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A NOTE ON NEAR COMPLETELY PRIME IDEAL RINGS OVER $\sigma(*)$-RINGS

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Abstract. In this paper, Let R be a ring and σ an endomorphism of R. Recall that R is said to be a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies that $a \in P(R)$ for $a \in R$, where $P(R)$ is the prime radical of R. We also recall that a ring R is said to be a completely prime ideal ring (CPI-ring) if every prime ideal of R is completely prime and a near completely prime ideal ring (NCPI-ring) if every minimal prime ideal of R is completely prime Bhat [6].

In this paper we give a relation between $\sigma(*)$-ring and near completely prime ideal ring and also prove that if R is a Noetherian ring and σ an endomorphism of R such that R is $\sigma(*)$-ring then $S(R) = R[x;\sigma]$ is a Noetherian near completely prime ideal ring.

Keywords: Ore extension; minimal prime ideal; completely prime ideal; near completely prime ideal; $\sigma(*)$-ring.

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1. Introduction

We use the notation as in Bhat [6] , but to make the paper self contained, we have the following:

A ring R means an associative ring with identity $1 \neq 0$. $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Z}$ denotes the ring of integers and $\mathbb{N}$ denotes the set of positive integers unless other wise stated. Let R be a ring. The set of prime ideals of R is denoted by $Spec(R)$, the set of minimal prime ideals of R is denoted

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by $MinSpec(R)$ and the set of completely prime ideals of R is denoted by $C.Spec(R)$. The prime radical of R is denoted by $P(R)$.

Let R be a ring, σ an endomorphism of R and δ a σ-derivation of R. Recall that the Ore extension

$$R[x;\sigma,\delta] = \{f = \sum_{i=0}^{n} x^i a_i, a_i \in R,\, n \in \mathbb{N}\}$$

with usual addition and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We denote $R[x;\sigma,\delta]$ by $O(R)$. If I is an ideal of R such that I is σ-stable (i.e. $\sigma(I) = I$) and is also δ-invariant (i.e. $\delta(I) \subseteq I$), then clearly $I[x;\sigma,\delta]$ is an ideal of $O(R)$, and we denote it as usual by $O(I)$. In case σ is the identity map, we denote the ring of differential operators $R[x;\delta]$ by $D(R)$. If J is an ideal of R such that J is δ-invariant (i.e. $\delta(J) \subseteq J$), then clearly $J[x;\delta]$ is an ideal of $D(R)$, and we denote it as usual by $D(J)$. In case δ is the zero map, we denote $R[x;\sigma]$ by $S(R)$. If K is an ideal of R such that K is σ-stable (i.e. $\sigma(K) = K$), then clearly $K[x;\sigma]$ is an ideal of $S(R)$, and we denote it as usual by $S(K)$.

2. Preliminaries

Prime ideals of Ore extensions

Goodearl and Warfield proved in (2ZA) of [10] that if R is a commutative Noetherian ring, and if σ is an automorphism of R, then an ideal I of R is of the form $P \cap R$ for some prime ideal P of $R[x,x^{-1};\sigma]$ if and only if there is a prime ideal S of R and a positive integer m with $\sigma^m(S) = S$, such that $I = \cap\sigma^i(\mathrm{S}), i = 1,2,...,m$. They proved in Theorem (2.22) of [10] that if δ is a derivation of a commutative Noetherian ring R which is also an algebra over $\mathbb{Q}$ and P is a prime ideal of $R[x;\delta]$, then $P \cap R$ is a prime ideal of R and if S is a prime ideal of R with $\delta(S) \subseteq S$, then $S[x;\delta]$ is a prime ideal of $R[x;\delta]$. Gabriel proved in [9] that if R is a right Noetherian ring which is also an algebra over $\mathbb{Q}$ and P is a prime ideal of $R[x;\delta]$, then $P \cap R$ is a prime ideal of R.

It is also proved that if R is a right Noetherian ring, then we know that $MinSpec(R)$ is finite by Theorem (2.4) of Goodearl and Warfield [10] and for any automorphism σ of R, $P \in MinSpec(R)$ implies that $\sigma^j(P) \in MinSpec(R)$ for all positive integers j.

Completely prime ideals

We have discussed some known facts about the prime ideals of Ore extensions. We shall now discuss some special types of prime ideals that play a key role in the notions introduced in this paper. These types of prime ideals include completely prime ideals and minimal prime ideals.

Recall that an ideal P of a ring R is completely prime if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$. In commutative sense completely prime and prime have the same meaning. We also note that every completely prime ideal of a ring R is a prime ideal, but the converse need not be true.

The following example shows that a prime ideal need not be a completely prime ideal.

Example 2.1. The zero ideal in the ring of $n \times n$ matrices is a prime ideal, but it is not completely prime.

Example 2.2.(Example 1.1 of Bhat [4]) Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} = M_2\mathbb{Z}$. If p is a prime number, then the ideal $P = M_2(p\mathbb{Z})$ is a prime ideal of R, but is not completely prime, since for $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we have $ab \in P$, even though $a \notin P$ and $b \notin P$.

There are examples of rings (noncommutative) in which prime ideals are completely prime.

Example 2.3. (Example 1.2 of Bhat [4]) Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$. Then $P_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$, $P_2 = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ and $P_3 = \begin{pmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{pmatrix}$ are prime ideals of R. Now all these are completely prime also.

A relation between the completely prime ideals of a ring R and those of $O(R)$ has been proved in Theorem (2.4) by Bhat [4] (states in Theorem 2.2).

Minimal prime ideals

Goodearl and Warfield [10] A minimal prime ideal in a ring R is any prime ideal of R that does not properly contain any other prime ideals.

For example if R is a prime ring, then 0 is a minimal prime ideal of R.

J. Krempa has investigated the relation between minimal prime ideals and completely prime ideals of a ring R. With this he proved the following:

Theorem (2.2) of Krempa [11] For a ring R the following conditions are equivalent:

(1) R is reduced.

(2) R is semiprime and all minimal prime ideals of R are completely prime.

(3) R is a subdirect product of domains.

Completely Prime Ideal Rings(CPI-rings)

Definition 2.4. Bhat [6] Let R be a ring. Then R is said to be completely prime ideal ring (CPI-ring) if every prime ideal of R is completely prime. For example a commutative ring is a CPI-ring.

Definition 2.5. Bhat [6] Let R be a ring. Then R is said to be near completely prime ideal ring (NCPI-ring) if every minimal prime ideal of R is completely prime. For example a reduced ring is a near completely primal ring.

It is known that

(1) if P is a prime ideal of a ring R, then $P[x]$ is a prime ideal of $R[x]$. (Brewer and Heinzer [7])

(2) for any ring R, an ideal P of $R[x]$ is prime if and only if $P \cap R$ is a prime ideal of R and

  (a) either $P = (P \cap R)[x]$

  (b) or P is maximal amongst ideals I of $R[x]$ such that $I \cap R = P \cap R$. (Ferrero [8])

$\sigma(*)$-rings

Recall that in Krempa [11], a ring R is called σ-rigid if there exists an endomorphism of R with the property that $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. In [12], Kwak defines a $\sigma(*)$-ring R to be a ring in which $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$.

Example 2.6. (Example 2 of Kwak [12]) Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Let $\sigma : R \to R$ be defined by $\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that σ is an endomorphism of R and that R is a $\sigma(*)$-ring.

We note that the above ring is not σ-rigid. For let $0 \neq a \in F$. Then
$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \sigma\Big(\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\Big) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Example 2.7. (Example 1.3 of Bhat [5]) Let $R = \mathbb{C}$, the field of complex numbers. Then $\sigma : R \to R$ defined by $\sigma(a + \iota b) = a - \iota b$ is an automorphism of R and R is a $\sigma(*)$-ring. Bhat has proved the following result in [3]:

Let R be a Noetherian ring and σ an automorphism of R such that R is a $\sigma(*)$-ring. Then $S(R) = R[x; \sigma]$ is also a $\sigma(*)$-ring.

It has been also proved that if σ is an automorphism of R, then it can be extended to an automorphism (say $\overline{\sigma}$) of $R[x; \sigma]$ such that $\sigma(x) = x$.

3. Main results

We now state the main result of this paper in the form of the following Theorem:

Theorem (A) Let R be a Noetherian ring and σ an automorphism of R such that R is a $\sigma(*)$-ring. Then $S(R) = R[x; \sigma]$ is Noetherian near completely prime ideal ring. This is proved in Theorem (3.6.).

Towards the proof of the above Theorem, we require the following:

Lemma 3.1. *Let R be a ring and σ be an automorphism of R.*

(1) *If P is a prime ideal of $S(R)$ such that $x \notin P$, then $P \cap R$ is a prime ideal of R and $\sigma(P \cap R) = P \cap R$.*

(2) *If U is a prime ideal of R such that $\sigma(U) = U$, then $S(U)$ is a prime ideal of $S(R)$ and $S(U) \cap R = U$.*

Proof. The proof follows on the same lines as in Lemma (10.6.4) of McConnell and Robson [13].

Theorem 3.2. *Let R be a ring. Let σ an automorphism of R and δ a σ-derivation of R. Then:*

(1) *For any completely prime ideal P of R with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $O(P) = P[x; \sigma, \delta]$ is a completely prime ideal of $O(R)$.*

(2) *For any completely prime ideal P of $O(R), U \cap R$ is a completely prime ideal of R.*

Proof. See Theorem 2.4 of Bhat [4].

Theorem 3.3. (Hilbert Basis Theorem): *Let R be a right/left Noetherian ring. Let σ be an automorphism of R and δ a σ-derivation of R. Then the ore extension $O(R) = R[x; \sigma, \delta]$ is right/left Noetherian.*

Proof. See Theorem (1.12) of Goodearl and Warfield [10].

Completely Prime ideals of polynomial rings over $\sigma(*)$-rings

Theorem 3.4.(Theorem 2.4 of Bhat and Kumari [3]) *Let R be a Noetherian ring, and σ an automorphism of R. Then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is a completely prime ideal of R.*

Proof. Let R be a Noetherian ring such that $\sigma(U) = U$ and U is completely prime ideal of R. Let $a \in R$ be such that $a\sigma(a) \in P(R) = \cap_{i=1}^{n} U_i$, where U_i are the minimal primes of R. Now for each $i, \sigma(a) \in U_i$. Now $\sigma(a) \in U_i = \sigma(U_i)$ implies that $a \in U_i$. Therefore $a \in P(R)$. Hence R is a $\sigma(*)$-ring.

Conversely, suppose that R is a $\sigma(*)$-ring. Then Proposition (2.1) of Bhat [2] implies that $P(R)$ is a completely semiprime ideal of R and $\sigma(U) = U$ for all $U \in MinSpec(R)$.

Now suppose that $U = U_1$ is not completely prime. Then there exists $a, b \in R/U$ with $ab \in U$. Let c be any element of $b(U_2 \cap U_3 \cap ... \cap U_n)a$. Then $c^2 \in \cap_{i=1}^{n} U_i = P(R)$ implies $b(U_2 \cap U_3 \cap ... \cap U_n)a \subseteq U$. Therefore $bR(U_2 \cap U_3 \cap ... \cap U_n)Ra \subseteq U$ and, as U is prime, $a \in U, U_i \subseteq U$ for some $i \neq 1$ or $b \in U$. None of these can occur, so U is completely prime.

Proposition 3.5. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R. Then $\delta(U) \subseteq U$ for all $U \in MinSpec(R)$.*

Proof. See Proposition (2.1) of Bhat [2].

We are now in a position to prove Theorem A:

Theorem 3.6. *Let R be a Noetherian ring and σ an automorphism of R such that R is a $\sigma(*)$-ring. Then $S(R) = R[x;\sigma]$ is Noetherian near completely prime ideal ring.*

Proof. R is Noetherian implies $S(R)$ is Noetherian by Hilbert Basis Theorem (Theorem 3.3). Let P be a minimal prime ideal of $S(R) = R[x;\sigma]$. Now Lemma (3.1) implies that $P \cap R \in MinSpec(R)$ and $\sigma(P \cap R) = P \cap R$ and $(P \cap R)[x;\sigma] = P$. Now R is Noetherian $\sigma(*)$-ring implies that $P \cap R$ is completely prime ideal by Theorem (3.4) . Now Theorem (3.2) implies that $(P \cap R)[x;\sigma] = P$ is completely prime. Therefore $R[x;\sigma]$ is near completely prime ideal ring.

Question 3.7. *Let R be a Noetherian ring and σ an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R. Is $O(R) = R[x;\sigma,\delta]$ a Noetherian near completely prime ideal ring?*

Question 3.8. *Let R be a Noetherian ring and σ an automorphism of R such that R is a near completely prime ideal ring. Is $S(R) = R[x;\sigma]$ a Near completely prime ideal ring?*

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