) then we can first exchange those y's, which are linearly independent. This yield the table

	y_I	x_{II}
$x_I =$	B_I	B_{II}
$y_{II}=$	C_I	C_{II}

Reading this table we find that in order for $y_{II} = C_I y_I + C_{II} x_{II}$ to be linearly dependent on y_I, C_{II} must be zero. But if $C_{II} = 0$ then y_{II} cannot be carried to the top of the table, a contradiction. Hence, all the y's must be linearly independent.

Suppose that all the equations are linearly independent (by Theorem 2.1, $m \leq n$) and that some y's cannot be exchanged to the top of the table (blocked by zero). Then rearranging and relabeling if necessary, we get a table of the form

	y_I	x_{II}

$x_I =$	B_I	B_{II}
$y_{II} =$	C	0

Reading this table we get $y_{II} = Cy_I$ to be linearly dependent on y_I which contradicts the hypothesis of the linear independence of the dependent variables. Hence, we must be able to convert all the dependent variables into independent variables. □

Theorem 2.3. Let A be an $n \times n$ matrix. Then the system $y = Ax$ can be converted into $x = By$ by n Jordan exchanges if and only if A is non singular. Moreover, we can determine whether A is nonsingular after at most n exchanges.

Proof. If A is singular then the rows of A are linearly dependent. By Theorem 2.3 some of the y's cannot be exchanged to the top of the table which makes it impossible for the system $y = Ax$ to be inverted to $x = By$. Moreover, this can be determined in less than n exchanges since some of the y's will remain on the side of the table.

If A is nonsingular , then the rows of A are linearly independent. By Theorem 2.3 all the y's can be exchanged to the top of the table. Since the number of y's is n, the system $y = Ax$ will be inverted to $x = By$ after exactly n exchanges. Hence, in both cases, we are able to determine whether or not the matrix is nonsingular with n exchanges at most.□

It is possible to use Jordan exchange to compute the rank of the matrix $\begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$.

To do so, we can write A in the form of a table and use Jordan exchange to carry all the independent variables to the top of that table, that is until all the y's are carried to the top or we are blocked by table of the form

	y_I	x_{II}
$x_I =$		
$y_{II} =$		0

By Theorem 2.2 the rank of the matrix A is the number of y's that can be carried to the top of the table. Moreover, we can use the last table to write the y's remaining on the side of the table as a linear combination of the y's that have been exchanged.

Theorem 2.4. If the system $Ax = b$ has a solution then the rank of A is equal to the rank of the augmented matrix $[A, b]$.

Proof. If we let $y = Ax - b = 0$ then we can write the system in a table of the form

	x	1
y	A	-b

y_I x_{II} 1

$x_I =$			d_I
y_I	C	0	d_{II}

Where $y_I = (y_1, y_2, ..., y_k)^t$ and $y_{II} = (y_{k+1}, y_{k+2}, ...y_n)^t$. Reading this table we get $y_{II} = Cy_I + 0x_{II} + d_{II} = 0$ which is true if and only if $d_{II} = 0$. This means that y_{II} must be linearly dependent on y_I. Hence the rank of $[A, b] = k$ which is the rank of A. □

Theorem 2.5. Let A be an $m \times n$ matrix. Then $Ax = b$ has no solution, exactly one solution, or infinitely many solutions.

Proof. Let $y = Ax - b = 0$. then we can write the system in a table of the form

	x	1
y	A	-b

and use Jordan exchange to carry as many y's as possible to the top of the table. At this stage, we consider the following two cases:

Case 1. $m \leq n$.(more x's than y's)

If all the y's can be exchanged and $m < n$, the final table will be of the form

	y	x_{II}	1
$x_I =$	B	C	d

Reading this table we find that if we choose x_{II} arbitrarily (there is no restriction on $x_I = By + Cx_{II} + d$ will also take on arbitrary values giving infinitely many solutions to the system. If $m = n$, then the final table will be

	y	1
$x =$	B	d

Which has the unique solution $x = By + d = B0 + d = d$. On the other hand, if some of the y's cannot be exchanged, the final table will be of the form

	y_I	x_{II}	1
$x_I =$	B	C	D
$y_I =$	D	O	d_{II}

Reading this table we get $y_{II} = Dy_I + 0x_{II} + d_{II}$. If $d_{II} = 0$ then choosing x_{II} arbitrarily we get infinitely many solutions to the system. However, If $d_{II} \neq 0$ we get $y_{II} = d_{II} \neq 0$ which means that the system is inconsistent, thus has no solution.

Case 2. $m > n$ (more y's than x's)

If some of the x's remain unexchanged, the final table will be

$y_I \ x_{II} \ 1$

$x_I =$			d_I
$y_{II}=$	C	O	d_{II}

As in the previous case, this table gives either no solution if $d_{II} \neq 0$ or infinitely many solution if $d_{II} = 0$.
On the other hand, if all x's can be exchanged the final table will be of the form

	y_I	1
$x_I =$	B	d_I
$y_{II} =$	C	d_{II}

Reading this table we find that if $d_{II} = 0$ then $x = By_I + d_I = d_I$ is the only solution. If $d_{II} \neq 0.$ we get $y_{II} = Cy_I + d_{II} = d_{II} \neq 0.$, which means that the system is inconsistent. Finally, we conclude that in both cases the system $Ax = b$ has no solution, one solution, or infinitely many solutions. □

Theorem 2.6. Let A be an $n \times n$ matrix. If the rank of A is k, then the system $Ax = b$ has $(n - k)$ parameter family of solutions.

Proof. If the rank of A is k, then A has k independent rows. Relabeling and Rearranging if necessary, we can exchange $y_1, ..., y_k$ to obtain the table

	y_I	x_{II}	1
$x_I =$			d_I
$y_{II} =$	C	O	d_{II}

where $y_I = (y_1, ..., y_k)^t$ and $x_{II} = (x_{k+1}, ..., x_n)^t$. Reading this table we find that in order for the system to have a solution, we must have $d_{II} = 0$. But if this is true, we can clearly see that x_{II} can be chosen arbitrarily, yielding $(n-k)$ parameter of solutions. □ **Theorem 2.7.** Every homogeneous system $Ax = 0$ with more unknowns than equations (more x's than y's) have a nontrivial solution.

Proof. Let A be an $m \times n$ matrix, $m < n$. The system $Ax = 0$ can be written in a table of the form

	x	1
0=	A	0

After performing all the possible Jordan exchanges, we obtain one of the following cases:

Case 1. (All zeros can be exchanged)

Since $m < n$, if all the zeros on the left can be exchanged then some x's will remain on top and we obtain the table

	0	x_{II}	1
$x_I =$			0

Case 2. (some zeros will not be exchanged)

	0	x_{II}	1
$x_I =$			0
$0 =$		0	0

Since the rightmost column remains zero throughout the process (See rules for Jordan exchange) the system will be consistent in both cases and x_{II} can be chosen arbitrarily yielding a nontrivial solution. □

Corollary 2.8. Let A be an $m \times n$ matrix. Then $Ax = b$ has a unique solution if all the xs can be exchanged to the side of the table.

Proof. Assume that the system has a unique solution while some of the xs remain on the top of the table. This means that the table will be of the form

	y_I	x_{II}	1
$x_I =$			d_I
$y_{II} =$		0	0

Reading this table we find that x_{II} can be chosen arbitrarily yielding infinitely many solutions, which contradicts our assumption.□

We give here a few algorithms using Jordan exchange for solving the system $Ax = b$, where A is an $m \times n$ matrix.

Algorithm 1. Write the system in a table of the form

	x
$b =$	A

After performing all the possible Jordan exchanges, we get one of the cases shown in Theorem 2.4. Then using the same theorem we can determine whether the system has no solution, one solution or infinitely many solutions.

Algorithm 2. Write the system in a table of the form

	x	1
$0 =$	A	-b

and exchange zeros and xs without computing the transformed pivot column. That is, after each exchange over a_{rs} suppress column s. This suppression of column s is justified since the dependent variable on the top of this column will be identically zero. After all the possible exchanges have been performed, we use Theorem 2.4 to determine whether or not the system is solvable.

Algorithm 3. This algorithm is the same as the second one except that after each exchange we suppress not only the pivot column, but the pivot row as well. At the end of the process, we use the last table to obtain the value of one of the unknowns, and then the values of the remaining unknowns are computed by back substitution in the previous tables.

Appendix:

Illustration 1. (for Jordan Exchange)

In the table

	x_1	x_2	x_3
$y_1 =$	1	-2	3
$y_2 =$	-1	1	(2)
$y_3 =$	2	-1	1

A Jordan exchange with the pivot element being (2,3) (the bracketed entry in the second row and third column) will exchange the roles of y_2 and x_3 and lead to the table

	x_1	x_2	y_2
$y_1 =$	5/2	-7/2	3/2
$x_3 =$	1/2	-1/2	1/2
$y_3 =$	5/2	-3/2	1/2

Illustration 2. (for matrix Inverse)

Let $A = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ writing A in table form we get

	x_1	x_2
$y_1 =$	(1)	3
$y_2 =$	1	4

A Jordan exchange with pivot (1,1) yields the table

	y_1	x_2
$x_1 =$	1	-3
$y_2 =$	1	(1)

Another Jordan exchange with pivot (2,2) yields the table

	y_1	y_2
$x_1 =$	4	-3
$x_2 =$	-1	1

The process ends here and the Inverse is $A^{-1} = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$

Note. Usually the final matrix requires some rearrangements of some of its rows and columns when some of the pivots are not diagonal entries.

Illustration 3. (for matrix Inverse)
Let $A = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}$ Writing A in table form and using Jordan exchange we get the following sequence of tables

	x_1	x_2
$y_1 =$	2	(1)
$y_2 =$	3	1

	x_1	y_1
$x_2 =$	-2	1
$y_2 =$	(1)	1

	y_2	y_1
$x_2 =$	-2	3
$x_1 =$	1	-1

Note that the final table does not represent the inverse of A. However, if we rearrange its rows and columns we get the equivalent table

	y_1	y_2
$x_1 =$	-1	1
$x_2 =$	3	-2

which represents the inverse of A.

Illustration 4. (For the rank of a matrix)

Consider the matrix $A = \begin{pmatrix} 2 & -1 & 3 & 4 \\ 1 & 0 & 2 & -3 \\ 5 & -2 & 8 & 5 \end{pmatrix}$

Writing A in table form and using Jordan exchange we get the following sequence of tables

	x_1	x_2	x_3	x_4
$y_1 =$	2	-1	3	4
$y_2 =$	(1)	0	2	-3
$y_3 =$	5	-2	8	5

	y_2	x_2	x_3	x_4
$y_1 =$	2	(-1)	-1	10
$x_1 =$	1	0	-2	3
$y_3 =$	5	-2	-2	20

	y_2	y_1	x_3	x_4
$x_2 =$	2	-1	-1	10
$x_1 =$	1	0	-2	3
$y_1 =$	1	2	0	0

Reading the final table we find that y_3 cannot be carried to the top of the table (blocked by zeros). Moreover, we get y_3 as a linear combination of y_1 and y_2; $y_3 = 2y_1 + y_2$. Hence the maximum number of independent rows is two, which is the rank of A.

Illustration 5. (For algorithm 1)

Consider the system $x_1+x_2 = 3; x_1-x_2 = 1$ Writing the system in table form and using Jordan exchange we get the following tables

	x_1	x_2
3=	(1)	1
1 =	1	-1

	3	x_2
$x_1=$	1	-1
$1 =$	1	(-2)

	3	1
$x_1=$	1/2	1/2
$x_2=$	1/2	-1/2

Reading the final table we get $x_1 = 1/2(3) + 1/2(1) = 2$;

$x_2 = 1/2(3) + (-1/2)(1) = 1$

Hence, $x = (2, 1)^t$ is the only solution to the system.

Illustration 6. (for Algorithm 2)

Consider the system

$$x_1 - x_2 + x_3 = -1$$
$$-x_1 - x_2 - x_3 = 1$$
$$2x_1 - 2x_2 + x_3 = -1$$

Writing the system in the form of a table we get

	x_1	x_2	x_3	1
$0 =$	(1)	-1	1	1
$0 =$	-1	-1	-1	-1
$0 =$	2	-2	1	1

Perform a Jordan exchange with pivot (1,1) and suppressing the first column we get the table

	x_1	x_2	x_3	1
$0 =$	(1)	-1	1	1
$0 =$	-1	-1	-1	-1
$0 =$	2	-2	1	1

Perform a Jordan exchange with pivot (1,1) and suppressing the first column we get the table

	x_2	x_3	1
$x_1 =$	1	-1	-1
$0 =$	-2	0	0
$0 =$	0	(-1)	-1

Perform another Jordan exchange with pivot (3,2) and suppressing the second column we get the table

	x_2	1
$x_1 =$	1	0
$0 =$	(-2)	0
$x_3 =$	0	1

Finally, performing a Jordan exchange with (2,1) and suppressing the first column we get the table

	1
$x_1 =$	0
$x_2 =$	0
$x_3 =$	-1

Reading the final table we get $x_1 = 0, x_2 = 0, x_3 = -1$

Hence the system has the unique solution $x = (0, 0, -1)^t$

Illustration 7 (for Algorithm 3)

Consider the system $x_1 - x_2 + 2x_3 = 2$

$x_1 + x_2 + x_3 = -1$

$x_1 + x_2 - x_3 = -3$

Writing the system in the form of a table we get

	x_1	x_2	x_3	1
$0 =$	(1)	-1	2	-2
$0 =$	1	1	1	1
$0 =$	1	1	-1	3

Performing a Jordan exchange with pivot (1,1) and suppressing the first row and first column we get the table

	x_2	x_3	1
$0 =$	(2)	-1	3
$0 =$	2	-3	5

Performing another Jordan exchange with pivot (1,1) and suppressing the first row and the first column we get

	x_3	1
$0 =$	-2	2

Reading the last table we get $0 = -2x_3 + 2 \Rightarrow x_3 = 1$. Substituting for x_3 and using the first row of the second table we get $0 = 2x_2 - x_3 + 3 \Rightarrow x_2 = -1$. Finally, the first table yields $0 = x_1 - x_2 + 2x_3 - 2 \Rightarrow x_1 = -1$. Hence the system has the unique solution $x = (-1, -1, 1)^t$.

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