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CR-SUBMANIFOLDS OF A NEARLY HYPERBOLIC KENMOTSU MANIFOLD ADMITTING A QUARTER SYMMETRIC NON-METRIC CONNECTION

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Abstract. We consider a nearly hyperbolic Kenmotsu manifold with a quarter symmetric non-metric connection and study CR- submanifolds of a nearly hyperbolic Kenmotsu manifold with quarter symmetric non-metric connection. We also study parallel distributions on nearly hyperbolic Kenmotsu manifold with quarter symmetric non-metric connection and find the integrability conditions of some distributions on nearly hyperbolic Kenmotsu manifold with quarter symmetric non-metric connection.

Keywords: CR-submanifolds, nearly hyperbolic Kenmotsu manifold, quarter symmetric non-metric connection, integrability conditions, parallel distribution.

2000 AMS Subject Classification: 53D05, 53D25, 53D12

1. Introduction

The notion of CR-submanifolds of a Kaehler manifold as generalization of invariant and anti-invariant submanifolds was introduced and studied by A. Bejancu in ([1], [2]). Since then, several papers on Kaehler manifolds were published. CR-submanifolds of Sasakian manifold was studied by C.J. Hsu in [5] and M. Kobayashi in [18]. CR-submanifolds of Kenmotsu manifold was studied by A. Bejancu and N. Papaghuic in [4]. Later, several geometers (see, [9], [12] [13], [15] [16]) enriched the study of CR-

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submanifolds of almost contact manifolds. The almost hyperbolic (f, g, η, ξ)-structure was defined and studied by Upadhyay and Dube in [17]. Dube and Bhatt studied CR-submanifolds of trans-hyperbolic Sasakian manifold in [10]. On the other hand, S. Golab introduced the idea of semi-symmetric and quarter symmetric connections in [8]. CR-submanifolds of LP-Sasakian manifold with quarter symmetric non-metric connection were studied by the first author and S.K. Lovejoy Das in [11]. CR-submanifolds of a nearly hyperbolic Sasakian manifold admitting a semi-symmetric semi-metric connection were studied by the first author, M.D. Siddiqi and S. Rizvi in [14]. In this paper, we study some properties of CR-submanifolds of a nearly hyperbolic Kenmotsu manifold with a quarter symmetric non-metric connection.

2. **Preliminaries**

Let $\bar{M}$ be an n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure-(ϕ, ξ, η, g), where a tensor ϕ of type $(1,1)$, a vector field ξ, called structure vector field and η, the dual 1-form of ξ satisfying the followings

(2.1) $\phi^2 X = X + \eta(X)\boldsymbol{\xi}, \qquad g(X, \xi) = \eta(X),$

(2.2) $\eta(\xi) = -1, \quad \phi(\xi) = 0, \quad \eta o \phi = 0,$

(2.3) $g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y)$

for any X, Y tangent to M [17]. In this case

(2.4) $g(\phi X, Y) = -g(\phi Y, X).$

An almost hyperbolic contact metric structure-(ϕ, ξ, η, g) on $\bar{M}$ is called hyperbolic Kenmotsu manifold [7] if and only if

(2.5) $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$

for all X, Y tangent to $\bar{M}$. On a hyperbolic Kenmotsu manifold $\bar{M}$, we have

(2.6) $\nabla_X \xi = X + \eta(X)\xi$

for a Riemannian metric g and Riemannian Connection ∇.

Further, an almost hyperbolic contact metric manifold $\bar{M}$ on (ϕ, ξ, η, g) is called nearly-hyperbolic Kenmotsu [7] if

(2.7) $(\nabla_X \phi)Y + (\nabla_Y \phi)X = -\eta(X)\phi Y - \eta(Y)\phi X.$

Now, Let M be a submanifold immersed in $\bar{M}$. The Riemannian metric induced on M is denoted by the same symbol g. Let $TM \; and \; T^{\perp}M$ be the Lie algebra of vector fields

tangential to M and normal to M respectively and ∇^* be the induced Levi-Civita connection on M, then the Gauss and Weingarten formulas are given respectively by

(2.8) $\nabla_X Y = \nabla_X^* Y + h(X, Y),$

(2.9) $\nabla_X N = -A_N X + \nabla_X^{\perp} N$

for any $X, Y \in TM \; and \; N \in T^{\perp}M$, where $\nabla^{\perp}$ is a connection on the normal bundle $T^{\perp}M, h$ is the second fundamental form and A_N is the Weingarten map associated with N as

(2.10) $g(A_N X, Y) = g(h(X, Y), N)$

for any $x \in M \; and \; X \in T_x M$. We write

(2.11) $X = PX + QX,$

where $PX \in D \; and \; QX \in D^{\perp}$.

Similarly, for N normal to M, we have

(2.12) $\phi N = BN + CN,$

where BN $(resp. CN)$ is the tangential component $(resp. normal \; component)$ of ϕN.

Now, we define a quarter symmetric non-metric connection by

(2.13) $(\bar{\nabla}_X Y) = \nabla_X Y + \eta(Y)\phi X$

such that

$$(\bar{\nabla}_X g)(Y, Z) = -\eta(Y) g(\phi X, Z) - \eta(Z) g(\phi X, Y).$$

Using (2.13) and (2.7), we have

(2.14) $(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = -\eta(X)\phi Y - \eta(Y)\phi X - \eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\boldsymbol{\xi}.$

An almost hyperbolic contact manifold $\bar{M}$ satisfying (2.14) is called nearly hyperbolic Kenmotsu manifold with quarter symmetric non-metric connection.

For a nearly hyperbolic Kenmotsu manifold with quarter symmetric non-metric connection, we have

(2.15) $\overline{\nabla}_X \boldsymbol{\xi} = -\boldsymbol{\alpha}(\phi X) + \beta(X - \eta(X)\xi) + \phi X.$

Gauss and Weingarten formula for nearly hyperbolic Kenmotsu manifold with quarter symmetric non-metric connection are given respectively by

(2.16) $\overline{\nabla}_X Y = \nabla_X Y + h(X, Y),$

(2.17) $\overline{\nabla}_X N = -A_N X + \nabla_X^{\perp} N.$

Definition 2.1. An m-dimensional submanifold M of an n-dimensional nearly hyperbolic Kenmotsu manifold $\bar{M}$ is called a CR-submanifold if there exists a differentiable distribution $D: x \longrightarrow D_x$ on M satisfying the following conditions:

i. D is invariant, that is $\phi D_x \subset D_x \; for\; each\; x \in M.$
ii. The complementary orthogonal distribution $D^{\perp}$of D is anti-invariant, that is$\emptyset D_x^{\perp} \subset T_x^{\perp} M.$

If $\dim D_x^{\perp} = 0$ $(resp., \dim D_x = 0)$, then the CR-Submanifold is called an invariant (resp., anti-invariant) submanifold. The distribution D $(resp., D^{\perp})$ is called the horizontal (resp., vertical) distribution. Also, the pair $(D, D^{\perp})$ is called $\xi -$ horizontal (resp. vertical) $if\; \xi_X \in D_X (resp., \xi_X \in D_X^{\perp})$.

3. Some basic lemmas

Lemma 3.1. *If M be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\bar{M}$ with quarter symmetric non-metric connection. Then*

(3.1) $-\eta(X)\phi PY - \eta(Y)\phi PX - \eta(X)PY - \eta(Y)PX - 2\eta(X)\eta(Y)P\xi + P\phi(\nabla_X Y)$

$+P\phi(\nabla_Y X) = P\nabla_X(\phi PY) + P\nabla_Y(\phi PX) - PA_{\phi QY}X - PA_{\phi QX}Y,$

(3.2) $-\eta(X)QY - \eta(Y)QX + 2\eta(X)\eta(Y)Q\xi + 2Bh(X, Y) = Q\nabla_X(\phi PY) + Q\nabla_Y(\phi PY)$

$$-Q\nabla_{\phi QY}X - Q\nabla_{\phi QX}Y,$$

(3.3) $$-\eta(X)\phi QY - \eta(Y)\phi QX - \phi Q(\nabla_X Y) + \phi Q(\nabla_Y X) + 2Ch(X,Y) = h(X,\phi PY)$$
$$+h(Y,\phi PX) + \nabla_X^{\perp}(\phi QY) + \nabla_Y^{\perp}(\phi QX)$$

for any $X, Y \in T(M)$.

Proof. From (2.11), we have

$$\phi Y = \phi PY + \phi QY.$$

Differentiating covariantly and using (2.16) and (2.17), we get

$$(\overline{\nabla}_X\phi)Y + \phi(\nabla_X Y) + \phi h(X,Y) = \nabla_X(\phi PY) + h(X,\phi PY) - A_{\phi QY}X + \nabla_X^{\perp}(\phi QY).$$

Interchanging X *and* Y, we have

$$(\overline{\nabla}_Y\phi)X + \phi(\nabla_Y X) + \phi h(Y,X) = \nabla_Y(\phi PX) + h(Y,\phi PX) - A_{\phi QX}Y + \nabla_Y^{\perp}(\phi QX).$$

Adding above two equations, we obtain

$$(\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)X + \phi(\nabla_X Y) + \phi(\nabla_Y X) + 2\phi h(Y,X) = \nabla_X(\phi PY) + \nabla_Y(\phi PX)$$
$$+ h(X,\phi PY) + h(Y,\phi PX) - A_{\phi QY}X - A_{\phi QX}Y + \nabla_X^{\perp}(\phi QY) + \nabla_Y^{\perp}(\phi QX).$$

Using (2.14) in above equation, we get

(3.4) $$-\eta(X)\phi Y - \eta(Y)\phi X - \eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\boldsymbol{\xi} + \phi(\nabla_X Y) + \phi\emptyset(\nabla_Y X) +$$
$$2\phi h(Y,X) = \nabla_X(\phi PY) + \nabla_Y(\phi PX) + h(X,\phi PY) + h(Y,\phi PX) - A_{\phi QY}X -$$
$$A_{\phi QX}Y + \nabla_X^{\perp}(\phi QY) + \nabla_Y^{\perp}(\phi QX).$$

Comparing tangential, vertical and normal components from both sides of (3.4), we get the desired results.

Hence lemma is proved. □

Lemma 3.2. *IfM be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold* $\overline{M}$ *with quarter symmetric non-metric connection. Then*

(3.5) $$2(\overline{\nabla}_X\phi)Y = -\eta(X)\phi Y - \eta(Y)\phi X - \eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi + \nabla_X\phi Y$$

$$+h(X,\phi Y)-\nabla_Y\phi X-h(Y,\phi X)-\phi[X,Y],$$

(3.6) $$2(\bar{\nabla}_Y\phi)X=-\eta(X)\phi Y-\eta(Y)\phi X-\eta(X)Y-\eta(Y)X-2\eta(X)\eta(Y)\xi-\nabla_X\phi Y$$

$$-h(X,\phi Y)+\nabla_Y\phi X+h(Y,\phi X)+\phi[X,Y]$$

for any $X,Y\in D$.

Proof. From Gauss formula (2.16), we get

(3.7) $$\bar{\nabla}_X\phi Y-\bar{\nabla}_Y\phi X=\nabla_X\phi Y+h(X,\phi Y)-\nabla_Y\phi X-h(Y,\phi X).$$

Also, we have

(3.8) $$\bar{\nabla}_X\phi Y-\bar{\nabla}_Y\phi X=(\bar{\nabla}_X\phi)Y-(\bar{\nabla}_Y\phi)X+\phi[X,Y].$$

From (3.7) and (3.8), we have

(3.9) $$(\bar{\nabla}_X\phi)Y-(\bar{\nabla}_Y\phi)X=\nabla_X\phi Y+h(X,\phi Y)-\nabla_Y\phi X-h(Y,\phi X)-\phi[X,Y].$$

Adding (3.9) and (2.14), we get

$$2(\bar{\nabla}_X\phi)Y=-\eta(X)\phi Y-\eta(Y)\phi X-\eta(X)Y-\eta(Y)X-2\eta(X)\eta(Y)\xi+\nabla_X\phi Y$$

$$+h(X,\phi Y)-\nabla_Y\phi X-h(Y,\phi X)-\phi[X,Y].$$

Subtracting (2.14) from (3.9), we get

$$2(\bar{\nabla}_Y\phi)X=-\eta(X)\phi Y-\eta(Y)\phi X-\eta(X)Y-\eta(Y)X-2\eta(X)\eta(Y)\xi-\nabla_X\phi Y$$

$$-h(X,\phi Y)+\nabla_Y\phi X+h(Y,\phi X)+\phi[X,Y].$$

Hence lemma is proved. □

Corollary 3.3. *If* M *be a* $\xi-vertical$ *CR-submanifold of a nearly hyperbolic Kenmotsu manifold* $\bar{M}$ *with quarter symmetric non-metric connection. Then*

$$2(\bar{\nabla}_X\phi)Y=\nabla_X\phi Y-\nabla_Y\phi X+h(X,\phi Y)-h(Y,\phi X)-\phi[X,Y],$$

and $$2(\bar{\nabla}_Y\phi)X=\nabla_Y\phi X-\nabla_X\phi Y+h(Y,\phi X)-h(X,\phi Y)+\phi[X,Y]$$

for any $X,Y\in D$.

Lemma 3.4. *If M be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\bar{M}$ with quarter symmetric non-metric connection. Then*

(3.10) $$2(\bar{\nabla}_Y\phi)Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y - \phi[Y,Z] - \eta(Y)\phi Z - \eta(Z)\phi Y$$
$$-\eta(Y)Z - \eta(Z)Y - 2\eta(Y)\eta(Z)\boldsymbol{\xi},$$

(3.11) $$2(\bar{\nabla}_Z\phi)Y = A_{\phi Z}Y - A_{\phi Y}Z + \nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z - \phi[Y,Z] - \eta(Y)\phi Z - \eta(Z)\phi Y$$
$$-\eta(Y)Z - \eta(Z)Y - 2\eta(Y)\eta(Z)\xi$$

for any $Y,Z \in D^{\perp}$.

Proof. Let $Y,Z \in D^{\perp}$. From Weingarten formula (2.17), we get

(3.12) $$\bar{\nabla}_Y\phi Z - \bar{\nabla}_Z\phi Y = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y.$$

Also, we have

(3.13) $$\bar{\nabla}_Y\phi Z - \bar{\nabla}_Z\phi Y = (\bar{\nabla}_Y\phi)Z - (\bar{\nabla}_Z\phi)Y + \phi[Y,Z].$$

From (3.12) and (3.13), we obtain

(3.14) $$(\bar{\nabla}_Y\phi)Z - (\bar{\nabla}_Z\phi)Y = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y - \phi[Y,Z].$$

For nearly hyperbolic Kenmotsu manifold, we have

(3.15) $$(\bar{\nabla}_Y\phi)Z - (\bar{\nabla}_Z\phi)Y = -\eta(Y)\phi Z - \eta(Z)\phi Y - \eta(Y)Z - \eta(Z)Y - 2\eta(Y)\eta(Z)\xi.$$

Adding (3.14) and (3.15), we get

$$2(\bar{\nabla}_Y\phi)Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y - \phi[Y,Z] - \eta(Y)\phi Z - \eta(Z)\phi Y$$
$$-\eta(Y)Z - \eta(Z)Y - 2\eta(Y)\eta(Z)\xi.$$

Subtracting (3.14) from (3.15), we get

$$2(\bar{\nabla}_Z\phi)Y = A_{\phi Z}Y - A_{\phi Y}Z + \nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z - \phi[Y,Z] - \eta(Y)\phi Z - \eta(Z)\phi Y$$
$$-\eta(Y)Z - \eta(Z)Y - 2\eta(Y)\eta(Z)\xi$$

for any $Y,Z \in D^{\perp}$.

Hence lemma is proved. □

Corollary 3.5. *If M be a $\xi -$ horizontal CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\bar{M}$ with quarter symmetric non-metric connection. Then*

$$2(\bar{\nabla}_Y\phi)Z = A_{\phi Y}Z - A_{\phi Z}Y + \nabla_Y^{\perp}\phi Z - \nabla_Z^{\perp}\phi Y - \phi[Y,Z],$$

and

$$2(\bar{\nabla}_Z\phi)Y = A_{\phi Z}Y - A_{\phi Y}Z + \nabla_Z^{\perp}\phi Y - \nabla_Y^{\perp}\phi Z - \phi[Y,Z]$$

for any $Y, Z \in D^{\perp}$.

Lemma 3.6. *If M be a CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\bar{M}$ with quarter symmetric non-metric connection. Then*

(3.16) $$2(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y] - \eta(X)\phi Y - \eta(Y)\phi X$$
$$-\eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi,$$

(3.17) $$2(\bar{\nabla}_Y\phi)X = -A_{\phi Y}X - \nabla_X^{\perp}\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y] - \eta(X)\phi Y - \eta(Y)\phi X$$
$$-\eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi$$

for any $X \in D$ and $Y \in D^{\perp}$.

Proof. Let $X \in D, Y \in D^{\perp}$

From Gauss and Weingarten formulae, we have

(3.18) $$\bar{\nabla}_X\phi Y - \bar{\nabla}_Y\phi X = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X).$$

Also, we have

(3.19) $$\bar{\nabla}_Y\phi X - \bar{\nabla}_X\phi Y = (\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X + \phi[X,Y].$$

From (3.18) and (3.19), we get

(3.20) $$(\bar{\nabla}_X\phi)Y - (\bar{\nabla}_Y\phi)X = -A_{\phi Y}Z + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y].$$

Also, for nearly hyperbolic Kenmotsu manifold we have

(3.21) $$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = -\eta(X)\phi Y - \eta(Y)\phi X - \eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi.$$

Adding (3.20) and (3.21), we obtain

$$2(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y] - \eta(X)\phi Y - \eta(Y)\phi X$$

$$-\eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi.$$

Subtracting (3.20) from (3.21), we find

$$2(\bar{\nabla}_Y\phi)X = -A_{\phi Y}X - \nabla_X^{\perp}\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y] - \eta(X)\phi Y - \eta(Y)\phi X$$

$$-\eta(X)Y - \eta(Y)X - 2\eta(X)\eta(Y)\xi.$$

Hence lemma is proved. □

Corollary 3.7. *If M be a $\xi -$ horizontal CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\bar{M}$ with quarter symmetric non-metric connection. Then*

$$2(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y] - \eta(X)\phi Y - \eta(X)Y,$$

$$2(\bar{\nabla}_Y\phi)X = -A_{\phi Y}X - \nabla_X^{\perp}\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y] - \eta(X)\phi Y - \eta(X)Y$$

for any $X \in D$ and $Y \in D^{\perp}$.

Corollary 3.8. *If M be a $\xi -$ vertical CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\bar{M}$ with quarter symmetric non-metric connection. Then*

$$2(\bar{\nabla}_X\phi)Y = -A_{\phi Y}X + \nabla_X^{\perp}\phi Y - \nabla_Y\phi X - h(Y,\phi X) - \phi[X,Y] - \eta(Y)\phi X - \eta(Y)X,$$

and

$$2(\bar{\nabla}_Y\phi)X = -A_{\phi Y}X - \nabla_X^{\perp}\phi Y + \nabla_Y\phi X + h(Y,\phi X) + \phi[X,Y] - \eta(Y)\phi X - \eta(Y)X$$

for $X \in D$ and $Y \in D^{\perp}$.

4. Parallel distribution

Definition 4.1. The horizontal (resp., vertical) distribution $D(resp., D^{\perp})$ is said to be Parallel [3] with respect to the connection on M if $\nabla_X Y \in D$ $(resp., \nabla_Z W \in D^{\perp})$ for any vector field $X, Y \in D$ $(resp., W, Z \in D^{\perp})$.

Theorem 4.2. *Let M be a* $\xi-vertical$ *CR-submanifold of a nearly hyperbolic Kenmotsu manifold* $\bar{M}$ *with quarter symmetric non-metric connection. Then*

(4.1) $h(X,\phi Y)=h(Y,\phi X)$

for $any\ X,Y\in D$.

Proof. Using parallelism of horizontal distribution D, we have

(4.2) $\nabla_X(\phi Y)\in D \quad and\ \nabla_Y\phi X\in D \quad for\ any\ X,Y\in D$.

From (3.2), we have

(4.3) $2Bh(X,Y)=0$,

for any $X,Y\in D$.

Also, from (2.12) we have

(4.4) $\phi h(X,Y)=Bh(X,Y)+Ch(X,Y)$.

Using (4.3) in (4.4), we get

(4.5) $\phi h(X,Y)=Ch(X,Y)$.

Next, from (3.3) we have

$$h(X,\phi Y)+h(Y,\phi X)=2Ch(X,Y).$$

Using (4.5) in above equation, we have

(4.6) $h(X,\phi Y)+h(Y,\phi X)=2\phi h(X,Y)$.

Replacing $X\ to\ \phi X$, we obtain

(4.7) $h(\phi X,\phi Y)+h(Y,X)=2\phi h(\phi X,Y)$.

Now, replacing $Y\ to\ \phi Y$ in (4.6), we get

(4.8) $h(X,Y)+h(\phi Y,\phi X)=2\phi h(X,\phi Y)$.

From (4.7) and (4.8), we have

$$h(X,\phi Y)=h(Y,\phi X).$$

Hence theorem is proved. □

Theorem 4.3. *Let M be a $\xi-vertical$ CR −submanifold of a nearly hyperbolic Kenmotsu manifold $\bar{M}$ with quarter symmetric non-metric connection. If the distribution $D^{\perp}$ is parallel with respect to the connection on M. Then*

(4.9) $A_{\phi Y}Z+A_{\phi Z}Y\in D^{\perp}$

for any $Y,Z\in D^{\perp}$.

Proof. Let $Y,Z\in D^{\perp}$, then using Weingarten formula (2.17), we obtain

(4.10) $(\bar{\nabla}_Y\phi)Z+\phi(\bar{\nabla}_YZ)=-A_{\phi Z}Y+\nabla_Y^{\perp}\phi Z.$

Using (2.16) in (4.10), we have

(4.11) $(\bar{\nabla}_Y\phi)Z=-A_{\phi Z}Y+\nabla_Y^{\perp}\phi Z-\phi(\nabla_YZ)-\phi h(Y,Z).$

Interchanging Y and Z, we have

(4.12) $(\bar{\nabla}_Z\phi)Y=-A_{\phi Y}Z+\nabla_Z^{\perp}\phi Y-\phi(\nabla_ZY)-\phi h(Z,Y).$

Adding (4.11) and (4.12), we get

(4.13) $(\bar{\nabla}_Y\phi)Z+(\bar{\nabla}_Z\phi)Y=-A_{\phi Z}Y-A_{\phi Y}Z+\nabla_Y^{\perp}\phi Z+\nabla_Z^{\perp}\phi Y-\phi(\nabla_YZ)-\phi(\nabla_ZY)$

$$-2\phi h(Y,Z).$$

Using (2.14) in (4.13), we obtain

(4.14) $-\eta(Z)\phi Y-\eta(Y)\phi Z-\eta(Y)Z-\eta(Z)Y+2\eta(Y)\eta(Z)\xi=-A_{\phi Y}Z-A_{\phi Z}Y$

$$+\nabla_Y^{\perp}\phi Z+\nabla_Z^{\perp}\phi Y-\phi(\nabla_YZ)-\phi(\nabla_ZY)-2\phi h(Y,Z).$$

Taking inner product with $X\in D$ in (4.14), we get

$$g\left(A_{\phi Y}Z+A_{\phi Z}Y,X\right)=0.$$

This implies that

$$A_{\phi Y}Z+A_{\phi Z}Y\in D^{\perp}$$

for any $Y,Z\in D^{\perp}$.

Hence theorem is proved. □

Definition 4.4. A CR-submanifold is said to be mixed-totally geodesic if $h(X,Z)=0$ for all$X \in D$ and $Z \in D^{\perp}$.

Definition 4.5. A Normal vector field $N \neq 0$ is called $D-parallel$ normal section if $\nabla^{\perp}_{X}N=0$ for all $X \in D$.

Theorem 4.6. *Let M be a mixed totally geodesic $\xi-vertical$ CR-submanifold of a nearly hyperbolic Kenmotsu manifold $\bar{M}$ with quarter symmetric non-metric connection. Then the normal section $N \in \phi D^{\perp}$ is $D-parallel$ if and only if $\nabla_X \phi N \in D$ for all $X \in D$.*

Proof. Let $N \in \phi D^{\perp}$, then from (3.2) we have

(4.15) $2Bh(X,Y)=Q\nabla_Y(\phi X)-QA_{\phi Y}X.$

Using definition of mixed geodesic CR-submanifold, we have

(4.16) $Q\nabla_Y(\phi X)-QA_{\phi Y}X=0.$

(4.17) $Q\nabla_Y\phi X=0$

for $X \in D$.

In particular, we have

(4.18) $Q\nabla_Y X=0.$

Using (4.18) in (3.3), we get

$$\phi Q\nabla_X Y=\nabla^{\perp}_{X}\phi QY.$$

That is, $\phi Q\nabla_X(\phi N)=\nabla^{\perp}_{X}N.$ Then by definition of parallelism of N, we have

$$\phi Q\nabla_X(\phi N)=0.$$

Consequently, we get

$$\nabla_X(\phi N) \in D \ for \ all \ X \in D.$$

Converse part is easy consequence of (4.20). □

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