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SPACELIKE – TIMELIKE INVOLUTE – EVOLUTE CURVE COUPLE ON DUAL LORENTZIAN SPACE

SÜMEYYE GÜR[1]& SÜLEYMAN ŞENYURT[2,*]

[1]Ege University Faculty of Science, Department of Mathematics, İzmir/ Turkey

[2]Ordu University Faculty of Arts and Science, Department of Mathematics Ordu/ Turkey

Abstract: In this paper, we have defined the involute curves of the dual timelike curve M_1 in dual Lorentzian space D_1^3. We have seen that the dual involute curve M_2 must be a dual spacelike curve with a dual spacelike (or timelike) binormal vector. The relationship between the Frenet frames of the spacelike – timelike involute – evolute dual curve couple have been found and some new characterizations related to the couple of the dual curve have been given.

Keywords: Dual Lorentzian space, dual involute – evolute curve couple, dual Frenet frames.

2000 AMS Subject Classifications:53A04, 53B30

1.Introduction

The concept of the involute of a given curve is well-known in 3-dimensional Euclidean space IR^3, [7,8,9,11,14]. Some basic notions of Lorentzian space are given [3,12,17,19]. M_1 is a timelike curve then the involute curve M_2 is a spacelike curve with a spacelike or timelike binormal.On the other hand, it has been investigated that the involute and evolute curves of the spacelike curve M_1 with a spacelike binormal in Minkowski 3-space and it has been seen that the involute curve M_2 is timelike, [4,5]. The involute curves of the spacelike curve M_1 with a timelike binormal is defined in Minkowski 3-space IR_1^3, [2,15,16]. Lorentzian angle is defined in [13].

2. Preliminaries

*Corresponding author

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W. K. Clifford introduced dual numbers as the set

$ID=\left\{\hat{\lambda}=\lambda+\varepsilon\lambda^{*}\,\middle|\,\lambda,\lambda^{*}\in IR,\ \ \varepsilon^{2}=0\ \text{ for }\ \varepsilon\neq 0\right\}$, [6]. Product, addition, division and absolute value operations are defined on *ID* like below, respectively:

$$\left(\lambda+\varepsilon\lambda^{*}\right)+\left(\beta+\varepsilon\beta^{*}\right)=\left(\lambda+\beta\right)+\varepsilon\left(\lambda^{*}+\beta^{*}\right),$$

$$\left(\lambda+\varepsilon\lambda^{*}\right)\left(\beta+\varepsilon\beta^{*}\right)=\lambda\beta+\varepsilon\left(\lambda\beta^{*}+\lambda^{*}\beta\right),$$

$$\frac{\lambda+\varepsilon\lambda^{*}}{\beta+\varepsilon\beta^{*}}=\frac{\lambda}{\beta}+\varepsilon\left(\frac{\lambda^{*}}{\beta}-\frac{\lambda\beta^{*}}{\beta^{2}}\right),$$

$$\left|\lambda+\varepsilon\lambda^{*}\right|=\left|\lambda\right|.$$

$ID^{3}=\left\{\vec{A}=\vec{a}+\varepsilon\vec{a}^{*}\,\middle|\,\vec{a},\vec{a}^{*}\in IR^{3}\right\}$ .The elements of ID^{3} are called dual vectors. On this set addition and scalar product operations are respectively

$$\begin{aligned}\oplus: & ID^{3}\times ID^{3}\to ID^{3}\\ & \left(\vec{A},\vec{B}\right)\ \to\vec{A}\oplus\vec{B}=\vec{a}+\vec{b}+\varepsilon\left(\vec{a}^{*}+\vec{b}^{*}\right)\end{aligned}$$

$$\begin{aligned}\odot: & ID\times ID^{3}\to ID^{3}\\ & \left(\tilde{\lambda},\vec{A}\right)\ \to\tilde{\lambda}\odot\vec{A}=\left(\lambda+\varepsilon\lambda^{*}\right)\odot\left(\vec{a}+\varepsilon\vec{a}^{*}\right)=\lambda\vec{a}+\varepsilon\left(\lambda\vec{a}^{*}+\lambda^{*}\vec{a}\right)\end{aligned}$$

The set $\left(ID^{3},\oplus,\odot\right)$ is a module over the ring $\left(ID,+,\cdot\right)$ and it is denoted by $\left(ID-Modul\right)$.

The Lorentzian inner product of dual vectors $\vec{A},\vec{B}\in ID^{3}$ is defined by

$$\left\langle\vec{A},\vec{B}\right\rangle=\left\langle\vec{a},\vec{b}\right\rangle+\varepsilon\left(\left\langle\vec{a},\vec{b}^{*}\right\rangle+\left\langle\vec{a}^{*},\vec{b}\right\rangle\right)$$

by means of the Lorentzian inner product, where $\vec{a}=(a_1,a_2,a_3)$ and $\vec{b}=(b_1,b_2,b_3)\in IR^3$ and the Lorentzian inner product is

$$\left\langle \vec{a},\vec{b}\right\rangle = -a_1b_1 + a_2b_2 + a_3b_3 .$$

Therefore, ID^3 with the Lorentzian inner product $\left\langle \vec{A},\vec{B}\right\rangle$ is called 3-dimensional dual Lorentzian space and denoted by of $ID_1^3=\left\{\vec{A}=\vec{a}+\varepsilon\vec{a}^{*}\,\middle|\,\vec{a},\vec{a}^{*}\in IR_1^3\right\}$.For $\vec{A}\neq 0$, the norm of $\vec{A}=\vec{a}+\varepsilon\vec{a}^{*}\in ID_1^3$ is defined by

$$\left\|\vec{A}\right\| = \sqrt{\left|\left\langle \vec{A},\vec{A}\right\rangle\right|} = \left\|\vec{a}\right\| + \varepsilon\frac{\left\langle \vec{a},\vec{a}^{*}\right\rangle}{\left\|\vec{a}\right\|}\ ,\ \left\|\vec{a}\right\|\neq 0\ .$$

For $\vec{A},\vec{B}\in ID_1^3$, the dual Lorentzian cross product is defined by

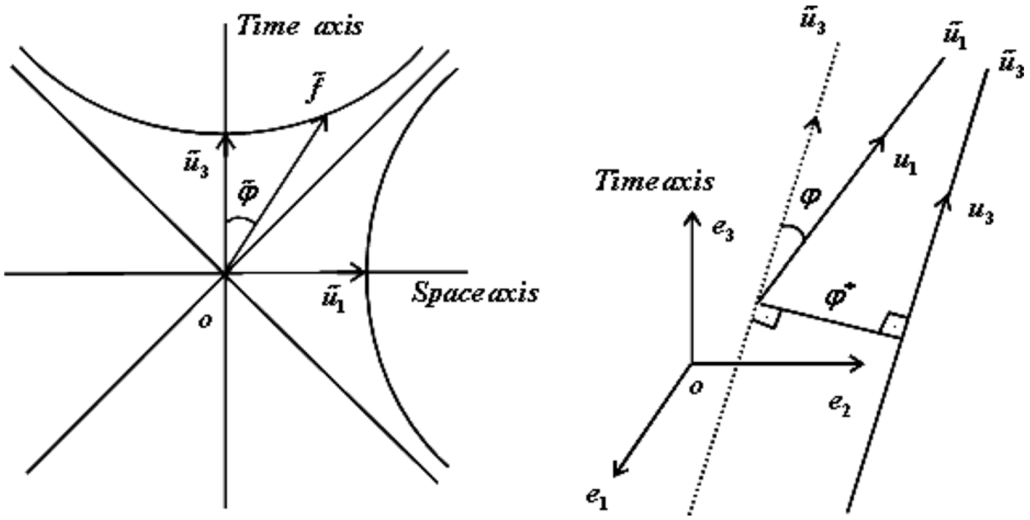
$$\vec{A}\wedge\vec{B}=\vec{a}\wedge\vec{b}+\varepsilon\left(\vec{a}\wedge\vec{b}^{*}+\vec{a}^{*}\wedge\vec{b}\right)$$

by means of the Lorentzian cross-product, such that for every $\vec{a}$, $\vec{b}\in IR_1^3$ the Lorentzian cross product is

$$\vec{a}\wedge\vec{b}=\left(a_3b_2-a_2b_3,a_1b_3-a_3b_1,a_1b_2-a_2b_1\right),[10].$$

The dual Frenet trihedron of the differentiable curve M in dual space ID_1^3 and instantaneous dual rotation vector have given in [1,20].The dual angle between $\vec{A}$ and $\vec{B}$ is $\tilde{\varphi}=\varphi+\varepsilon\varphi^{*}$ where φ is the angle between two directed lines that $\vec{A}$ and $\vec{B}$ represent in IR_1^3, respectively and φ^{*} is the shortest distance between these lines. See the Fig.1. In addition, the following equations are true for the dual angle, $\tilde{\varphi}$.

$$\begin{cases}\sinh\left(\varphi+\varepsilon\varphi^{*}\right)=\sinh\varphi+\varepsilon\varphi^{*}\cosh\varphi\\ \cosh\left(\varphi+\varepsilon\varphi^{*}\right)=\cosh\varphi+\varepsilon\varphi^{*}\sinh\varphi.\end{cases}$$



a) ***b)***

Fig.2. **a)** The dual hyperbolic angle $\tilde{\varphi} = \varphi + \varepsilon\varphi^*$ between dual timelike unit vectors $\tilde{u}_3$ and $\tilde{f}$, and the Lorentzian geometrical interpretation of this angle, $\tilde{\varphi}$.
b) The geometrical representation of $\tilde{\varphi}$.

The dual Lorentzian sphere and the dual hyperbolic sphere of 1 radius in IR_1^3 are defined by

$$S_1^2 = \left\{ A = a + \varepsilon a_0 \,\middle|\, \|A\| = (1,0); a, a_0 \in IR_1^3, \text{ and } a \text{ is spacelike} \right\},$$

$$H_0^2 = \left\{ A = a + \varepsilon a_0 \,\middle|\, \|A\| = (1,0); a, a_0 \in IR_1^3, \text{ and } a \text{ is timelike} \right\},$$

respectively [19].

Lemma 2. 1. Let X and Y be nonzero Lorentz orthogonal vectors in ID_1^3 If X is timelike, then Y is spacelike, [13].

Lemma 2. 2. Let X ,Y be positive (negative) timelike vectors in ID_1^3. Then $\langle X, Y \rangle \leq \|X\|\|Y\|$ is valid if and only if X and Y are linearly dependent, [13].

Lemma 2.3.i) Let X and Y be positive (negative) timelike vectors in ID_1^3. There is a unique nonnegative dual number $\Phi(X,Y)$,such that

$$\langle X,Y\rangle = \|X\|\|Y\|\cosh\Phi(X,Y)$$

where $\Phi(X,Y)$ is the Lorentzian timelike dual angle between X and Y.

ii) Let X and Y be spacelike vectors in ID_1^3 that span a spacelike vector subspace. Then wehave $|\langle X,Y\rangle| \leq \|X\|\|Y\|$ .Hence, there is a unique dual number $\Phi(X,Y)$ between 0 and π,such that

$$\langle X,Y\rangle = \|X\|\|Y\|\cos\Phi(X,Y)$$

where $\Phi(X,Y)$ is the Lorentzian spacelike dual angle between X and Y.

iii) Let X and Y be spacelike vectors in ID_1^3 that span a timelike vector subspace. Then we have $|\langle X,Y\rangle| \geq \|X\|\|Y\|$. Hence, there is a unique positive dual number $\Phi(X,Y)$, such that

$$\langle X,Y\rangle = \|X\|\|Y\|\cosh\Phi(X,Y)$$

where $\Phi(X,Y)$ is the Lorentzian timelike dual angle between X and Y.

iv) Let *X* be a spacelike vector and *Y* a positive timelike vector in ID_1^3. Then there is a unique nonnegative dual number $\Phi(X,Y)$ is the Lorentzian timelike dual angle between *X* and *Y*, such that

$$\langle X,Y\rangle = \|X\|\|Y\|\sinh\Phi(X,Y), \text{ [13].}$$

Let $\{T,N,B\}$ be the dual Frenet trihedron of the differentiable curve M.in the dual space ID_1^3 and $T = t + \varepsilon t^*$, $N = n + \varepsilon n^*$ and $B = b + \varepsilon b^*$ be the tangent, the principal normal and the binormal vectors of M, respectively. Depending on the causal character of the curve M, we have an instantaneous dual rotation vector :***i)*** Let M be a unit speed timelike dual space curve with the dual curvature $\kappa = k_1 + \varepsilon k_1^*$ and the dual torsion $\tau = k_2 + \varepsilon k_2^*$.The Frenet vectors *T, N,B* of *M* are timelike vector, spacelike vector, spacelike vector, respectively, such that

$$T \wedge N = -B \quad , \quad N \wedge B = T \quad , \quad B \wedge T = -N. \qquad (2.1)$$

From here,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, [18]. \tag{2.2}$$

(2.2) leaves the real and dual components

$$\begin{cases} \begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & -k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \\ \begin{bmatrix} t^{*\prime} \\ n^{*\prime} \\ b^{*\prime} \end{bmatrix} = \begin{bmatrix} 0 & k_1^* & 0 \\ k_1^* & 0 & -k_2^* \\ 0 & k_2^* & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} + \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & -k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix}. \end{cases}$$

The Frenet instantaneous rotation vector W of the timelike curve is given by

$$W = \tau T - \kappa B, [17] \tag{2.3}$$

(2.3) leaves the real and dual components

$$\begin{cases} w = k_2 t - k_1 b \\ w^* = k_2^* t + k_2 t^* - k_1^* b - k_1 b^* \end{cases}$$

Let $\Phi = \varphi + \varepsilon\varphi^*$ be a Lorentzian timelike dual angle between the spacelike binormal unit vector B and the Frenet instantaneous dual rotation vector W. Then, $C = c + \varepsilon c^*$ is the unit dual vector in direction of W:

a) If $|\kappa| > |\tau|$, W is a spacelike vector. In this case, we can write

$$\begin{cases} \kappa = \|W\| \cosh\Phi \\ \tau = \|W\| \sinh\Phi \end{cases}, \quad \|W\|^2 = \langle W, W \rangle = \kappa^2 - \tau^2 \tag{2.4}$$

and

$$C = \sinh\Phi T - \cosh\Phi B. \tag{2.5}$$

b) If $|\kappa| < |\tau|$, W is a timelike vector. In this case, we can write

$$\begin{cases} \kappa = \|W\|\sinh\Phi \\ \tau = \|W\|\cosh\Phi \end{cases} , \quad \|W\|^2 = -\langle W, W\rangle = -\left(\kappa^2 - \tau^2\right) \tag{2.6}$$

and

$$C = \cosh\Phi T - \sinh\Phi B\ . \tag{2.7}$$

ii) Let M be a unit speed dual spacelike space curve with the spacelike binormal. The Frenet vevtors *T*, *N*,*B* of *M* are spacelike vector, timelike vector, spacelike vector, respectively, such that

$$T \wedge N = -B \ , \ N \wedge B = -T \ , \ B \wedge T = N. \tag{2.8}$$

From here,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, [18]. \tag{2.9}$$

(2.9) leaves the real and dual components

$$\begin{cases} \begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \\ \begin{bmatrix} t^{*\prime} \\ n^{*\prime} \\ b^{*\prime} \end{bmatrix} = \begin{bmatrix} 0 & k_1^* & 0 \\ k_1^* & 0 & k_2^* \\ 0 & k_2^* & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} + \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix} \end{cases}$$

and the Frenet instantaneous rotation vector for the spacelike curve is given by

$$W = -\tau T + \kappa B, [17] \tag{2.10}$$

(2.10) leaves the real and dual components

$$\begin{cases} w = -k_2 t + k_1 b \\ w^* = -k_2^* t - k_2 t^* + k_1^* b + k_1 b^* \end{cases}$$

Let $\Phi = \varphi + \varepsilon\varphi^*$ be the dual angle between B and W. If B and W spacelike vectors that span a spacelike vector subspace, we can write

$$\begin{cases} \kappa = \|W\|\cos\Phi \\ \tau = \|W\|\sin\Phi \end{cases} , \quad \|W\|^2 = \langle W, W \rangle = \kappa^2 + \tau^2 \tag{2.11}$$

and

$$C = -\sin\Phi T + \cos\Phi B \ . \tag{2.12}$$

iii) Let M be a unit speed dual spacelike space curve. The Frenet vectors T, N,B of M are spacelike vector, timelike vector, spacelike vector, respectively, such that

$$T \wedge N = B \ , \ N \wedge B = -T \ , \ B \wedge T = -N. \tag{2.13}$$

From here,

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \ \ [18]. \tag{2.14}$$

The equation, (2.14) leaves the real and dual components

$$\begin{cases} \begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \\ \begin{bmatrix} t^{*\prime} \\ n^{*\prime} \\ b^{*\prime} \end{bmatrix} = \begin{bmatrix} 0 & k_1^* & 0 \\ -k_1^* & 0 & k_2^* \\ 0 & k_2^* & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} + \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix} \end{cases}$$

and the Frenet instantaneous dual rotation vector W of the spacelike curve is given by

$$W = -\tau T + \kappa B, [17] \tag{2.15}$$

The equation (2.15) leaves the real and dual components

$$\begin{cases} w = k_2 t - k_1 b \\ w^* = k_2^* t + k_2 t^* - k_1^* b - k_1 b^* \end{cases}$$

Let $\Phi = \varphi + \varepsilon\varphi^*$ be the Lorentzian timelike dual angle between B and W :

a)If $|\kappa| < |\tau|$, W is a spacelike vector. In this case, we can write

$$\begin{cases} \kappa = \|W\| \sinh \Phi \\ \tau = \|W\| \cosh \Phi \end{cases} , \quad \|W\|^2 = \langle W, W \rangle = \tau^2 - \kappa^2 \tag{2.16}$$

and

$$C = \cosh \Phi T - \sinh \Phi B \ . \tag{2.17}$$

b)If $|\kappa| > |\tau|$, W is a timelike vector. In this case, we can write

$$\begin{cases} \kappa = \|W\| \cosh \Phi \\ \tau = \|W\| \sinh \Phi \end{cases} , \quad \|W\|^2 = -\langle W, W \rangle = -\left(\tau^2 - \kappa^2\right) \tag{2.18}$$

and

$$C = \sinh \Phi T - \cosh \Phi B \ . \quad (2.19)$$

3.Main Results

Definition 3.1. Let $M_1 : I \to ID_1^3 \quad M_1 = M_1(s)$ be the unit speed dual timelike curve and $M_2 : I \to ID_1^3 \ M_2 = M_2(s)$ be the unit speed dual curve. If the tangent vector of curve M_1 is orthogonal to the tangent vector of M_2, M_1 is called evolute of curve M_2 and M_2 is called involute of M_1. Thus, the dual involute – evolute curve couple is denoted by (M_2, M_1). Since the tangent vector of M_1 is timelike, the tangent vector of M_2 must be spacelike vector. So, M_2 is a spacelike curve and (M_2, M_1) is called *"spacelike – timelike involute – evolute dual curve couple"* .

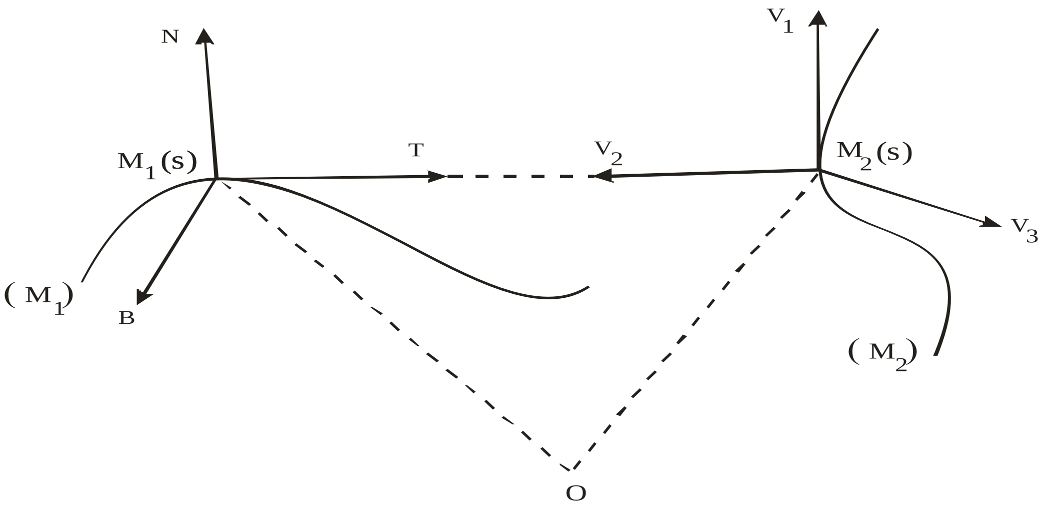


Fig. 2. Involute – evolute curve couple.

Theorem 3.1: Let (M_2, M_1) be the spacelike – timelike involute – evolute dual curve couple. Let $\{T, N, B\}$ and $\{V_1, V_2, V_3\}$ be the dual Frenet frames of M_1 and M_2, respectively. The dual distance between M_1 and M_2 at the corresponding points is

$$d\left(M_1(s), M_2(s)\right) = |c_1 - s| - \varepsilon c_2 \, , \ c_1, c_2 = \text{constant}$$

Proof: If M_2 is the dual involute of M_1, we can write from the Fig. 2

$$M_2(s) = M_1(s) + \lambda T(s), \lambda = \lambda_1 + \varepsilon \lambda_1^* \in ID \quad (3.1)$$

Differentiating (3.1) with respect to s, we have

$$V_1 \frac{ds^*}{ds} = \left(1+\lambda'\right)T + \lambda\kappa N$$

where s and s^* are arc parameter of M_1 and M_2, respectively. Since the direction of T is orthogonal to the direction of V, we obtain

$$\lambda' = -1.$$

From here, it can be easily seen that

$$\lambda = \left(c_1 - s\right) + \varepsilon c_2 \tag{3.2}$$

Furthermore, the dual distance between the points $M_1\left(s\right)$ and $M_2\left(s\right)$

$$\begin{aligned} d\left(M_1\left(s\right), M_2\left(s\right)\right) &= \sqrt{\left|\left\langle \lambda T\left(s\right)\ \lambda T\left(s\right)\right\rangle\right|} \\ &= \left|\lambda_1\right| - \varepsilon\lambda_1^* \ . \end{aligned}$$

Since $\lambda_1 = \left(c_1 - s\right)$, $\lambda_1^* = c_2$, we have

$$d\left(M_1\left(s\right), M_2\left(s\right)\right) = \left|c_1 - s\right| - \varepsilon c_2 . \tag{3.3}$$

Theorem 3.2. Let (M_2, M_1) be the spacelike – timelike involute – evolute dual curve couple. Let $\left\{T, N, B\right\}$ and $\left\{V_1, V_2, V_3\right\}$ be the dual Frenet frames of M_1 and M_2,respectively. Since the dual curvature of M_2 is $P = p + \varepsilon p^*$,we have

$$P^2 = \frac{\mp\left(k_2^2 - k_1^2\right)}{\left(c_1 - s\right)^2 k_1^2} \mp \varepsilon\left[\frac{2k_2\left(k_1 k_2^* - k_1^* k_2\right)}{\left(c_1 - s\right)^2 k_1^3} - \frac{2c_2\left(k_2^2 - k_1^2\right)}{\left(c_1 - s\right)^3 k_1^2}\right].$$

Where the dual curvature of M_1 is $\kappa = k_1 + \varepsilon k_1^*$

Proof: Differentiating (3.1) with respect to s, we get

$$\frac{dM_2}{ds^*}\frac{ds^*}{ds^?}=\frac{dM_1}{ds}+\frac{d\lambda}{ds}T+\lambda\frac{dT}{ds}$$

or

$$V_1\frac{ds^*}{ds}=T-T+\lambda\kappa N=\lambda\kappa N.$$

From here, we can write

$$V_1=N \tag{3.4}$$

and

$$\frac{ds^*}{ds}=\lambda\kappa\ .$$

By differentiating the last equation and using (2.2), we obtain

$$\frac{dV_1}{ds^*}\frac{ds^*}{ds^?}=\frac{dN}{ds}=\kappa T-\tau B$$

or

$$PV_2=\frac{1}{\lambda\kappa}\left(\kappa T-\tau B\right).$$

From here, we have

$$P^2=\mp\frac{\left(\tau^2-\kappa^2\right)}{\lambda^2\kappa^2} \tag{3.5}$$

From the fact that $P=p+\varepsilon p^*$, $\lambda=\lambda_1+\varepsilon\lambda_1^*$, $\kappa=k_1+\varepsilon k_1^*$ and $\tau=k_2+\varepsilon k_2^*$, we get

$$P^2=\frac{\mp\left(k_2^2+2\varepsilon k_2k_2^*-k_1^2-2\varepsilon k_1k_1^*\right)}{\left(\lambda_1^2+2\varepsilon\lambda_1\lambda_1^*\right)\left(k_1^2+2\varepsilon k_1k_1^*\right)}$$

$$=\mp\frac{\left(k_2^2-k_1^2\right)}{\lambda_1^2k_1^2}\mp\varepsilon\left[\frac{2k_2\left(k_1k_2^*-k_1^*k_2\right)}{\lambda_1^2k_1^3}-\frac{2\lambda_1^*\left(k_2^2-k_1^2\right)}{\lambda_1^3k_1^2}\right].$$

From here, by using $\lambda_1 = (c_1 - s)$, $\lambda_2 = c_2$, we obtain

$$P^2 = \mp \frac{\left(k_2^2 - k_1^2\right)}{\left(c_1 - s\right)^2 k_1^2} \mp \varepsilon \left[\frac{2k_2\left(k_1 k_2^* - k_1^* k_2\right)}{\left(c_1 - s\right)^2 k_1^3} - \frac{2c_2\left(k_2^2 - k_1^2\right)}{\left(c_1 - s\right)^3 k_1^2} \right]. \tag{3.6}$$

Theorem 3.3. Let (M_2, M_1) be the spacelike – timelike involute – evolute dual curve couple and $\{T, N, B\}$ and $\{V_1, V_2, V_3\}$ be Frenet frames of M_1 and M_2, respectively. The dual torsion $\tau = k_2 + \varepsilon k_2^*$ of M_1 and the dual torsion $Q = q + \varepsilon q^*$ of M_2 is satisfy the following equation:

$$Q = \frac{k_1 k_2' - k_1' k_2}{\left|k_1^2 - k_2^2\right| k_1 \left|c_1 - s\right|} + \varepsilon \left[\frac{k_1\left(k_1 k_2^{*\prime} - k_1' k_2^*\right) + k_2\left(k_1^* k_1' - k_1^{*\prime} k_1\right)}{\left|k_1^2 - k_2^2\right| \left|c_1 - s\right| k_1^2} \right].$$

Proof: By differentiating (3.1) three times with respect to s, we get

$$\begin{aligned} M_2' &= \lambda\kappa N \\ M_2'' &= \lambda\kappa^2 T + \left(\lambda\kappa' - \kappa\right) N - \lambda\kappa\tau B \\ M_2''' &= \left(3\lambda\kappa\kappa' - 2\kappa^2\right) T + \left(\lambda\kappa^3 + \lambda\kappa\tau^2 - 2\kappa' + \lambda\kappa''\right) N + \left(2\kappa\tau - 2\lambda\kappa'\tau - \lambda\kappa\tau'\right) B \end{aligned}$$

The vector product of M_2' and M_2'' is

$$M_2' \wedge M_2'' = -\lambda^2\kappa^2\tau T + \lambda^2\kappa^3 B = \lambda^2\kappa^2\left(-\tau T + \kappa B\right) \tag{3.7}$$

From here, we obtain

$$\left\| M_2' \wedge M_2'' \right\|^2 = \left|\lambda\right|^4 \left|\kappa\right|^4 \left|\kappa^2 - \tau^2\right| \tag{3.8}$$

and

$$\det\left(M_2', M_2'', M_2'''\right) = \lambda^3\kappa^3\left(\kappa\tau' - \kappa'\tau\right). \tag{3.9}$$

Substituting by (3.8) and (3.9) values into $Q = \dfrac{\det\left(M_2', M_2'', M_2'''\right)}{\left\|M_2' \wedge M_2''\right\|^2}$, we get

$$Q = \frac{\left(\kappa\tau' - \kappa'\tau\right)}{|\lambda|\kappa\left|\kappa^2 - \tau^2\right|} \tag{3.10}$$

and then, substituting $Q = q + \varepsilon q^*$, $\lambda = \lambda_1 + \varepsilon\lambda_1^*$, $\kappa = k_1 + \varepsilon k_1^*$ and $\tau = k_2 + \varepsilon k_2^*$ into the last equation, we have

$$Q = \frac{k_1 k_2' - k_1' k_2}{|\lambda_1| k_1 \left|k_1^2 - k_2^2\right|} + \varepsilon\left[\frac{k_1\left(k_1 k_2^{*\prime} - k_1' k_2^*\right) + k_2\left(k_1^* k_1' - k_1^{*\prime} k_1\right)}{|\lambda_1| k_1^2 \left|k_1^2 - k_2^2\right|}\right]$$

By the fact that $\lambda_1 = \left(c_1 - s\right)$, we get

$$Q = \frac{k_1 k_2' - k_1' k_2}{|c_1 - s| k_1 \left|k_1^2 - k_2^2\right|} + \varepsilon\left[\frac{k_1\left(k_1 k_2^{*\prime} - k_1' k_2^*\right) + k_2\left(k_1^* k_1' - k_1^{*\prime} k_1\right)}{|c_1 - s| k_1^2 \left|k_1^2 - k_2^2\right|}\right]. \tag{3.11}$$

Theorem 3.4. Let (M_2, M_1) be the spacelike – timelike involut – evolut dual curve couple, $\{T, N, B\}$ and $\{V_1, V_2, V_3\}$ be the dual Frenet frames of M_1 and M_2, respectively and $\Phi = \varphi + \varepsilon\varphi^*$ be the Lorentzian dual timelike angle between binormal vector B and W. For (M_2, M_1) dual curve couple, the following equations is obtained:

1) If W is spacelike,

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cosh\Phi & 0 & \sinh\Phi \\ -\sinh\Phi & 0 & \cosh\Phi \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

leaves the real and dual components

$$\begin{cases} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\cosh\varphi & 0 & \sinh\varphi \\ -\sinh\varphi & 0 & \cosh\varphi \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \\ \begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix} = \varphi^* \begin{bmatrix} 0 & 0 & 0 \\ -\sinh\varphi & 0 & \cosh\varphi \\ -\cosh\varphi & 0 & \sinh\varphi \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ -\cosh\varphi & 0 & \sinh\varphi \\ -\sinh\varphi & 0 & \cosh\varphi \end{bmatrix} \begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix} \end{cases}$$

2) If W is timelike,

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \sinh\Phi & 0 & -\cosh\Phi \\ -\cos\Phi & 0 & \sinh\Phi \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

leaves the real and dual components

$$\begin{cases} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \sinh\varphi & 0 & -\cosh\varphi \\ -\cosh\varphi & 0 & \sinh\varphi \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} \\ \begin{bmatrix} v_1^* \\ v_2^* \\ v_3^* \end{bmatrix} = \varphi^* \begin{bmatrix} 0 & 0 & 0 \\ \cosh\varphi & 0 & -\sinh\varphi \\ -\sinh\varphi & 0 & \cosh\varphi \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ \sinh\varphi & 0 & -\cosh\varphi \\ -\cosh\varphi & 0 & \sinh\varphi \end{bmatrix} \begin{bmatrix} t^* \\ n^* \\ b^* \end{bmatrix} \end{cases}$$

Proof: 1) From (2.4), (3.4) and (3.8), we have

$$\left\| M_2' \wedge M_2'' \right\| = \lambda^2 \kappa^2 \|W\|. \tag{3.12}$$

By using (3.7) and (3.12) and from the fact that $V_3 = \dfrac{M_2' \wedge M_2''}{\left\| M_2' \wedge M_2'' \right\|}$, we obtain

$$V_3 = -\frac{\tau}{\|W\|} T + \frac{\kappa}{\|W\|} B,$$

Substituting (2.4) into the last equation, we obtain

$$V_3 = -\sinh\Phi T + \cosh\Phi B\,. \tag{3.13}$$

Since $V_2 = V_3 \wedge V_1$, it can be easily seen that

$$V_2 = -\cosh\Phi T + \sinh\Phi B\,. \tag{3.14}$$

Considering (3.4), (3.13) and (3.14) according to dual components, the following equations are obtained:

$$\begin{cases} V_1 = n + \varepsilon n^* \\ V_2 = \left(-\cosh\varphi t + \sinh\varphi b\right) + \varepsilon\left[\left(-\cosh\varphi t^* + \sinh\varphi b^*\right) + \varphi^*\left(-\sinh\varphi t + \cosh\varphi b\right)\right] \\ V_3 = \left(-\sinh\varphi t + \cosh\varphi b\right) + \varepsilon\left[\left(-\sinh\varphi t^* + \cosh\varphi b^*\right) + \varphi^*\left(-\cosh\varphi t + \sinh\varphi b\right)\right] \end{cases} \tag{3.15}$$

Writing (3.15) in matrix form, the proof is completed.

2)From (3.4) , (3.7) and (3.12), we obtain

$$V_1 = N$$

and

$$V_3 = -\frac{\tau}{\|W\|}T + \frac{\kappa}{\|W\|}B\,.$$

Substituting (2.6) into the last equation, we get

$$V_3 = -\cosh\Phi T + \sinh\Phi B, \tag{3.16}$$

$$V_2 = \sinh\Phi T - \cosh\Phi B\,. \tag{3.17}$$

Considering (3.4), (3.16) and (3.17) according to dual components, we obtain following equations:

$$\begin{cases} V_1 = n + \varepsilon n^* \\ V_2 = \left(\sinh \varphi t - \cosh \varphi b\right) + \varepsilon \left[\left(\sinh \varphi t^* - \cosh \varphi b^*\right) + \varphi^* \left(\cosh \varphi t - \sinh \varphi b\right) \right] \\ V_3 = \left(-\cosh \varphi t + \sinh \varphi b\right) + \varepsilon \left[\left(-\cosh \varphi t^* + \sinh \varphi b^*\right) + \varphi^* \left(-\sinh \varphi t + \cosh \varphi b\right) \right] \end{cases} \tag{3.18}$$

Writing (3.18) in matrix form, the proof is completed.

Theorem 3.5. Let (M_2, M_1) be the spacelike – timelike involute – evolute dual curve couple and $W = w + \varepsilon w^*$ and $\overline{W} = \overline{w} + \varepsilon \overline{w}^*$ be the dual Frenet instantaneous rotation vectors of M_1 and M_2 respectively. Thus,

1)If W is spacelike,

$$\overline{W} = \frac{-\varphi' n - w}{\left|c_1 - s\right| k_1} + \varepsilon \left(\frac{-\varphi' n^* - \varphi^{*\prime} n - w^*}{\left|c_1 - s\right| k_1} + \frac{k_1^* \left(\varphi' n + w\right)}{\left|c_1 - s\right| k_1^2} \right),$$

2) If W is timelike,

$$\overline{W} = \frac{-\varphi' n + w}{\left|c_1 - s\right| k_1} + \varepsilon \left(\frac{-\varphi' n^* - \varphi^{*\prime} n + w^*}{\left|c_1 - s\right| k_1} + \frac{k_1^* \left(\varphi' n - w\right)}{\left|c_1 - s\right| k_1^2} \right).$$

Proof: 1)From (2.10), we can write

$$\overline{W} = -QV_1 + PV_3$$

using (3.4), (3.5), (3.10) and (3.13) , we have

$$\overline{W} = \frac{1}{\left|\lambda\right| \kappa} \left(-\frac{\kappa \tau' - \kappa' \tau}{\left|\kappa^2 - \tau^2\right|} N + \sqrt{\left|\tau^2 - \kappa^2\right|} \left(-\sinh \Phi T + \cosh \Phi B\right) \right).$$

Substituting (2.4) into the last equation, we obtain

$$\overline{W} = \frac{1}{|\lambda|\kappa}\left(-\frac{\kappa\tau' - \kappa'\tau}{|\kappa^2 - \tau^2|}N - W\right)$$

and then, we get

$$\overline{W} = \frac{1}{|\lambda|\kappa}\left(-\Phi' N - W\right). \tag{3.19}$$

Considering (3.19) according to dual components and substituting $\lambda_1 = (c_1 - s)$ into (3.19), we leaves the real and dual components

$$\begin{cases} \overline{w} = \dfrac{-\varphi' n - w}{|c_1 - s| k_1} \\ \overline{w^*} = \dfrac{-\varphi' n^* - \varphi^{*\prime} n - w^*}{|c_1 - s| k_1} + \dfrac{k_1^*(\varphi' n + w)}{|c_1 - s| k_1^2}. \end{cases} \tag{3.20}$$

2) From (2.15), the dual Frenet instantaneous rotation vector of M_2 is

$$\overline{W} = QV_1 - PV_3$$

Using (3.4), (3.5), (3.10) and (3.16), we have

$$\overline{W} = \frac{1}{|\lambda|\kappa}\left(\frac{\kappa\tau' - \kappa'\tau}{|\kappa^2 - \tau^2|}N - \sqrt{|\tau^2 - \kappa^2|}\left(-\cosh\Phi T + \sinh\Phi B\right)\right).$$

Substituting (2.6) into the last equation, we obtain

$$\overline{W} = \frac{1}{|\lambda|\kappa}\left(\frac{\kappa\tau' - \kappa'\tau}{|\kappa^2 - \tau^2|}N + W\right)$$

and then, we get

$$\overline{W} = \frac{1}{|\lambda|\kappa}\left(-\Phi' N + W\right). \tag{3.21}$$

Considering (3.21) according to dual components and substituting $\lambda_1 = (c_1 - s)$ into (3.21), we leaves the real and dual components

$$\begin{cases} \overline{w} = \dfrac{-\varphi' n + w}{|c_1 - s| k_1} \\ \\ \overline{w^*} = \dfrac{-\varphi' n^* - \varphi^{*\prime} n + w^*}{|c_1 - s| k_1} + \dfrac{k_1^* (\varphi' n - w)}{|c_1 - s| k_1^2}. \end{cases} \tag{3.22}$$

Theorem 3.6. Let (M_2, M_1) be the spacelike – timelike involute – evolute dual curve couple and $C = c + \varepsilon c^*$ and $\overline{C} = \overline{c} + \varepsilon \overline{c}^*$ be unit dual vectors of W and $\overline{W}$, respectively. Thus,

i) If W is spacelike,

$$\overline{C} = \left(\frac{\varphi'}{\sqrt{|k_1^2 - k_2^2 + \varphi'^2|}} n + \frac{\sqrt{|k_1^2 - k_2^2|}}{\sqrt{|k_1^2 - k_2^2 + \varphi'^2|}} c\right) + \varepsilon\left(\frac{\varphi' n^* + \varphi^{*\prime} n + \sqrt{|k_1^2 - k_2^2|} c^*}{\sqrt{|k_1^2 - k_2^2 + \varphi'^2|}}\right),$$

ii) If W is timelike,

$$\overline{C} = \left(-\frac{\varphi'}{\sqrt{|k_1^2 - k_2^2 + \varphi'^2|}} n + \frac{\sqrt{|k_1^2 - k_2^2|}}{\sqrt{|k_1^2 - k_2^2 + \varphi'^2|}} c\right) + \varepsilon\left(\frac{-\left(\varphi' n^* + \varphi^{*\prime} n\right) + \sqrt{|k_1^2 - k_2^2|} c^*}{\sqrt{|k_1^2 - k_2^2 + \varphi'^2|}}\right).$$

Proof: i) From the fact that the unit dual vector of $\overline{W}$ is $\overline{C} = \dfrac{\overline{W}}{\left\|\overline{W}\right\|}$, we obtain

$$\overline{C} = \frac{-\Phi' N - W}{\sqrt{\left|\kappa^2 - \tau^2 + \Phi'^2\right|}}$$

or

$$\overline{C} = -\frac{-\Phi'}{\sqrt{\left|\kappa^2 - \tau^2 + \Phi'^2\right|}} N - \frac{\sqrt{\left|\kappa^2 - \tau^2\right|}}{\sqrt{\left|\kappa^2 - \tau^2 + \Phi'^2\right|}} C. \quad (3.24)$$

Considering (3.24) according to dual components, we see that

$$\overline{C} = \left(-\frac{\varphi'}{\sqrt{\left|k_1^2 - k_2^2 + \varphi'^2\right|}} n - \frac{\sqrt{\left|k_1^2 - k_2^2\right|}}{\sqrt{\left|k_1^2 - k_2^2 + \varphi'^2\right|}} c \right) + \varepsilon \left(-\frac{\varphi' n^* + \varphi^{*\prime} n + \sqrt{\left|k_1^2 - k_2^2\right|} c^*}{\sqrt{\left|k_1^2 - k_2^2 + \varphi'^2\right|}} \right). \quad (3.25)$$

ii) Substituting (3.21) into (3.23) , we obtain

$$\overline{C} = \frac{-\Phi' N + W}{\sqrt{\left|\kappa^2 - \tau^2 + \Phi'^2\right|}}$$

or

$$\overline{C} = -\frac{-\Phi'}{\sqrt{\left|\kappa^2 - \tau^2 + \Phi'^2\right|}} N + \frac{\sqrt{\left|\kappa^2 - \tau^2\right|}}{\sqrt{\left|\kappa^2 - \tau^2 + \Phi'^2\right|}} C. \quad (3.26)$$

Considering (3.26) according to dual components, we see that

$$\overline{C} = \left(-\frac{\varphi'}{\sqrt{\left|k_1^2 - k_2^2 + \varphi'^2\right|}} n + \frac{\sqrt{\left|k_1^2 - k_2^2\right|}}{\sqrt{\left|k_1^2 - k_2^2 + \varphi'^2\right|}} c \right) + \varepsilon \left(\frac{-\varphi' n^* - \varphi^{*\prime} n + \sqrt{\left|k_1^2 - k_2^2\right|} c^*}{\sqrt{\left|k_1^2 - k_2^2 + \varphi'^2\right|}} \right). \quad (3.27)$$

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