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CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS THROUGH CONTRAST OF GENERALIZED ORDER STATISTICS

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Abstract: In this paper a general form of continuous probability distribution is characterized through conditional expectations of contrast of generalized order statistics, conditioned on a non-adjacent generalized order statistics. Further, some of its deductions are also discussed.

Keywords: Characterization; Conditional expectation; Continuous distributions; Generalized order statistics; Order statistics and record statistics.

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1. Introduction

The concept of generalized order statistics has been introduced as a unified approach to a variety of models of ordered random variables (Kamps, 1995), such as ordinary order statistics, sequential order statistics, progressive type II censoring record values and Pfeifer's records .Generalized order statistics serve as a common approach to a structural similarities and analogies.

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (iid) random variables $(rv's)$ with absolutely continuous distribution function (df) $F(x)$ and the probability density function (pdf) $f(x)$, $x \in (\alpha, \beta)$. Let $n \in N$, $n \geq 2$, $k > 0$

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$\tilde{m}=(m_1,\ldots,m_{n-1})\in\Re^{n-1}$, $M_r=\sum_{j=r}^{n-1}m_j$ such that $\gamma_r=k+(n-r)+M_r>0$ or all $r\in\{1,\ldots,n-1\}$. Then $X(r,n,\tilde{m},k)$, $r=1,2,\ldots,n$ are called *gos* if their joint density function is given by (Kamps, 1995)

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\prod_{i=1}^{n-1}[1-F(x_i)]^{m_i}f(x_i)[1-F(x_n)]^{k-1}f(x_n) \tag{1.1}$$

For $F^{-1}(0)<x_1\le\cdots\le x_n<F^{-1}(1)$.

Choosing the parameters appropriately (Cramer, 2002),

Table 1.1: Variants of the generalized order statistics

		$\gamma_n=k$	γ_r	m_r
i)	Sequential order statistics	α_n	$(n-r+1)\alpha_r$	$(\gamma_r-\gamma_{r+1}-1)$
ii)	Ordinary order statistics	1	$n-r+1$	0
iii)	Record statistics	1	1	-1
iv)	Progressively type II censored order statistics	R_n+1	$n-r+1+\sum_{j=r}^{n}R_j$	R_r
v)	Pfeifer's record statistics	β_n	β_r	$(\beta_r-\beta_{r+1}-1)$

Here we consider two cases:

Case I: $m_1=m_2=\cdots=m_{n-1}=m$.

Case II: $\gamma_i \neq \gamma_j$, $i, j = 1, 2, \cdots, n-1$, $i \neq j$.

For Case I, the *pdf* of $X(r,n,m,k)$ is given by (Kamps, 1995)

$$f_{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) \tag{1.2}$$

and the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$ is given by

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{c_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m \; g_m^{r-1}(F(x))$$

$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s - 1} f(x) f(y), \alpha < x < y < \beta \tag{1.3}$$

where

$$\bar{F}(x) = 1 - F(x)$$

$$c_{s-1} = \prod_{i=1}^{s} \gamma_i$$

$$\gamma_i = k + (n-i)(m+1), \tag{1.4}$$

$$h_m(x) = \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\ -\log(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = \int_0^x (1-t)^m dt = h_m(x) - h_m(0), \; x \in [0,1).$$

The conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \le r < s \le n$

is given by

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y \mid x) = \frac{c_{s-1}}{(s-r-1)! c_{r-1}} \frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}}{[\bar{F}(x)]^{\gamma_r + 1}}$$

$$\times [\bar{F}(y)]^{\gamma_s - 1} f(y), \; \alpha < x < y < \beta \tag{1.5}$$

For Case II, the *pdf* of $X(r,n,\tilde{m},k)$ is given by (Kamps and Cramer, 2001)

$$f_{X(r,n,\tilde{m},k)}(x) = c_{r-1}\, f(x)\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i - 1} \tag{1.6}$$

and the joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$, $1 \le r < s \le n$ is

$$f_{X(r,n,\tilde{m},k),\, X(s,n,\tilde{m},k)}(x,y) = c_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i}$$

$$\times\left[\sum_{i=1}^{r} a_i(r)(\bar{F}(x))^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)}\,\frac{f(y)}{\bar{F}(y)} \tag{1.7}$$

where

$$\gamma_i = k + n - i + M_i\,, \tag{1.8}$$

$$a_i(r) = \prod_{\substack{j=1 \\ j \ne i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}\ ,\ \gamma_i \ne \gamma_j\,,\ 1 \le i \le r \le n$$

and $$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \ne i}}^{s} \frac{1}{(\gamma_j - \gamma_i)}\ ,\ \gamma_i \ne \gamma_j\ ,\ r+1 \le i \le s \le n$$

Thus, the conditional *pdf* of $X(s,n,\tilde{m},k)$ given $X(r,n,\tilde{m},k) = x$, $1 \le r < s \le n$

$$f_{X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k)}(y \mid x) = \frac{c_{s-1}}{c_{r-1}} \sum_{i=r+1}^{s} a_i^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i} \frac{f(y)}{[\bar{F}(y)]},\ x \le y \tag{1.9}$$

Characterization results of distributions based on generalized order statistics are given by Ahsanullah (1995), Kamps (1995), Kamps and Gather (1997), Ahsanullah and Nevzorov (2001), Cramer *et al.* (2003 a, b), Khan and Alzaid (2004) and Khan *et al.* (2006) amongst others.

In particular, Keseling and Kamps (2003), Cramer and Kamps (2001), Cramer *et al.* (2003b) and Khan *et al*. (2011) have characterized distributions using conditional spacing of generalized order statistics. An attempt has been made here to characterize distributions through contrast of conditional expectation of generalized order statistics, extending the earlier known results.

Now for $m_i = m_j = m$, we have

$$a_i(r) = \frac{1}{(m+1)^{r-1}}(-1)^{r-i}\frac{1}{(i-1)!\,(r-i)!}$$

and $$a_i^{(r)}(s) = \frac{1}{(m+1)^{s-r-1}}(-1)^{s-i}\frac{1}{(i-r-1)!(s-i)!}.$$

Thus, case II reduces to case I. Therefore, we will characterize the distribution for case II only and then deduce it for case I.

2. Main result (Characterization of distributions when $\gamma_i \neq \gamma_j$, $i \neq j$.)

Theorem 2.1: Let X be an absolutely continuous random variable with the *df* $F(x)$ and the *pdf* $f(x)$ on the support (α, β), where α and β may be finite or infinite. Then for $1 \le r < s < t \le n$,

$$\sum_{l=s}^{t} b_l E[h\{X(l,n,\tilde{m},k)\} \mid X(r,n,\tilde{m},k) = x] = \frac{1}{a}\sum_{l=s}^{t} b_l \sum_{j=r+1}^{l}\frac{1}{\gamma_j} \tag{2.1}$$

if and only if

$$F(x) = 1 - e^{-ah(x)} \quad , \; a > 0 \tag{2.2}$$

where b_l are real numbers $s \le l \le t$, satisfying $\sum_{l=s}^{t} b_l = 0$, for all $b_l \neq 0$ and $h(x)$ is a monotonic increasing and differentiable function of x.

Proof: First we will prove (2.2) implies (2.1). We have, (Khan and Alzaid, 2004)

$$E[h\{X(l,n,\tilde{m},k)\} \mid X(r,n,\tilde{m},k) = x] = h(x) + \frac{1}{a}\sum_{j=r+1}^{l}\frac{1}{\gamma_j}$$

Therefore,

$$\sum_{l=s}^{t} b_l E[h\{X(l,n,\tilde{m},k)\} \mid X(r,n,\tilde{m},k)=x] = \sum_{l=s}^{t} b_l \left[h(x) + \frac{1}{a}\sum_{j=r+1}^{l}\frac{1}{\gamma_j}\right]$$

$$= \frac{1}{a}\sum_{l=s}^{t} b_l \sum_{j=r+1}^{l}\frac{1}{\gamma_j}, \text{ as } \sum_{i=s}^{t} b_l = 0$$

Hence the 'if' part.

For the sufficiency part. We have

$$\sum_{l=s}^{t} b_l E[h\{X(l,n,\tilde{m},k)\} \mid X(r,n,\tilde{m},k)=x] = b \tag{2.3}$$

where $b = \frac{1}{a}\sum_{l=s}^{t} b_l \sum_{j=r+1}^{l}\frac{1}{\gamma_j}$

Therefore,

$$\sum_{l=s}^{t} b_l \frac{c_{l-1}}{c_{r-1}} \sum_{i=r+1}^{l} a_i^{(r)}(l)\int_x^\beta h(y)\frac{[\bar{F}(y)]^{\gamma_i-1}}{[\bar{F}(x)]^{\gamma_i}} f(y)dy = b \tag{2.4}$$

Integrating by parts, we get

$$\sum_{l=s}^{t} b_l \frac{c_{l-1}}{c_{r-1}} \sum_{i=r+1}^{l} a_i^{(r)}(l)\int_x^\beta h'(y)\frac{[\bar{F}(y)]^{\gamma_i}}{[\bar{F}(x)]^{\gamma_i}} dy$$

$$-\sum_{l=s}^{t} b_l \frac{c_{l-1}}{c_{r-1}} \sum_{i=r+1}^{l} a_i^{(r)}(l)(\gamma_i - 1)\int_x^\beta h(y)\frac{[\bar{F}(y)]^{\gamma_i-1}}{[\bar{F}(x)]^{\gamma_i}} f(y)dy = b$$

Using the relation $c_r = \gamma_{r+1} c_{r-1}$ and $a_i^{(r+1)}(l) = (\gamma_{r+1} - \gamma_i)\, a_i^{(r)}(l)$, we get

$$\sum_{l=s}^{t} b_l \frac{c_{l-1}}{c_{r-1}} \sum_{i=r+1}^{l} a_i^{(r)}(l)\int_x^\beta a\, h'(y)\frac{[\bar{F}(y)]^{\gamma_i}}{[\bar{F}(x)]^{\gamma_i}} dy = 0, \text{ as } \sum_{l=r}^{s} b_l = 0 \tag{2.5}$$

Since,

$$\frac{c_{l-1}}{c_{r-1}} \sum_{i=r+1}^{l} a_i^{(r)}(l)\int_x^\beta \frac{[\bar{F}(y)]^{\gamma_i-1}}{[\bar{F}(x)]^{\gamma_i}} f(y)dy = 1, \text{ in view of (1.9).}$$

Therefore,

$$\sum_{l=s}^{t} b_l \frac{c_{l-1}}{c_{r-1}} \sum_{i=r+1}^{l} a_i^{(r)}(l)\int_x^\beta \frac{[\bar{F}(y)]^{\gamma_i-1}}{[\bar{F}(x)]^{\gamma_i}} f(y)dy = 0 \tag{2.6}$$

Comparing (2.5) and (2.6), we get

$$a\,h'(y)\bar{F}(y) = f(y)$$

That is,

$$F(y) = 1 - \exp[-ah(y)]\ ,a > 0$$

and hence the Theorem.

Remark 2.1: Putting $b_t = 1$ and $b_s = -1$ in Theorem 2.1, we get the result as obtained by Khan *et al*. (2011).

The result can be deduced for order statistics and records as well as for case I.

Table 2.1: Examples based on the distribution function $F(x) = 1 - e^{-a\,h(x)}\,, a > 0$

Distribution	$F(x)$	a	$h(x)$
Exponential	$1 - e^{-\theta x}$ $0 < x < \infty$	θ	x
Weibull	$1 - e^{-\theta x^p}$ $0 < x < \infty$	θ	x^p
Pareto	$1 - \left(\frac{x}{a}\right)^{-p}$ $a < x < \infty$	p	$\log\left(\frac{x}{a}\right)$
Lomax	$1 - (1+x)^{-k}$ $0 < x < \infty$	k	$\log(1+x)$
Gompertz	$1 - \exp[-\frac{\lambda}{\mu}(e^{\mu x} - 1)]$ $0 < x < \infty$	$\frac{\lambda}{\mu}$	$e^{\mu x} - 1$
Beta of the I kind	$1 - (1-x)^p$ $0 < x < 1$	p	$-\log(1-x)$
Beta of the II kind	$1 - (1+x)^{-1}$ $0 < x < \infty$	1	$\log(1+x)$

Extreme value I	$1-\exp[-e^{x}]$ $-\infty < x < \infty$	1	e^{x}
Log logistic	$1-(1+x^{c})^{-1}$ $0 < x < \infty$	1	$\log(1+x^{c})$
Burr Type IX	$1-\left[\frac{c\{(1+e^{x})^{k}-1\}}{2}+1\right]^{-1}$ $-\infty < x < \infty$	1	$\log\left[\frac{c\{(1+e^{x})^{k}-1\}}{2}+1\right]$
Burr Type XII	$1-(1+x^{c})^{-k}$ $0 < x < \infty$	k	$\log(1+x^{c})$

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