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ON DYNAMICS OF CHAOTIC EVOLUTION IN FOOD CHAIN MODEL

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Abstract: In this paper, A discrete-time food chain model of three species has been introduced comprising of a set of three nonlinear difference equations. Evolutionary dynamics of this system have investigated and regular as well as chaotic attractors have been obtained for certain parameter values. Bifurcation diagrams have been drawn by varying parameters along coordinate axes and the motion has been analyzed. Numerical calculations have been carried out for calculations of Lyapunov exponents, topological entropies and correlation dimension. The investigation is then further extended to obtain FLI, SALI and DLI for regular and chaotic evolution of the food chain model.

Keywords: Food chain model, Chaotic attractor, Topological entropy, Correlation dimension, Lyapunov exponents.

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1. Introduction

Studies on food chains and webs in various ecosystem environment show growing

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interest in areas of applied mathematics related to biology. Food chain modeling provides challenges in the fields of both theoretical ecology and applied mathematics. As the realistic models are nonlinear, their evolutionary behavior may show regular as well as chaotic fluctuations depending upon the variations of parameters of the system. Observational facts reveal that the ecological food webs or food chains typically, contain several layers such that the consumers which eat from the bottom resource layer are the prey of another predator. Thus, the prey-predator models mentioned above can easily be extended with a "top predator" that lives on the predator population. Taking into account the interacting behavior of the species, models of food chains and webs can be represented through differential equations or by difference equations as appearing in recent literature.

Articles on dynamics of prey-predator systems under different ecological conditions have been investigated by many researchers in the past, e.g. Holling (1965), May (1974), Freedman and Waltman (1977), Freedman and So (1985), Hastings and Powell (1991), Klebanoff and Hastings (1994), McCann and Yodzis (1994), Deng (2001), El-Owaidy et al.(2001); Letellier and Aziz-Alaoui (2002) and many others. These studies established very interesting results also.

Some recent articles suggest that models of food chain written in difference equations are more appropriate than continuous equation when the interactions of species are non-overlapping generations, Ivanchikov and Nedorezov (2011, 2012), Liu and Xu, (2003). A recent article on three species food chain with discrete equations was introduced by Elsadany (2012) where existence of local steady states and their stability were widely discussed. Bifurcation diagrams have been obtained by varying certain parameter and appearance of Hopf bifurcation under certain conditions has also been shown. Chaotic attractors and calculation of Lyapunov exponents are also done in this article.

The objective of the present work to conduct more extensive study of the long term evolutionary dynamics of the discrete food chain model proposed by Elsadany (2012) for three distinct insects and introduce certain measure of chaos, (e.g. calculation of Lyapunov exponents, topological entropies and correlation dimensions), when the system evolve chaotically. Also, as part of further extension, we wish to apply the indicators FLI, SALI and DLI our model.

2. Preliminaries

Descriptions of Chaos Measuring Tools: Lyapunov exponents, Topological Entropies, Correlation Dimensions:

(i) Lyapunov Exponents (Lyapunov Numbers):

Lyapunov exponents are dynamical measure of exponential divergence in a deterministic system of orbits which started with slightly different initial conditions and are capable to characterize deterministic chaos. Chaos in the system features to the highly sensitive dependence on initial conditions. It is an effective tool for identification of regularity and chaos in the system and can be explained in the following ways:

Lyapunov Numbers and Lyapunov Exponents:

Consider any one dimensional map defined in some interval (a, b)

$$x_{n+1} = f(x_n) \qquad (2.1)$$

and its two orbits starting at x_0 and $x_0 \pm \delta_0$, where δ_0 is very small. Expanding $f(x_0 + \delta_0)$ by taylors series, the distance between the orbits after one iteration be given by

$$\delta_1 = | f'(x_0) | / \delta_0 = M0\, \delta_0 . \qquad (2.2)$$

M_o is known as first step magnification factor. Similarly, at the second iteration, the distance between the orbits can be written as

$$\delta_2 = | f'(x_1) |/ \delta 1 = M1\, \delta_1 = M_1 M_0 \delta_0 \qquad (2.3)$$

Continuing in this manner, separation between the orbits at n^{th} iteration is

$$\delta_n = | f'(x_{n-1}) | / \quad \delta_{n-1} \quad = \quad M_{n-1} \delta_{n-1} = M_{n-1} M_{n-2} \ldots M_0 \delta_0 \qquad (2.4)$$

The product $M_0\ M_1\ M_2\ \ldots..M_{n-1}$ is the accumulation of magnification factors. It is meaningful to consider an average of it and the most convenient is the geometric average

$$(M_0\, M_1\, M_2 \ldots\ldots M_{n-1})^{1/n}$$

Taking log, one obtains the arithmetic average

$$\lambda = \ln (M_0\, M_1\, M_2 \ldots\ldots M_{n-1})^{1/n}$$

$$= 1/n\, (\ln M_0 + \ln M_1 + \ln M_2 + \ldots + \ln M_{n-1})$$

$$= 1/n\, (\ln (|\, f'(x_0)\,|) + \ln (|\, f'(x_1)\,|) + \ln (|\, f'(x_2)\,|) + \ldots + \ln (|\, f'(x_{n-1})\,|)) \qquad (2.5)$$

Then, the condition of stability of a implies:

If average magnification is less than 1, the orbit is stable and if it is greater than 1 the orbit is unstable, i.e. $\lambda < 0$ => stable orbit and $\lambda > 0$ => unstable orbit. For accurate result, one should take the iterations n as large as possible. This leads to the following definition of Lyapunov exponents:

Def. 1: **Lyapunov exponents** of a smooth map f on R with x_0 as initial point be defined as

$$\lambda(x_0) = \lim_{n \to \infty} 1/n\, [\ln (|\, f'(x_0)\,|) + \ln (|\, f'(x_1)\,|) + \ln (|\, f'(x_2)\,|) + \ldots + \ln (|\, f'(x_{n-1})\,|)]$$

provided the limit exists. Lyapunov number is the exponent of Lyapunov exponent and is given by

$$L(x_0) = e^{\lambda(x_0)} \qquad (2.6)$$

Def. 2: A bounded orbit $\{x_0, x_1, x_2, x_3, \ldots, x_n\}$ of the map f on R is called chaotic if following conditions are satisfied:

(a) $\{x_0, x_1, x_2, x_3, \ldots, x_n\}$ is not asymptotically periodic.

(b) No $\lambda(x_0)$ is exactly equal to zero.

(c) $\lambda(x_0) > 0$ or equivalently, $L(x_0) > 1$.

From above definition, a clear interpretation for Lyapunov exponent is given as: it is the measure of loss of information during the process of iteration.

For higher dimensional system, we can generalize the above one dimensional case to higher dimension and obtain

$$\lambda(X_0, U_0) = \lim_{n \to \infty} \frac{1}{n} \log \left\| \coprod_{t=0}^{n-1} J(X_t) U_0 \right\|, \quad (2.7)$$

and

$$\left\| X_n - Y_n \right\| \approx e^{\lambda(X_0, U_0) n},$$

where $X \in \measuredangle^n$, F: $\measuredangle^n \to \measuredangle^n$, $U_0 = X_0 - Y_0$ and J is the Jacobian matrix of map F.

Quantitatively, two trajectories in phase space with initial separation δx_0 diverge if

$$|\delta x(t)| \approx e^{\lambda t} |\delta x(0)| \quad (2.8)$$

where $\lambda > 0$ is the Lyapunov exponent. The system described by the map f be *regular* as long as $\lambda \leq 0$ and *chaotic* when $\lambda > 0$.

(ii) Topological Entropy

The usefulness of Lyapunov exponents are limited because of the following important features, Gribble (1995):

- Lyapunov exponents are local in nature and are not necessarily constant throughout the evolution and so ergodicity is also required to characterize chaos.
- As per their definitions, Lyapunov exponents are time dependent and this leads to a serious drawback for systems arising from relativistic considerations.

A chaotic attractor is composed of a complex pattern. To investigate chaotic behavior in a wide variety of systems evolving time, an alternate replacement of Lyapunov exponents which could be more reliable and acceptable as indicator is the topological entropy, Balmforth et al (1964), Adler et al (1965), Bowen (1970), Boyarsky et al (1991) and Iwai (1998). Topological entropy describes the *rate of mixing* of a dynamical system. It has a relationship to both Lyapunov exponents, through the dependence of rate, and to the ergodicity, because of the association of mixing. For a system having non-zero topological entropy, the rate of mixing must be exponential which reminiscent of of Lyapunov exponents. But such exponentiality of mixing is not relative to time, but rather to the number of discrete steps through which the system has evolved. Positivity of Lyapunov exponent and topological entropy are characteristic of chaos. A mathematical definition of topological entropy can be obtained from the book by Nagashima and Baba (2005).

Topological entropy h(f) for a map f defined in a close interval **I** = [a, b], is closely related to Li and Yorke chaos, Nagashima and Baba (2005), and measures the complexity of the map f.

If f be a continuous map from **I** to **I** and if α be an open initial cover of **I**, then the topological entropy h(f) can be described by the supremum, sup h(α,f), for all the covers of interval **I** such that

$$h(\alpha, f) = \lim_{n\to\infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} f^{-1} \alpha \right), \qquad (2.9)$$

then

$$h(f) = \sup h(\alpha, f). \qquad (2.10)$$

When the map f is piecewise-monotonic over I, the topological entropy can be determined by the lap number, $lap(f^n)$ of the iterated map f^n as follows :

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log lap(f^n). \qquad (2.11)$$

The lap number of f grows with n in general. If the growth obeys the power law,

$$lap(f^n) \sim k\, n^{\alpha},$$

then by (2.10),

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log(kn^{\alpha}) = \lim_{n\to\infty} \frac{\alpha}{n} \log n = 0. \qquad (2.12)$$

However, if it grows exponentially, $lap(f^n) \sim k\,\alpha^n$, $(\alpha > 1)$, then

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log(k\alpha^n) = \log \alpha. \qquad (2.13)$$

This shows that h(f) is determined by the way $lap(f^n)$ increases.

In case of super stable periodic orbits, the method of structure matrix **M** can be employed. (e.g. Nagashima and Baba, page 131, 2005). If λ_{max} be the largest eigenvalue of **M**, then the topological entropy can be obtained as

$$h = \log(\lambda_{max}) \tag{2.17}$$

(iii) Correlation Dimension

As stated, chaos may exist in nonlinear systems during evolution and that can be seen easily by observing the bifurcation diagrams. A chaotic set, an strange attractor, has fractal structure. Correlation dimension gives a *measure of dimensionality* of the chaotic set. Being one of the characteristic invariants of nonlinear system dynamics, the correlation dimension actually gives a measure of complexity for the underlying attractor of the system. To determine correlation dimension we use statistical method. It is a very practical and efficient method than other methods, like box counting etc. The procedure to obtain correlation dimension follows the following steps,Martelli (1999):Consider an orbit $O(\mathbf{x}_1) = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \ldots\}$, of a map f: U → U, where U is an open bounded set in $\measuredangle^n$. To compute correlation dimension of $O(x_1)$, for a given positive real number r, we form the correlation integral, Grassberger and Procaccia (1983),

$$C(r) = \lim_{n\to\infty} \frac{1}{n(n-1)} \sum_{i\neq j}^{n} H\left(r - \left\| x_i - x_j \right\|\right), \tag{2.18}$$

where

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases},$$

is the unit-step function, (Heaviside function). The summation indicates the the number of pairs of vectors closer to r when $1 \leq i, j \leq n$ and $i \neq j$. C(r) measures the density of pair of distinct vectors $\mathbf{x_i}$ and $\mathbf{x_j}$ that are closer to r.

The correlation dimension D_c of $O(\mathbf{x_1})$ is defined as

$$D_c = \lim_{r\to 0} \frac{\log C(r)}{\log r} \tag{2.19}$$

To obtain D_c, log C(r) is plotted against log r and then we find a straight line fitted to this curve. The y- intercept of this straight line provides the value of the correlation dimension D_c.

(iv) Chaos Indicators FLI, SALI and DLI:

For the description of indicators FLI, SALI and DLI, we refer the article by Saha and Budhraja (2007), Yuasa and Saha (2008), Saha and Tehri (2010). However, we must keep in mind the properties of these indicators as follows:

FLI'*s increase exponentially for chaotic orbits and linearly for regular orbits*.

SALI's *fluctuates around a non-zero value for ordered orbits while it tends to zero for chaotic orbits*.

DLI's , *form a definite pattern, then the motion is regular and if they are distributed randomly, (with no definite pattern), then the motion is chaotic*.

3. Main results

1. **A Three Dimension Food Chain Model**:

A discrete 3-dimensional food-chain model proposed by Elsadany, (2012), to study the ecosystems three interacting species, each with non-overlapping generations. The food chain is considered to describe the insect group of three fully different insects. These insects are a lowest-level prey x is preyed upon by a mid-level species y, which in turn, is preyed by a top-level predator z. The model be given by following non-dimensional form of difference equations

$$
\begin{aligned}
x_{n+1} &= a\, x_n (1-x_n) - b\, x_n y_n \\
y_{n+1} &= c\, x_n y_n - d\, y_n z_n \\
z_{n+1} &= r\, y_n z_n \qquad (3.1)
\end{aligned}
$$

The model contains five parameters a, b, c, d, r all are assumed to be positive. Each of these has significant interpretation as given in a recent article by Elsadany (2012). The stability of steady state solutions have also been explained there.

The system (3.1) has fixed points given by p_0^*, p_1^*, p_2^*, p_3^* and p_4^* with respective coordinates

$$(0,\ 0,\ 0),\ (\frac{a-1}{a},\ 0,\ 0),\ (\frac{1}{c},\ \frac{ac-a-c}{bc},\ 0),\ (0,\ \frac{1}{r},\ -\frac{1}{d})\ \text{and}\ (\frac{r(a-1)-b}{ar},\ \frac{1}{r},\ \frac{cr(a-1)-bc-ar}{adr}).$$

Also, the Jacobian matrix of system (3.1) is given by

$$J = \begin{pmatrix} a-2ax-by & -bx & 0 \\ cy & cx-dz & -dy \\ 0 & rz & ry \end{pmatrix}$$

Eigenvalues of matrix J obtained with respect to above fixed point determine their stability criteria. One can easily verify, as $a > 0$,the origin p_0^*: (0, 0, 0) is stable for $0 < a < 1$ and unstable for $a > 1$. If we fix $c = 3$, (as we have done in one case during our numerical calculations), the eigenvalues corresponding to p_1^* are given by $e_{11} = 0$, $e_{12} = 3(a - 1)/a$ and $e_{13} = 2 - a$. So, p_1^* is stable when $1 < a < \frac{3}{2}$. If we fix a, $a = 2.5$, (as $a < 3$) then for $c < \frac{3}{2}$, p_1^* is stable. Fixing $b= 3.7$, $c = 3.0$ and $r=3.8$, we obtain the eigenvalues corresponding to p_2^* , i.e. e_{21} , e_{22}, e_{23} and the fixed point p_2^* is stable or unstable according as the absolute values of these eigenvalues be less than or greater than unity. Followed by the work of Elsadany (2012), one may extend the stability analysis of $p3^*$ and p_4^* also.

When any one of parameters vary keeping others fixed, one observes bifurcations scenario of the model. Through such bifurcations regular and chaotic evolutions are clearly visible. For, first we have fixed $b = 3.7$, $c = 3$, $d = 3.5$, $r = 3.8$ and let a to vary from $a = 2.1$ to $a = 4.3$ and obtained bifurcation for six different ranges of values of a shown in Fig. 1. Note that, throughout our calculations, we have fixed three parameters as $b = 3.7$, $d = 3.5$ and $r = 3.8$ and in turn first let a to vary by fixing c and the let c to vary by fixing a. The bifurcations in thr later case are shown in Fig. 2 The process of bifurcation, in both the cases, produces some interesting features of regular and chaotic evolutions.

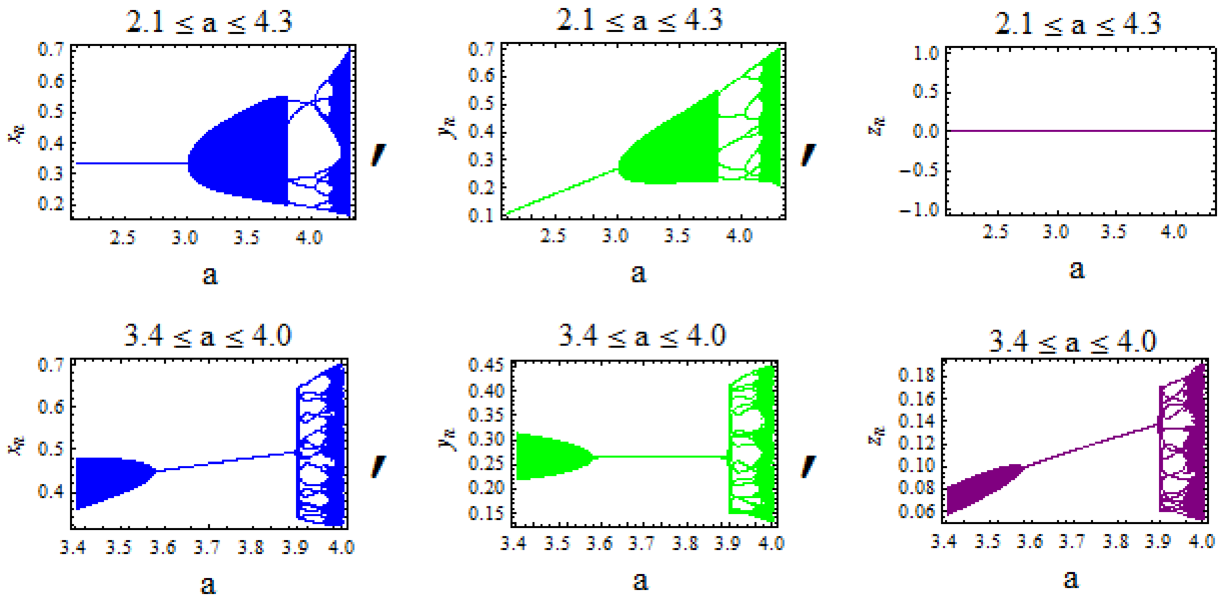

Fig. 1: Bifurcation of food chain model along the coordinate axes for two ranges of values of a when other parameters are fixed with b = 3.7, c = 3, d = 3.5 and r = 3.8.

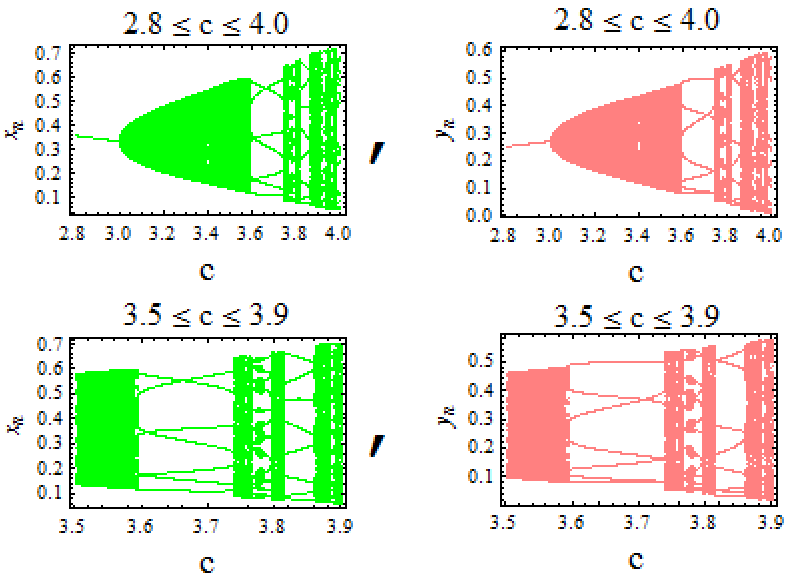

Fig.2: Bifurcation of food chain model along the coordinate axes for two ranges of values of c when other parameters are fixed with a = 3, b = 3.7, d = 3.5 and r = 3.8.

Attractors for regular and chaotic motions are drawn for map (1) and shown below in Fig. 3.

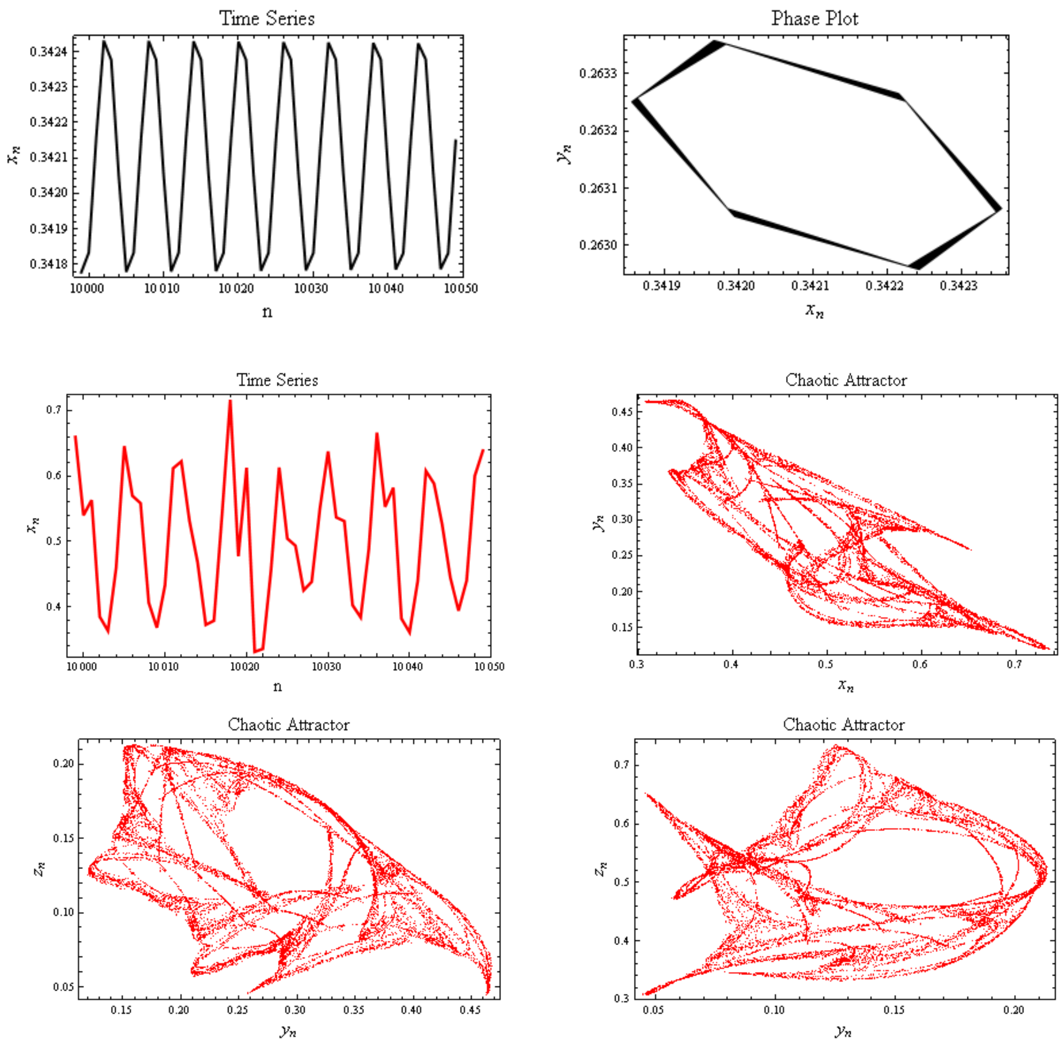

Fig. 3: Time series plots and attractors of food chain model (3.1); for regular case upper row figures are drawn for a = 3, b = 3.7, c = 3, d = 3.5, r = 3.8 and for lower row a = 4.1.

For fixed parameters b = 3.7, c = 4, d = 3.5, r = 3.8 when a is allowed to vary, one moves from regularity to chaos. This can be observed from the phase plots in x-y plane shown in Fig. 4.

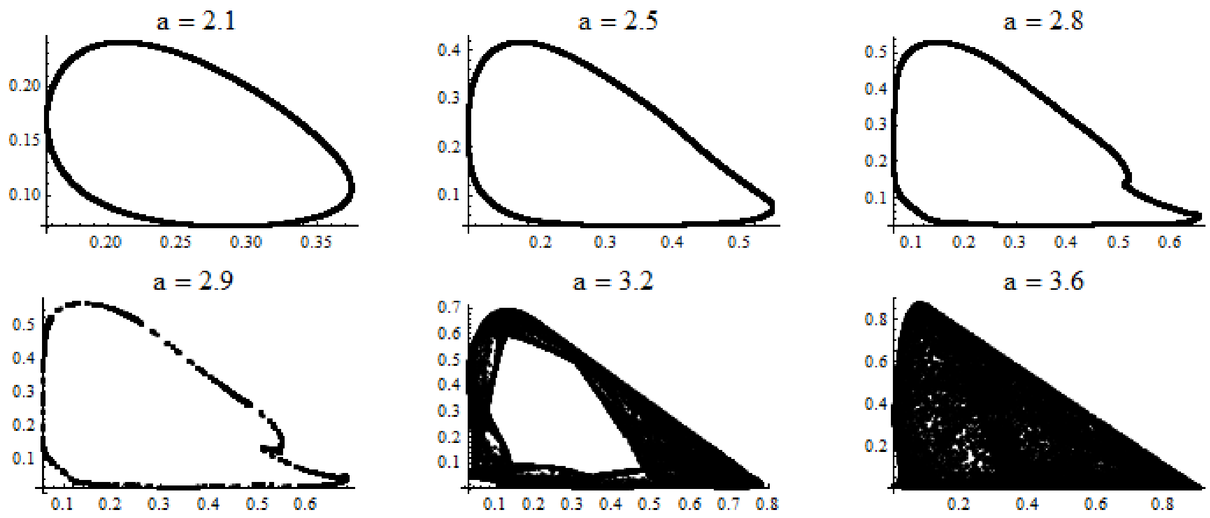

Fig. 4: Regular and chaotic attractors of food chain model with changing values of a when c = 4 and b = 3.7, d = 3.5, r = 3.8.

The last chaotic attractor in Fig. 4, with magnification looks like figure below, Fig. 5.

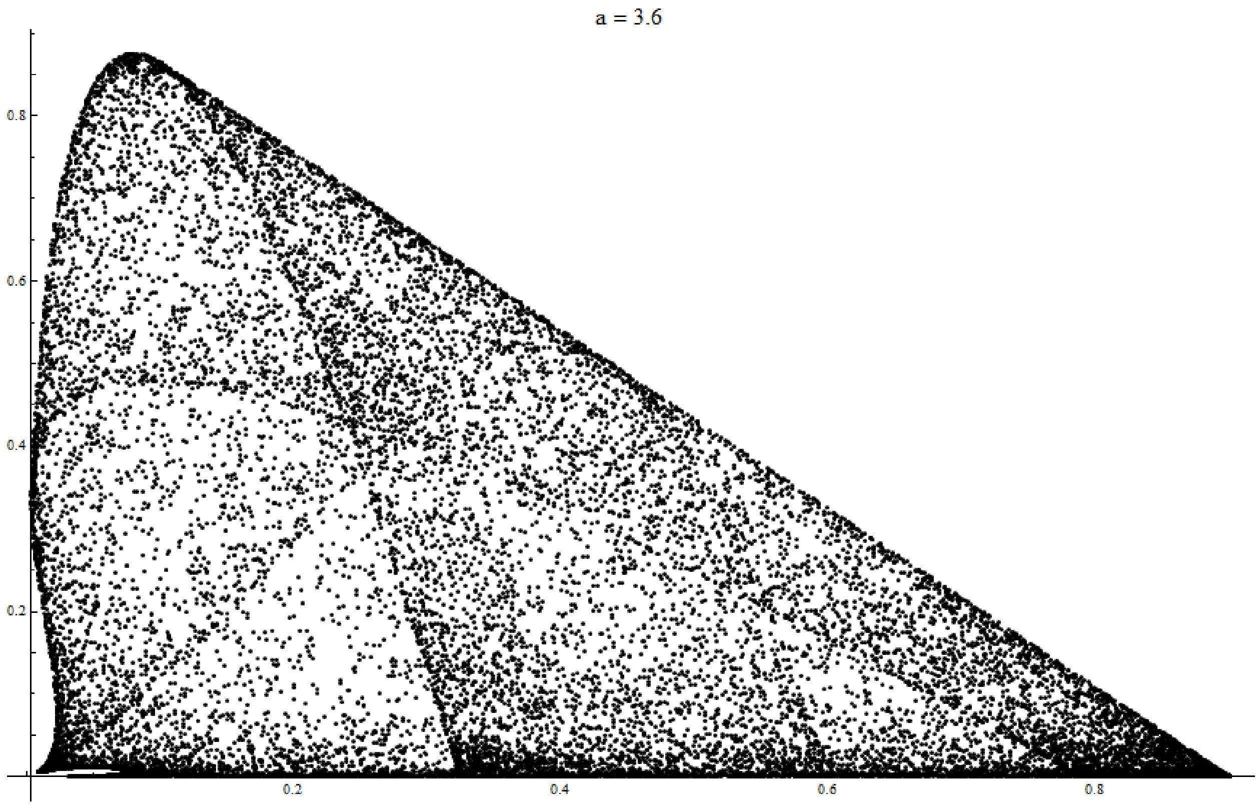

Fig. 5 : A chaotic attractor for parameter a = 3.6

5. Numerical Calculations of Lyapunov Exponents,Topological Entropies and Correlation Dimensions:

We have performed numerical simulations to obtain various results of evolution of the map (3.1). Mathematica codes have been written in appropriate form to perform calculations. First we have calculated the Lyapunov numbers and Lyapunov exponents (LCEs) and plotted their graphs as shown in Fig. 6.

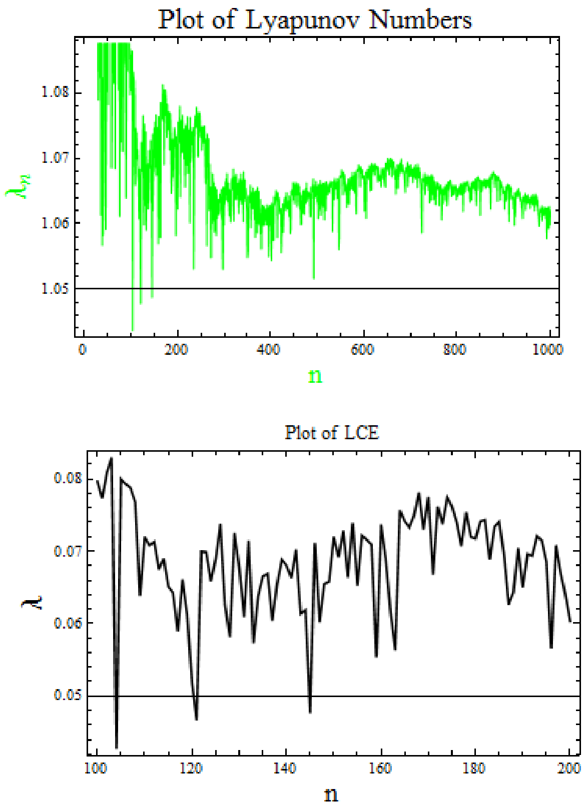

Fig. 6 : Plots of Lyapunov numbers and Lyapunov exponents of the food chain model for parameters a =4.1, b = 3.7, c = 3.0, d = 3.5 and r = 3.8 and with the initial condition (x_0, y_0, z_0) = (0.2, 0.1,0.1).

For the same condition and parameter values, for the long term evolution of the system, the evolution becomes totally chaotic as shown through the Lyapunov exponents plot in Fig.7.

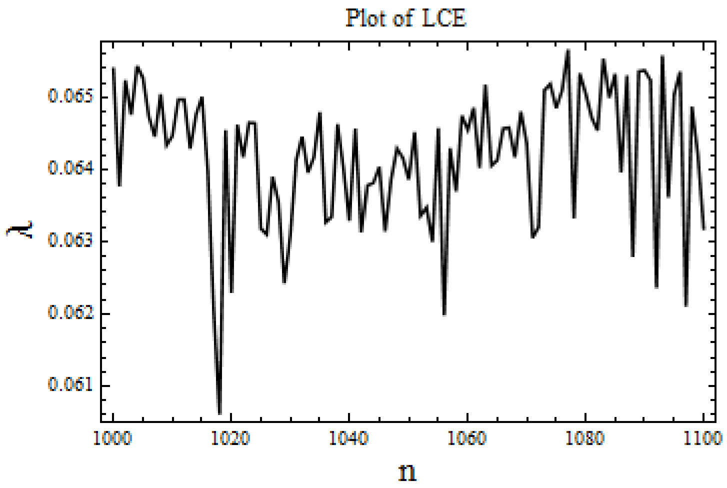

Fig. 7. Plot of Lyapunov exponents in the long term evolution.

Topological entropies for the food chain map have been calculated and plotted for different ranges of parameters a and c and their plots are shown in Fig. 9.

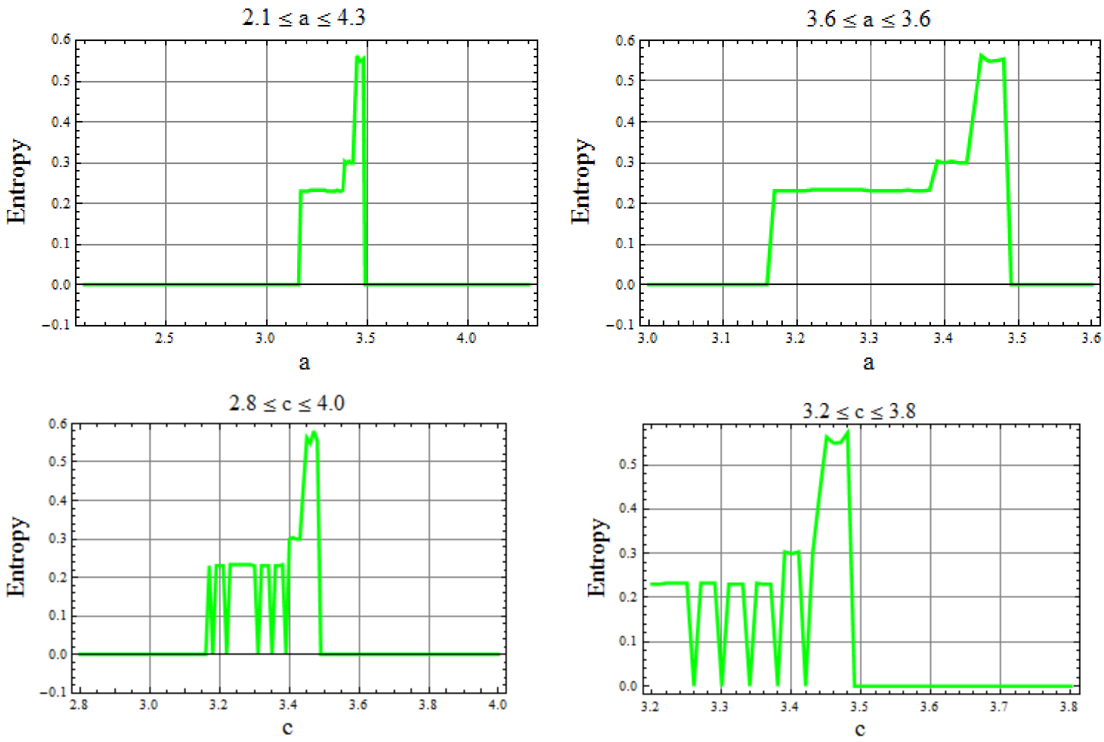

Fig. 8: Plots of topological entropies; upper row for fixed parameters b = 3.7, c = 4, d = 3.5, r = 3.8 and lower row for fixed parameters a = 3, b = 3.7, d = 3.5, r = 3.8.

Using the data obtained by evaluating the correlation integral, we have obtained the correlation curve for the chaotic evolution as shown in Fig. 9.

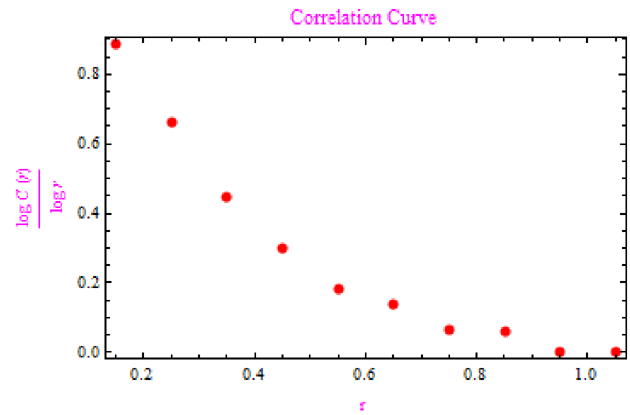

Fig. 9: Plot of correlation curve. Parameter values and initial conditions are same as in the case of Lyapunov exponents.

Using least square linear fit to the data of correlation integral we obtain the equation of the straight line fitting the data as

$$y = 0.832272 - 0.928877x \tag{4.1}$$

whose y intercept is 0.832272 ≈ 0.83. So, the correlation dimension of the chaotic attractor is approximately 0.83.

Plots of FLI, SALI, DLI for regular as well as chaotic cases for the above map are shown in Fig.10.

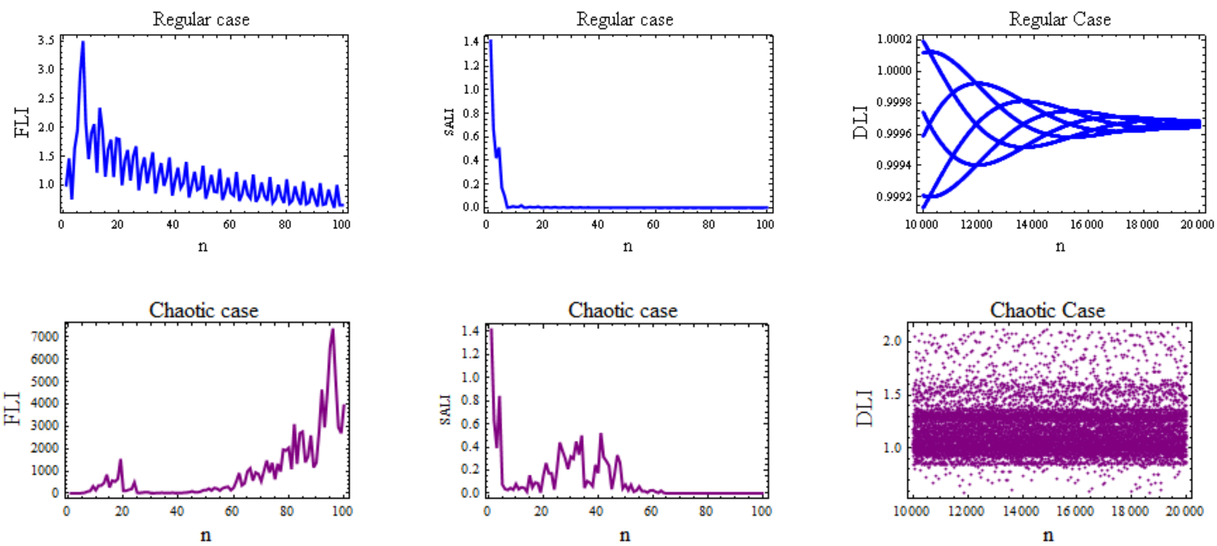

Fig.10: Plots of FLI, SALI and DLI for food chain model (3.1) for fixed parameters b = 3.7, c = 3, d = 3, r = 3.8 ; the upper row is for regular case when a = 3.0, and the lower row is for chaotic case when a = 4.1.

Discussion

We have studied the nonlinear behavior of a food chain system (3.1) together with certain measure for chaotic evolution. Bifurcation diagrams of this food chain model have been drawn by varying both of the parameters a and c while fixing other parameters in Fig. 1 and Fig. 2. These figures provide information regarding evolution with stable solutions as well as chaotic nature of nonlinear properties and limitation for parameter space. Regular and chaotic time series graphs and attractors have been drawn in Fig. 3 – Fig. 5. In order to impose certain measure of chaos, Numerical calculations have been carried out to obtain certain measure of chaos by calculating Lyapunov exponents and topological entropies and shown through Fig. 6 – Fig. 8. To measure the dimensionality of the chaotic attractor, numerical simulations have been extended to evaluate an correlation integral and collect appropriate correlation data. Plot of correlation data resulting in correlation curve is shown in Fig.9. Then, by using the method of least square linear fit, we have obtained the equation of the straight line approximately fitting the data given by equation (4.1) and its y-intercept provides the correlation dimension shown there. Finally, we have applied the recently introduced indicators, FLI, SALI and DLI to the food chain model for distinguishing regular and chaotic motion. Results obtained are shown in Fig. 10. It has been observed that the first two indicators do not perfectly working according to their definitions. However, the last indicator, DLI was seen to work perfectly for this model.

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