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ON SOME PROPERTIES FOR SPECTRAL RADIUS OF BRUALDI-LI MATRIX

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Abstract. Let B_{2m} denote the Brualdi-Li matrix,and let $\rho(B_{2m})$ denote the spectral radius of Brualdi-Li matrix. We obtain some properties of $\rho(B_{2m})$.

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1. INTRODUCTION

A tournament matrix of order n is a $(0,1)$ matrix T satisfying the equation $T+T^t = J-I$, where J is the all ones matrix, I is the identity matrix, and T^t is the transpose of T. The tournament matrices are inspired in the round robin competitions.Tournament matrices(and their generalizations)appear in a variety of combinatorial applications (e.g., in biology, sociology, statistics, and networks).

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Brualdi and Li matrix of order $2m$ is defined by

$$\mathscr{B}_{2m} = \begin{pmatrix} U_m & U_m^t \\ I + U_m^t & U_m \end{pmatrix},$$

where U_m is strictly lower triangular tournament matrix(all of whose entries below the main diagonal are equal to one). A matrix A of order n is said to be a reducible matrix if there exists a permutation matrix P such that

$$PAP^t = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix},$$

where A_1 and A_2 are square (non-vacuous), or if $n = 1$ and $A = O$. A matrix is called irreducible matrix if it is not reducible. The spectral radius of a matrix $A_{n\times n}$ defined as $\rho(A) = max\{|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|\}$, where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of $A_{n\times n}$. If an nonnegative matrix A is irreducible and it has exactly one eigenvalue of modulus $\rho(A)$,then the matrix is called a primitive matrix. Obviously,Brualdi and Li matrix $\mathscr{B}_{2m}$ $(m \geq 2)$is primitive matrix.

In 1983 Brualdi and Li conjectured that the maximal spectral radius for tournaments of order $2m$ is attained by the Brualdi-Li matrix [1]. This conjecture has recently be confirmed in [2]. The several interesting properties of Brualdi-Li matrix are studied. In this paper we investigative some properties of spectral radius for Brualdi-Li Matrix.

2. PRELIMINARIES

The notation and terminology used in this paper will basically follow those in [3].

Let $\mathbf{1}_m = (1, 1, \cdots, 1)^t_{m\times 1}$,$\mathbf{0}_m = (0, 0, \cdots, 0)^t_{m\times 1}$, and

$$U_m = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}_{m\times m},$$

where $m \geq 2$ is an integer.

Lemma 2.1[3] Let n be a nonnegative integer,and A be a primitive matrix of order n. Then

$$\lim_{k\to\infty}(\frac{A}{\rho})^k\mathbf{1}_n = S,$$

where $\rho=\rho(A)>0$,$S>0$ is a eigenvector of A corresponding to the eigenvalue of $\rho(A)$.

Let $b(2m,k)=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^k\mathbf{1}_{2m}$, $b_l(2m,k)=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^k\binom{\mathbf{1}_m}{\mathbf{0}_m}$,and $b_r(2m,k)=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^k\binom{\mathbf{0}_m}{\mathbf{1}_m}$, then
$b(2m,k+1)=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^{k+1}\mathbf{1}_{2m}=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^k\binom{(m-1)\mathbf{1}_m}{m\mathbf{1}_m}$
$=mb(2m,k)-b_l(2m,k)$
$=(m-1)b(2m,k)+b_r(2m,k)$.
It is easy to verify that the follow result.

Lemma 2.2 Let $k,m\geq 2$ be an integer,and $\rho=\rho(\mathscr{B}_{2m})$. Then
(1)$\lim_{k\to\infty}\sqrt[k]{b(2m,k)}=\rho$;
(2)$\lim_{k\to\infty}\frac{b(2m,k)}{b(2m,k-1)}=\rho$;
(3)$\lim_{k\to\infty}\frac{b_l(2m,k)}{b(2m,k)}=m-\rho$;
(4)$\lim_{k\to\infty}\frac{b_r(2m,k)}{b(2m,k)}=\rho-m+1$;
(5)$\lim_{k\to\infty}\frac{b_l(2m,k)}{b_r(2m,k)}=\frac{m-\rho}{\rho-m+1}$.

Let $\mathscr{B}_{2m}^k=\begin{pmatrix} B_{11}^{(k)} & B_{12}^{(k)} \\ B_{21}^{(k)} & B_{22}^{(k)} \end{pmatrix}$,and $b_{ij}(2m,k)=\mathbf{1}_m^tB_{ij}^{(k)}\mathbf{1}_m, i,j=1,2$, where $B_{11}^{(k)},B_{12}^{(k)},B_{21}^{(k)},B_{22}^{(k)}$ are matrices of order m.

Now that $b(2m,k+1)=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^{k+1}\mathbf{1}_{2m}=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^k\binom{(m-1)\mathbf{1}_m^t}{m\mathbf{1}_m^t}$
$=(m-1)(b_{11}(2m,k)+b_{21}(2m,k))+m(b_{12}(2m,k)+b_{22}(2m,k))$,
$b(2m,k+1)=\mathbf{1}_{2m}^t(\mathscr{B}_{2m}^t)^{k+1}\mathbf{1}_{2m}=\mathbf{1}_{2m}^t(\mathscr{B}_{2m}^t)^k\binom{m\mathbf{1}_m^t}{(m-1)\mathbf{1}_m^t}$
$=m(b_{11}(2m,k)+b_{12}(2m,k))+(m-1)(b_{21}(2m,k)+b_{22}(2m,k))$,
we have
$b_{11}(2m,k)=b_{22}(2m,k)$.
$b(2m,k+2)=\mathbf{1}_{2m}^t\mathscr{B}_{2m}^{k+2}\mathbf{1}_{2m}=(m\mathbf{1}_m^t,(m-1)\mathbf{1}_m^t)\mathscr{B}_{2m}^k\binom{(m-1)\mathbf{1}_m^t}{m\mathbf{1}_m^t}$
$=2(m-1)mb_{11}(2m,k)+m^2b_{12}(2m,k)+_(m-1)^2b_{21}(2m,k)$.
Leading to the following result.

Lemma 2.3 Let $k,m \geq 2$ be an integer. Then

(1) $$\begin{pmatrix} b(2m,k) \\ b(2m,k+1) \\ b(2m,k+2) \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 2m-1 & m & m-1 \\ 2(m-1)m & m^2 & (m-1)^2 \end{pmatrix} \begin{pmatrix} b_{11}(2m,k) \\ b_{12}(2m,k) \\ b_{21}(2m,k) \end{pmatrix};$$

(2) $$\begin{pmatrix} b_{11}(2m,k) \\ b_{12}(2m,k) \\ b_{21}(2m,k) \end{pmatrix} = \begin{pmatrix} -(m-1)m & 2m-1 & -1 \\ (m-1)^2 & -2m+2 & 1 \\ m^2 & -2m & 1 \end{pmatrix} \begin{pmatrix} b(2m,k) \\ b(2m,k+1) \\ b(2m,k+2) \end{pmatrix}.$$

Lemma 2.4 ([4]) Let $m \geq 2$ be an integer, $\rho = \rho(\mathscr{B}_{2m})$,and $(v_1, v_2, \ldots, v_m, w_1, w_2, \ldots, w_m)^t$ be a eigenvector of $\mathscr{B}_{2m}$ corresponding to the eigenvalue of $\rho(\mathscr{B}_{2m})$,where $\Sigma_{i=1}^m v_i + \Sigma_{i=1}^m w_i = 1$. Then

(1) $\rho = m - \Sigma_{i=1}^m w_i = m - 1 + \Sigma_{i=1}^m v_i$;

(2)$v_m = \frac{\rho+1-m}{\rho+1}$;

(3) $w_k - v_k = \frac{1-v_k(2\rho+1)}{\rho+1}$ and $v_k = \frac{1}{2\rho+1} - \frac{2(\rho+\frac{1-m}{2})^2+(1-m)(\frac{1+m}{2})}{\rho(2\rho+1)} \cdot (\frac{\rho+1}{\rho})^{2k+1}$,

$k = 1, 2, \cdots, m$.

3. Some properties for spectral radius of Brualdi-Li matrix

Theorem 3.1 Let $m \geq 2$ be an integer,and $\rho = \rho(\mathscr{B}_{2m})$.Then

(1)$\lim_{k\to\infty} \frac{b_{11}(2m,k)}{b(2m,k)} = -\rho^2 + (2m-1)\rho - m(m-1)$;

(2)$\lim_{k\to\infty} \frac{b_{12}(2m,k)}{b(2m,k)} = \rho^2 - 2(m-1)\rho + (m-1)^2$;

(3)$\lim_{k\to\infty} \frac{b_{21}(2m,k)}{b(2m,k)} = \rho^2 - 2m\rho + m^2$.

Proof By Lemma 2.3(1),$b_{11}(2m,k) = -m(m-1)b(2m,k) + (2m-1)b(2m,k+1) - b(2m,k+2)$, then $\frac{b_{11}(2m,k)}{b(2m,k)} = -m(m-1)\frac{b(2m,k)}{b(2m,k)} + (2m-1)\frac{b(2m,k+1)}{b(2m,k)} - \frac{b(2m,k+2)}{b(2m,k)}$. By Lemma 2.2(2),we have

$\lim_{k\to\infty} \frac{b_{11}(2m,k)}{b(2m,k)} = -m(m-1)\lim_{k\to\infty} \frac{b(2m,k)}{b(2m,k)} + (2m-1)\lim_{k\to\infty} \frac{b(2m,k+1)}{b(2m,k)} - \lim_{k\to\infty} \frac{b(2m,k+2)}{b(2m,k)}$

$= -m(m-1) + (2m-1)\rho - \lim_{k\to\infty}(\frac{b(2m,k+2)}{b(2m,k+1)} \cdot \frac{b(2m,k+1)}{b(2m,k)})$

$= -\rho^2 + (2m-1)\rho - m(m-1)$.

Using a similar approach,we have obtain (2)and (3).

Theorem 3.2 Let $m \geq 2$ be an integer,and $\rho = \rho(\mathscr{B}_{2m})$.Then

(1)$\lim_{k\to\infty} \frac{b_l(2m,k)}{b_l(2m,k-1)} = \lim_{k\to\infty} \frac{b_r(2m,k)}{b_r(2m,k-1)} = \rho$;

(2)$\lim_{k\to\infty}\frac{b_{ij}(2m,k)}{b_{ij}(2m,k-1)}=\rho$, for $i,j=1,2$.

Proof By Lemma 2.3(2),(3),
$\lim_{k\to\infty}\frac{b_l(2m,k)}{b_l(2m,k-1)}=\lim_{k\to\infty}(\frac{b_l(2m,k)}{b(2m,k)}\cdot(\frac{b_l(2m,k-1)}{b(2m,k-1)})^{-1}\cdot\frac{b(2m,k)}{b(2m,k-1)})$
$=(m-\rho)(m-\rho)^{-1}\rho$
$=\rho.$
Using a similar approach,we have obtain $\lim_{k\to\infty}\frac{b_r(2m,k)}{b_r(2m,k-1)}=\rho$ and (2).

Theorem 3.3 Let $m\geq 2$ be an integer, $\rho=\rho(\mathscr{B}_{2m})$,and $(v_1,v_2,\ldots,v_m,w_1,w_2,\ldots,w_m)^t$ be a eigenvector of $\mathscr{B}_{2m}$ corresponding to the eigenvalue of $\rho(\mathscr{B}_{2m})$,where $\Sigma_{i=1}^m v_i+\Sigma_{i=1}^m w_i=1$. Then

$$\Sigma_{i=1}^m i(w_i-v_i)=(m-\rho)^2.$$

Proof By Lemma 2.4,
$\Sigma_{i=1}^m i(w_i-v_i)$
$=\Sigma_{i=1}^m i(\frac{1-v_i(2\rho+1)}{\rho+1})$
$=\Sigma_{i=1}^m\frac{i}{\rho+1}-\frac{2\rho+1}{\rho+1}\Sigma_{i=1}^m iv_i$
$=\Sigma_{i=1}^m\frac{i}{\rho+1}-\frac{2\rho+1}{\rho+1}\Sigma_{i=1}^m i(\frac{1}{2\rho+1}-(\frac{2(\rho+\frac{1-m}{2})^2+\frac{(1-m)(1+m)}{2}}{\rho(2\rho+1)})(\frac{\rho+1)}{\rho})^{2i+1})$
$=\frac{(2\rho^2+2(1-m)\rho+1-m)}{\rho(\rho+1)}\sum_{i=1}^m i(\frac{\rho+1}{\rho})^{2i-1}.$

Now that $\sum_{k=1}^m kx^{k-1}=\frac{mx^{m+1}+1-(m+1)x^m}{(x-1)^2}$,thence
$\sum_{i=1}^m i(\frac{\rho+1}{\rho})^{2i-1}=\frac{m(1+\frac{1}{\rho})^{2m+3}-(m+1)(1+\frac{1}{\rho})^{2m+1}+1+\frac{1}{\rho})}{((1+\frac{1}{\rho})^2-1)^2}$
$=\frac{\rho^4}{(2\rho+1)^2}(m(1+\frac{1}{\rho})^{2m+3}-(m+1)(1+\frac{1}{\rho})^{2m+1}+1+\frac{1}{\rho}).$
We have

$\Sigma_{i=1}^m i(w_i-v_i)$
$=\frac{(2\rho^2+2(1-m)\rho+1-m)}{\rho(\rho+1)}(\frac{\rho^4}{(2\rho+1)^2})(m(1+\frac{1}{\rho})^{2m+3}-(m+1)(1+\frac{1}{\rho})^{2m+1}+1+\frac{1}{\rho})$
$=\frac{\rho^3}{2\rho+1}(\frac{(2\rho^2+2(1-m)\rho+1-m)}{\rho(2\rho+1)})(m(1+\frac{1}{\rho})^{2m+2}-(m+1)(1+\frac{1}{\rho})^{2m}+1)$
$=\frac{\rho^3}{2\rho+1}(\frac{(2\rho^2+2(1-m)\rho+1-m)}{\rho(2\rho+1)})(1+\frac{1}{\rho})^{2m-1}(m(1+\frac{1}{\rho})^3-(m+1)(1+\frac{1}{\rho}))+\frac{\rho^2(2\rho^2+2(1-m)\rho+1-m)}{(2\rho+1)^2}$
$=\frac{\rho^3}{(2\rho+1)}(\frac{1}{2\rho+1}-\frac{\rho+1-m}{\rho+1})(m(1+\frac{1}{\rho})^3-(m+1)(1+\frac{1}{\rho}))+\frac{\rho^2(2\rho^2+2(1-m)\rho+1-m)}{(2\rho+1)^2}$
$=\frac{1}{(2\rho+1)^2}(m+2m\rho-\rho^2)(m+2m\rho-2\rho^2-2\rho)+\frac{\rho^2(2\rho^2+2(1-m)\rho+1-m)}{(2\rho+1)^2}$

$$= \frac{1}{(2\rho+1)^2}(4\rho^4 - 4(2m-1)\rho^3 + (4m^2-8m+1)\rho^2 + 2m(2m-1)\rho + m^2)$$
$$= \frac{1}{(2\rho+1)^2}(4\rho^4 + 4\rho^3 + (4m^2+1)\rho^2 + 4m^2\rho + m^2 - (8m\rho^3 + 8m\rho^2 + 2m\rho))$$
$$= \frac{1}{(2\rho+1)^2}((4\rho^2+4\rho+1)\rho^2 + 4m^2\rho^2 + 4m^2\rho + m^2 - (4\rho^2+4\rho+1)2m\rho)$$
$$= (m-\rho)^2.$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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