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ON LINEAR ALGEBRA AND ITS APPLICATIONS

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Abstract. Jordan Exchange is a method for solving a given system of m linear equations in n unknowns by exchanging the roles of the dependent and independent variables. In this paper we give a new twist to the Jordan Exchange method and call the resulting technique modified Jordan Exchange method. Using this new technique we give different proofs for certain pertinent results concerning simplex method of linear programming. Further we present several illustrations in the Appendix section to elucidate the arising intricacies of the method.

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1. Introduction

Optimization is the act of obtaining the best result under given circumstances. In design, construction, and maintenance of any engineering system, engineers have to take many technological and managerial decisions at several stages. The ultimate goal of all such decisions is either to minimize the effort required or maximize the desired benefit. Since the effort required or the benefit desired in any practical situation can be expressed

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as a function of certain decision variables, optimization can be defined as the process of finding the conditions that give the maximum or minimum value of a function. There is no single method available for solving all optimization problems efficiently. Hence a number of optimization methods have been developed for solving different types of optimization problems.

The existence of optimization methods can be traced to the days of Newton, Lagrange and Cauchy. The development of differential calculus methods of optimization was possible because of the contributions of Newton and Leibnitz to calculus. The foundations of calculus of variations were laid by Bernouille, Euler, Lagrange and Weierstrass. The method of optimization for constrained problems, which involves the addition of unknown multipliers became known by the name of its inventor, Lagrange. Cauchy made the first application of the steepest descent method to solve unconstrained minimization problems. In spite of these early contributions, very little progress was made until the middle of the twentieth century, when high speed computers made implementation of the optimization procedures possible and stimulated further research on new methods. It is interesting to note that the major developments in the area of numerical methods of unconstrained optimization have been made in the United Kingdom only in the 1960s. The development of simplex method by Dantzig in 1947 for linear programming problems and the enunciation of principle of optimality in 1957 by Bellman for dynamic programming problems paved the way for development of the methods of constrained optimization. The work by Kuhn and Tucker in 1951 on the necessary and sufficient conditions for the optimal solution of programming problems laid the foundations for a great deal of research further. Motivated by the fascinating works of these big names we also attempt to introduce a new method for solving linear programming problems and in the process made some progress.

In [12] we have introduced Jordan exchange method and discussed its application to linear Algebra. In this paper we introduce modified Jordan exchange method and study its application for solving a given LPP using compact table representation.

Theorem 2.1 A modified Jordan Exchange with pivot element a_{rs} may be interpreted in the following two ways:

a) Primal Interpretation: For the system $y = A(-x)$, solve $y_r = \sum_{j=1}^{n} a_{rj}x_j$ for x_s and substitute for x_s in the remaining equations

b) Dual Interpretation: For the system $v = A^t u$, solve $v_s = \sum_{i=1}^{m} a_{is}v_i$ for u_r and substitute for u_r in the remaining equations

Proof. a) We have $y_r = a_{rs}x_s - \sum_{j=1}^{n} a_{rj}x_j$. This implies $X_s = \frac{1}{a_{rs}}(-y_r) + \sum_{j=1, j\neq s}^{n} \frac{a_{rj}}{a_{rs}}(-x_j)$. Hence, we get $y_i = -a_{is}x_s - \sum_{j=1,j\neq s}^{n} a_{ij}x_j = \frac{-a_{is}}{a_{rs}}(-y_r) + \sum_{j=1,j\neq s}^{n} a_{ij} - \frac{a_{rj}a_{is}}{a_{rs}}(-x_j)$.

b) We have $v_s = a_{rs}u_r + \sum_{i=1,i\neq r}^{m} a_{is}u_i$. This implies $u_r = \frac{1}{a_{rs}}v_s + \sum_{i=1}^{m} -\frac{a_{ij}u_i}{a_{rs}}$.
Hence, we get $v_j = a_{rj}u_r + \sum_{i=1,i\neq r}^{m} a_{ij}u_i = \frac{a_{rj}}{a_{rs}}(v_s) + \sum_{i=1,i\neq r}^{m} (a_{ij} - \frac{a_{rj}a_{is}}{a_{rs}})u_i$. ■

Theorem 2.2 Let A be an $m \times n$ matrix. Then row rank of A is equal to the column rank of A.

Proof. Let $k \leq m$ be the row rank of A. Then consider the system $y = Ax$ and its dual $v = -A^t u$. After k exchanges we get the table

V_I	x_I		
V_{II}	y_{II}		0

Where $v_I = (v_1, v_2, ...v_k)^t$ and $v_{II} = (v_{k+1}, v_{k+2}, ...v_n)^t$. Reading this table for the dual system we find that v_{II} is linearly dependent on v_I. Hence, the column rank of A is k which is the row rank of A.■

Consider the problem of solving a standard linear programming problem using the simplex method. We use the compact table and the modified Jordan exchange method to implement the simplex method.

Standard LPP

Consider the linear objective function $z = \sum_{i=1}^{n} p_2 x_i$ and the system of linear constraints $\sum_{j=1}^{n} a_{ij}x_j \leq a_i$, $i = 1, ..., m, x_j \geq 0, j = 1, ..., n$. This LPP can be stated in a compact way as follows: Maximize $Z = p^t x$ subject to $y = A(-x) + a \geq 0, x \geq 0$. If the function is a minimizing problem, we can make it a maximizing problem by taking the function

$z = -p^t x$ as our objective function, then negating its value after a maximal solution is reached.

The simplex method

The simplex method, introduced by George Dantzig in 1948, is one of the most successful algorithms for solving linear programs. The underlying concept of the simplex method is to start with a corner point of the feasible region then move to another corner point at which the value of the objective function will increase until a maximal solution is reached. If no feasible solution can be found (feasible region is empty) then the system of inequalities is inconsistent, the LPP has no solution.

Table representation

Consider the LPP maximize $z = p^t x$, subject to $y = A(-x) + a \geq 0, x \geq 0$. This LPP can be represented in the compact table, where the first m rows represent the system of

	$-x_1 \ldots - x_s \ldots - x_n$	
$y_1=$	$a_{11} \ldots a_{1s} \ldots a_{1n}$	a_1
.	.	.
.	.	.
.	.	.
$y_r=$	$a_{r1} \ldots a_{rs} \ldots a_{rn}$	a_r
.	.	.
.	.	.
.	.	.
$y_m=$	$a_{m1} \ldots a_{ms} \ldots a_{mn}$	a_m
z	$-p_1 \ldots - p_s \ldots - p_n$	0

constraints $y = A(-x) + a \geq 0$ and the bottom row represents the function $z = p^t x$. This table represents the origin in R^n because we assume all the independent variables on top of it to be zero. This means that if an entry in the last column is negative, say $a_k < 0$, then the constraint $y_k = oA_k + a_k > 0$ will be violated and the origin will not be feasible. We shall call phase I of the simplex method to be the process of searching for a feasible point if the origin is not feasible. On the other hand, if the origin is feasible then we will use the modified Jordan exchange to move to another feasible point until we reach an optimal solution. We shall call phase II of the simplex method to be the process of searching for an optimal solution that gives an initial feasible solution.

Phase II of the simplex method

At this stage, we shall begin to develop the rules by which we can implement the simplex method using the modified Jordan exchange. In order to do this, we must perform an exchange if and only if it increases the value of the objective function while maintaining feasibility. This in turn leads us to the task of choosing the pivot over which an exchange should be performed. For example,

Consider the LPP Max $z = -3x_1 + 6x_2$ subject to $y_1 = x_1 + 2x_2 + 1 \geq 0; y_2 = 2x_1 + x_2 \geq 0; y_3 = x_1 - x_2 + 1 \geq 0; y_4 = x_1 - 4x_2 + 13 \geq 0; y_5 = -4x_1 + x_2 + 23 \geq 0,\ x_1, x_2 \geq 0$.

Writing the LPP in table from we get

	$-x_1$	$-x_2$	1
$y_1=$	-1	-2	1
$y_2=$	-2	-1	0
$y_3=$	-1	1	1
$y_4 =$	-1	4	13
$y_5=$	4	-1	23
z	3	-6	0

The origin, which is represented by this table, is a feasible solution of the LPP. This is true because when all the independent variables are zero, all the constraints of the LPP are satisfied. At this stage, we shall begin to develop the rules by which we can implement the simplex method using the modified Jordan exchange. In order to do this, we must perform an exchange if and only if it increases the value of the objective function while maintaining feasibility. This in turn leads us to the task of choosing the pivot over which an exchange should be performed.

a) choice of Pivot column

We know that the original table represents the point $x_1 = 0, x_2 = 0$. Moreover, any point in the feasible region must be of the form $x_1 = \lambda_1 \geq 0, x_2 = \lambda_2 \geq 0$ in order to satisfy the constraints $x_1, x_2 \geq 0$. Therefore if we choose $x_1 = \lambda, x_2 = 0, \lambda > 0$ then the value of the objective function becomes $z = -3\lambda < 0, \lambda > 0$. For this choice, the value of the objective function decreases. On the other hand, if we choose $x_2 = \lambda, x_1 = 0, \lambda \geq 0$ then the value of the objective function becomes $z = 6\lambda > 0, \lambda > 0$. For this choice, the value of the objective function increases. Hence we must choose the column under x_2 to be our pivot column and set $x_2 = \lambda > 0$ while holding x_1 to level zero.

Note that if the coefficient corresponding to x_2 in the z row was non-negative then the value of the objective function will decrease when we make $x_2 > 0$. Hence our pivot column s must be a column with a negative bottom entry ($p_s > 0$). If no such column exist then the point represented by the table is an optimal solution to the LPP. On the other hand, if we have more than one column with negative bottom entries we can choose either one of them arbitrarily or pick the one with the smallest bottom entry.

b) Choice of Pivot row

Substituting for $x_1 = 0$ and $x_2 = \lambda$ in the LPP, we get $y_1 = 2\lambda + 1 \geq 0; y_2 = \lambda \geq 0;$ For all the constraints to be satisfied we must choose 1. This is necessary, because if we chose $\lambda > 1$ then the constraint $y_3 = -\lambda + 1$ will be violated that is, y_3 will be less than zero. Hence we must choose the row three to be our pivot row in order to maintain feasibility. Note that our pivot row is the row, which yields the minimum ratio for $\lambda(1 < 13/4)$. In general, if s is the pivot column then the pivot row r is defined by $r = \min\{\frac{b_i}{a_{is}} : a_{is} > 0, i = 1, ...m\}$ where b is the last column.

c) Pivoting

We have been that in order to increase the value the objective function, we must choose $x_1 = 0, x_2 > 0$ (choice of pivot column). We have also been that in order to maintain feasibility, we must choose $x_2 \leq 1$ (choice of pivot row by minimum ration test). These two conditions can be simultaneous satisfied by performing a modified Jordan exchange in which x_2 and y_3 are exchanged. Hence we get the tables:

Wring again the above rules for finding the pivot column and row, we choose the column under x_1 to be our next pivot column (negative bottom entry) and the row corresponding

	$-x_1$	$-x_2$	1
$y_1=$	-1	-2	1
$y_2=$	-2	-1	0
$y_3=$	-1	1	1
$y_4=$	-1	1	13
$y_5=$	4	-1	23
$Z=$	3	-6	0

	$-x_1$	$-y_3$	1
$y_1=$	-3	2	3
$y_2=$	-3	1	1
$y_3=$	-1	1	1
$y_4=$	3	-4	9
$y_5=$	3	1	24
$Z=$	- 3	6	6

to y_4 to be our next pivot row (minimum ration rest). This is equivalent to a modified Jordan exchange with pivot (4,2), which yields the table The simplex method ends have

	$-y_4$	$-y_3$	1
$y_1=$			12
$y_2=$			10
$x_2=$			4
$x_1=$			3
$y_5=$			15
$z=$	1	2	15

because all the elements in the bottom row are non negative which means that the values of z cannot be decreased by performing another exchange. Note that in this table y_3 and y_4 are now zero which yields $x_1 = 3, x_2 = 4$ and $z- = -3(3) + 6(4) = 15$. If we look at the graph of the system, we find that the simplex method, as we shall prove later, moves from one corner point to another corner point of the feasible region. We also find that the first exchange corresponds to moving along the positive x_2-axis, from the origin to the point (0,1) and that the second stage corresponds to moving along the line $y_3 = 0$, from the point (0,1) to the point (3,4). It is clear that the value of the objective function at the point (3,4) is bigger than its value at any other corner point of the feasible origin. Hence the maximal solution is reached at that point and the process is terminated.

Table Interpretation

A table is said to be 1) feasible if the point it represents belongs to the feasible region 2) Optimal if the point it represents is an optimal solution to the LP 3) unbounded if its is feasible and the objective function is unbounded. To determine whether a table is feasible, optimal, or unbounded we consider the LPP $\max z = p^t x$ subject to $y = A(-x) + a \geq 0$; $x \geq 0$. Writing the LPP in the table form we get

-x 1

y=	A	a
z=	-p	0

Then by applying the simplex method to the LPP we get after reordering and relabeling if necessary a table of the form

	-y_I $-x_{II}$	1
x_I=	B	b
y_{II}=		
z=	q	Q

At this stage we can use the above table to state and prove the conditions for feasibility optimality unboundedness.

1) feasibility: A table is feasible if the last column is non-negative.

Proof. If b=$b=\begin{pmatrix} b_I \\ b_{II} \end{pmatrix} \geq 0$ then $x_I = b_I \geq 0$ and $y_{II} = b_{II} \geq 0$. Since $y_I = x_{II} = 0$ by convention, we get $y \geq 0$ and $x \geq 0$ which is the condition for feasibility.

2) Optimality: A table is optimal if the last column and bottom row are non negative.

Proof. We have shown that if b 0 then the table is feasible. On the other hand in order for the objective function to be maximal, we must have $z = q\begin{pmatrix} -y_I \\ -x_{II} \end{pmatrix} + Q \geq \begin{pmatrix} -y'_I \\ -x'_{II} \end{pmatrix} + Q$ for all $y'_I, x'_{II} \geq 0$. Since $y_I = x_{II} = 0$ we must have $z = Q \geq q\begin{pmatrix} -y'_I \\ -x'_{II} \end{pmatrix} + Q$. But this is true iff $q\begin{pmatrix} -y'_I \\ -x'_{II} \end{pmatrix} \leq 0$ which yields $q \geq 0$ since $y'_I, x'_{II} \geq 0$. Hence the table is optimal if the last column and bottom row are non negative.

c) Unboundness:

A table is unbound if it is feasible and has a non-positive column with a negative bottom entry.

Proof. Suppose that a feasible table has a non-positive column k with a negative bottom entry and that the variable sitting on top of it is x_k. Then setting $x_k = \lambda \geq 0$ and the

rest of the independent variables to zero we get x
$\begin{pmatrix} x_I \\ y_{II} \end{pmatrix} = -\lambda B_k + b \geq 0$ since $B_k \leq 0$. Hence $z = -q_k\lambda \to \infty$. Thus z is unbounded.

d) phase I of the simplex method

In this phase we try to find a feasible point if the origin is not feasible. The occurs when one or more of the entries in the last column are negative. If a feasible point cannot be found then the LPP has no solution.

Theorem 2.3 Every feasible table represents a corner point of the feasible region.

Proof. since every BFS is a corner point it suffices to show that a feasible table represents a BFS consider the table.

	-x	1
y=	A	a
z=	-p	0

Since $y = A(-x) + a \Leftrightarrow 0 = -y + A(-x) + a$, wee can rewrite this table in the following form

	$-y$	-x	1
0=	I	A	a
z=	0	-p	0

Let $H = [I, A]$ then performing a modified Jordan exchange over hours we obtain the matrix $H' = PH$ where $P = \begin{pmatrix} 1... & \frac{h_{1s}}{h_{rs}}... & 0 \\ 0... & \frac{1}{h_{rs}}... & 0 \\ 0... & \frac{-h_{ms}}{h_{rs}}... & 1 \end{pmatrix}$ and $P^{-1} = \begin{pmatrix} 1... & h_{is}... & 0 \\ 0... & h_{rs}... & 0 \\ 0... & h_{ms}... & 1 \end{pmatrix}$. This is so because when we perform a modified Jordan exchange over h_{rs} all the columns of the identify matrix. I will remain unchanged except the one with the entry 1 in the pivot row. As for the remaining columns, we have $h_{rj} = 0$ which yields $h_{ij} = h_{ij} - \frac{h_{rj}h_{is}}{h_{rs}} = h_{ij}$. ■

Hence, after k exchange we get $H^{(k)} = p^k p^{k-1}...p^1 H$. If $x = (x_I, x_{II}), x_I \geq 0, x_{II} = 0$ is represented by $H^{(k)}$ then exactly n columns of I have been changed. Since the column of I are linearly independent (will remain so because the p's are nonsingular, we can find the

k independent columns of A that x are by multiplying $H^{(k)}$ by the P matrices in reverse order. Hence, x is a BFS II. ■

Degeneracy

A feasible table is said to be degenerate if the value of at least one of the entries in the last column is zero. This occurs when the simplex method does not yield a unique choice of pivot row at some step. Geometrically this means that two or more vertices of the feasible region merge, and the edge connecting there vertices contacts to a point. In other words the current vertex can be expressed a the intersection of two or more sets of hyper planes passing through that vertex.

Theoretically, the simplex method may cycle in the presence of degeneracy. This means that after some steps, each of which leads to some choice of n planes passing through a given vertex, we return to some previous choice and the process repeats. Although cycling seldom occurs in practice, the fact that it can occur has lead to the development of some methods to avoid it. One of these methods is the so called Blands rule. One can refer [1] for more details.

Appendix.

Illustration.1 Consider the system $y_1 = -2x_1 - x_2$ $y_2 = -x_1 - x_2$.

Writing the system in the form of the table and performing a modified Jordan exchange with pivot (1,2) we get

	$-x_1$	$-x_2$
$y_1=$	2	1
$y_2=$	1	1

	$-x_1$	$-y_1$
$x_2=$	2	1
$y_2=$	-1	-1

Illustration 2. Max $z = 2x$ subject to $y_1 = x - 40; y_2 = -x + 6 \geq 0; x \geq 0$.

Writing the system in table form we get

$-x_1$ $-x_2$

y_1=	-1	-4
y_2=	1	6
z=	-2	0

This table in not feasible because if we choose $x = 0$ we get $y_1 = -4 < 0$ which is a violation of the constraint $y_1 = x - 4 \geq 0$. It is also clear that if we choose $x > 6$ then we violate the constraint $y_2 = -x+6 \geq 0$. Hence, our feasible region is the set $\{x|4 \leq x \leq 6\}$ in R^1. Since the origin is not in the feasible region, it cannot be used as our starting point. To overcome this problem, we introduce an artificial variable 0 in each constraint which is violated when $x = 0$. Hence our LPP becomes

Max $z = 2x$ subject to $y_1 = x + \rho - 4 \geq 0$; $y_2 = -x + 6 \geq 0; x, \rho \geq 0$ Writing the new LPP in table form we get

y_1=	-1	-1	-4
y_2=	1	0	6
z=	-2	0	0

In the above we note that (i) the introduction of in the original LPP corresponds to adding a new column, headed by , to the original table.

(ii) in the column handed by , the entries corresponding to the violated constraints is ? while the entries corresponding to the unviolated constraints is 0 (iii) the original problem in R^1 has been extended to a problem in R^2, one which is always feasible for large enough values of ρ. If we look at the graph of the new LPP in the $x - \rho$ plane, we find that the point (0,0) ($\rho = 0$) is still infeasible but the point (0,4) (ρ=4) is feasible. We can get to the point (0,4) by moving along the positive ρ-axis that is increasing while holding x to level zero. This corresponds, in the new table to exchanging with y_1, since y_1, has the largest violation that is the entry corresponding to y_1 in the last column is the smallest entry in that column. Hence we get the tables:

	-x	ρ	1
y_1=	-1	-1	-4
y_2=	1	0	6
z=	-2	0	0

$\rightarrow$

	$-x$	$-y_1$	1
$y_1=$	-1	-1	-4
$y_2=$	1	0	6
$z=$	-2	0	0

The last table represents the point (0,4), which is feasible in the $x-\rho$ plane. But our objective is to find a feasible point in the original space R^1. This can be done by driving ρ down to zero (make it independent again) along the edge $y_1 = 0$, that is by increasing x while holding y_1 to level zero. Hence we exchange x and to get the table

	$-\rho$	$-y_1$	1
$x=$	1	-1	4
$y_2=$	-1	1	2
$z=$	2	-2	8

At this stage, we can drop the artificial variable ρ since we obtained a feasible point for the original LPP. Thus our final table becomes

$x=$	-1	4
$y_2=$	1	2
$z=$	-2	8

Using this feasible table we can proceed to solve the problem by going to phase II of the simplex method.

Illustration 3.

In illustration 2, we know beforehand that the feasible region was not empty. In general we introduce, along with $\rho \geq 0$, a new objective function $\omega = -\rho$ then try to maximize this new function. If the maximum of the new objective function is zero (ρ=0) then the original LPP is feasible. On the other hand, if the maximum of is not zero then the extended LPP is feasible but not the original LPP. To see how this works we consider the following LPP Maximize $z = 2x$ subject to $y_1 = -x + 4 \geq 0$ $y_2 = x - 6 \geq 0$.

As in illustration 2, we introduce in all the violated constraints. We also introduce $\omega = -\rho$ as our new objective function so that our LPP becomes Maximize $\omega = -\rho$ subject to $y_1 = -x + 4 \geq 0$ $y_2 = x + -6 \geq 0$

Writing the new LPP in the table form and using the modified Jordan exchange we get the tables The last table is optimal for ω and the optimal solution is $\omega = -\rho = -2 \neq 0$. Hence

	-x	-ρ	1
y_1=	1	0	4
y_2=	-1	-1	-6
z=	-2	0	0
ω	0	1	0

	-x	$-y_2$	1
y_1=	1	0	4
ρ=	1	-1	6
z=	-2	0	0
ω=	**-1**	1	-6

	$-y_1$	$-y_2$	
y_1=	1	0	4
ρ=	-1	-1	2
z=	2	0	8
ω=	1	1	-2

the original is not feasible since the maximum of ω is not zero. Note that ρ remained on the hide of the table after we reached an optimal solution for ω.

Illustration 4

When phase I is completed successfully, and a feasible point for the original LP, has been found we can continue with phase II of the simplex method. To see this works we consider the following LP

Max $z = -4x_1 - 5x_2$

Subject to $x_1 + x_2 + 1 \geq 0$

$x_1 + 2x_2 - 1 \geq 0$

$4x_1 + 2x_2 - 8 \geq 0$

-$x_1 - x_2 + 3 \geq 0$

-$x_1 + x_2 - 1 \geq 0$

$x_1, x_2 \geq 0$

Writing the LP in the table form we get

-x_1 -x_2 1

y_1=	-1	-1	1
y_2=	-1	-2	-1
y_3=	-4	-2	-8
y_4=	1	1	3
y_5=	1	-1	-1
z=	4	5	0

This table is not feasible because some of the entries in the last column are negative. Hence we introduce an artificial variable ρ in all the violated constraints and add a new objective function $\omega = -\rho$ so that our table becomes

	$-x_1$	$-x_2$	ρ	1
y_1=	-1	-1	0	1
y_2=	-1	-2	-1	-1
y_3=	-4	-2	-1	-8
y_4=	1	1	0	3
y_5=	1	-1	-1	-1
z=	4	5	0	0
ω	0	0	1	0

Since the third constraint has the largest violation (smallest entry in the last column) we perform our first modified. Jordan exchange over the third row and the third column (ρ-column). This yields the following table

	$-x_1$	$-x_2$	y_3	1
y_1=	-1	-1	0	1
y_2=	3	0	-1	7
ρ=	4	2	-1	8
y_4=	1	1	0	3
y_5=	5	1	-1	7
z=	4	5	0	0
ω	-2	1	1	-8

At this stage we try to maximize ω in order to reach a feasible solution in the original space. But in order to maximize ω, we must go to phase II of the simplex method with ω as our objective function. This leads to modified Jordan exchange over (4,2) (negative bottom entry and minimum ratio) which yields the table

	$-x_1$	$-y_4$	y_3	1
$y_1=$	0	1	0	1
$y_2=$	3	0	-1	7
$\rho=$	2	-2	-1	2
$x_2=$	1	1	0	3
$y_5=$	4	-1	-1	4
$z=$	-1	-5	0	-15
ω	-2	2	1	-2

Finally, we can move ρ back to the top by performing a modified Jordan exchange with (3,1). This yields the following table

	$-\rho$	$-y_4$	y_3	1
$y_1=$				
$y_2=$				
$x_1=$				
$x_2=$				
$y_5=$				
$z=$	1/2	-6	-1/2	-14
ω	1	0	0	0

As expected when we moved back to the top of the table, the value of the objective function ω became zero. Now we can drop ρ and ω and go to phase II of the simplex method with z as our objective function.

Geometric Interpretation of the simplex method

First we recall a few definitions. A set $S \subseteq R^n$ is said to be convex if for all $x_1, x_2 \in S, \alpha x_1 + (1-\alpha)x_2 \in S$ for all $0 \leq \alpha \leq 1$. Let $S \subseteq R^n$ be the set of solutions to the system $Ax = a, x \geq 0$. If $x \in S$, the set of column vectors that x uses is $\{Aj | x_j \geq 0, j = 1, ..., n\}$.

A feasible solution $x \in S$ is said to be a basic feasible solution (BFS) to the system if the set of column vectors of A that x uses is linearly independent. We now recall a few known results.

The set of feasible solutions to the LP is convex. Let $S \subseteq R^n$ be convex. A point $x \in S$ is said to be a corner point of S if it is impossible to express it as a convex combination of two distinct points in S.

Let S be the set of feasible solutions for the system $Ax = a$, $x \geq 0$. If $x \in S$ is a basic feasible solution of the system than x is a corner point of S.

Illustration 5

Consider the LP maximize $z = x_1 + 2x_2$ subject to

$y_1 = x_1 - 2x_2 + 2 \geq 0$

$y_2 = -x_2 + 1 \geq 0$

$y_3 = -x_1 + x_2 + 2 \geq 0$, $x_1, x_2 \geq 0$

Writing the LP in the form of a table we get

	-x_1	-x_2	1
y_1=	-1	2	2
y_2=	0	1	1
y_3=	1	-1	2
z=	-1	-2	0

It is clear that if we arbitrarily choose the second column to be our pivot column (note that we could have chosen the first column to be our pivot column as well) then our pivot choice will not be unique. This occurs because the minimum ratio test results in a tie between the first and second rows. Breaking tie arbitrarily, we choose the first row to be our pivot row. Hence by performing a modified Jordan exchange with pivot (1,2) our table becomes

-x_1 -y_1 1

x_2=	-1/2	1/2	1
y_2=	1/2	-1/2	0
y_3=	1/2	1/2	3
z=	-2	1	2

This table is degenerate since the second entry in the last column is zero. This, in turn, forces the second row to be the next pivot row (o is the minimum ratio) which yields the table

	-y_1	-y_2	1
x_2=	1	0	1
x_1=	2	-1	0
y_3=	-1	1	3
z=	4	-1	2

It is clear that the current table and the previous one represents the same vertex (0,1) see fig. And the value of the objective function is the same in both tables (z=2). Finally if we perform a modified Jordan exchange on the current table we get

	-y_2	-y_3	1
x_2=			1
x_1=			3
y_1=			3
z=	3	1	5

The simplex method terminates at this table since all the entries in the bottom row are non-negative. Moreover, the optimal solution is $x_1 = 3, x_2 = 1, z = 5$. Geometrically the LP has the following graph in the $x_1 - x_2$ plane. If we look at the feasible (shaded) region we find that the feasible vertex (0,1) can be viewed as the intersection of the lines $(x_1 = 0, y_1 = 0)$ or $(y_1 = 0, y_2 = 0)$. This clearly should that degeneracy occurs when more than n planes are involved in determining a single vertex in R^n.

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