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NUMERICAL RADIUS INEQUALITIES FOR THE CARTESIAN DECOMPOSITION OF BOUNDED LINEAR OPERATORS IN HILBERT SPACES

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Abstract. Let $B(H)$ be the space of C^*- algebra of all bounded linear operators on a complex Hilbert space H, and let $T \in B(H)$. In this article, we establish several numerical radius inequalities for the cartesian decomposition of T.

Keywords: Numerical Radius; Operator Norm; Cartesian Decomposition.

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1. Introduction

Let H be a complex Hilbert space with inner product $\langle .,. \rangle$, and let $B(H)$ be the space of C^*-algebra of all bounded linear operators on H. For $T \in B(H)$, let T^* denote the adjoint operator of T. Also, let $w(T)$ denote the numerical radius of T given by

$$w(T) = \sup\{|\langle Tx, x\rangle| : x \in H, ||x|| = 1\}.$$

It is well known that $w(.)$ is a norm on $B(H)$, which is equivalent to the usual operator norm $\| . \|$ defined, for $T \in B(H)$, by

$$\| T \| = \sup \{\|Tx\| : x \in H, \|x\| = 1\},$$

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where $\|Tx\| = \langle Tx, Tx\rangle^{\frac{1}{2}}$. More precisely, for $T \in B(H)$, [6] showed that

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1.1}$$

Several numerical radius inequalities that provide alternative upper bounds for $w(.)$ have received much attention from many authors. We refer the readers to [1], [6], and [7] for their history and significance, and [2], [3], [9], and [11] for recent developments in this area. For example, [9] proved that for $T \in H$,

$$w(T) \leq \frac{1}{2}\||T| + |T^*|\| \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{\frac{1}{2}}\right), \tag{1.2}$$

where $|T| = (T^*T)^{\frac{1}{2}}$ is the absolute value of T. [11] determine that

$$\frac{1}{4}\|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|. \tag{1.3}$$

If $B + iC$ is the Cartesian decomposition of T, then B and C are self-adjoint, and so since $T^*T + TT^* = 2\left(B^2 + C^2\right)$, we conclude, by using (1.3), that

$$\frac{1}{2}\left\|B^2 + C^2\right\| \leq w^2(T) \leq \left\|B^2 + C^2\right\|. \tag{1.4}$$

Recently, [5] generalized the inequality (1.4). In fact, [5] established that for $T \in B(H)$ with $T = A + iB$, and $r \geq 2$,

$$2^{\frac{-r}{2}-1}\||B + C|^r + |B - C|^r\| \leq w^r(T) \leq \frac{1}{2}\||B + C|^r + |B - C|^r\|. \tag{1.5}$$

Although some open problems related to the numerical radius inequalities for bounded linear operator still remain open, the investigation to establish numerical radius inequalities for several bounded linear operators has been started, (see for instance [4] and [6]). For example, If T_1, $T_2 \in B(H)$, [6] evidenced that

$$w(T_1T_2) \leq 4w(T_1)w(T_2).$$

Moreover, in the case $T_1T_2 = T_2T_1$, [6] verified that

$$w(T_1T_2) \leq 2w(T_1)w(T_2).$$

Very recently, for T_1, $T_2 \in B(H)$ and $r \geq 1$, [4] showed that

$$w^r(T_2^*T_1) \leq \frac{1}{2}\left\|(T_1^*T_1)^r + (T_2^*T_2)^r\right\|.$$

Moreover, for T_1, $T_2 \in B(H)$, $\alpha \in (0,1)$, and $r \geq 1$, [4] applied a different approach to obtain

$$w^{2r}(T_2^*T_1) \leq \left\|\alpha(T_1^*T_1)^{\frac{r}{\alpha}} + (1-\alpha)(T_2^*T_2)^{\frac{r}{1-\alpha}}\right\|.$$

The purpose of this paper is to establish various numerical radius inequalities for the cartesian decomposition of bounded linear operators on a complex Hilbert space. In particular, we use a tranquil approach to attain new upper bounds for numerical radius of $T \in B(H)$.

2. The Main Results

In this section, we establish and prove some numerical radius inequalities for the cartesian decomposition of bounded linear operators on a complex Hilbert space. The proofs of our sequels mainly depend on the following two well known inequalities.

Lemma 2.1. *Let $a, b \geq 0$ and $r \geq 1$. Then*

$$(a+b)^r \leq 2^{r-1}(a^r + b^r).$$

Obviously, the above lemma is obtained as a consequence of the classical Jensen's inequality concerning the convexity of the function $f(t) = t^r$ for $r \geq 1$.

The next lemma is reached by combining the spectral theorem for positive operators with Jensen's inequality. In particular, we have the following.

Lemma 2.2 [10]**.** *Let $T \in B(H)$ be a positive operator, and let $x \in H$ be a unit vector. Then*

$$\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle \text{ for } r \geq 1,$$

and

$$\langle T^r x, x\rangle \leq \langle Tx, x\rangle^r \text{ for } 0 < r \leq 1.$$

Let us use this lemma to prove our first result.

Theorem 2.3. *Let $T \in B(H)$, and let $B + iC$ be the Cartesian decomposition of T. Suppose that $r \geq 1$. Then*

$$w^{2r}(T) \leq 2^{r-1}\left(\max\{\|B\|^{2r}, \|C\|^{2r}\} + w^r(CB)\right). \tag{2.1}$$

Proof. For any vector $x \in H$ with $\|x\| = 1$, we have that

$$\begin{aligned}
|\langle Tx, x\rangle|^{4r} &= \left(\langle x, \langle x, Bx\rangle Bx + \langle x, Cx\rangle Cx\rangle\right)^{2r} \\
&\leq \|\langle x, Bx\rangle Bx + \langle x, Cx\rangle Cx\|^{2r} \\
&\leq \left(|\langle x, Bx\rangle|^2\|Bx\|^2 + |\langle x, Cx\rangle|^2\|Cx\|^2 + 2\,|\langle x, Bx\rangle|\,|\langle x, Cx\rangle|\,|\langle Bx, Cx\rangle|\right)^r \\
&\leq \left(|\langle x, Bx\rangle|^2\|Bx\|^2 + |\langle x, Cx\rangle|^2\|Cx\|^2 + \left(|\langle x, Bx\rangle|^2 + |\langle x, Cx\rangle|^2\right)|\langle Bx, Cx\rangle|\right)^r \\
&= \left[|\langle x, Bx\rangle|^2\left(\|Bx\|^2 + |\langle Bx, Cx\rangle|\right) + |\langle x, Cx\rangle|^2\left(\|Cx\|^2 + |\langle Bx, Cx\rangle|\right)\right]^r \\
&\leq |\langle Tx, x\rangle|^{2r}\left(\max\{\|Bx\|^2, \|Cx\|^2\} + |\langle Bx, Cx\rangle|\right)^r.
\end{aligned}$$

By this and Lemma 2.1, we obtain

$$|\langle Tx, x\rangle|^{2r} \leq 2^{r-1}\left(\max\{\|Bx\|^{2r}, \|Cx\|^{2r}\} + |\langle CBx, x\rangle|^r\right).$$

We finish the proof by taking the superemum over all unit vectors $x \in H$.

Following the same manner used in proving the above theorem, we achieve the following theorem.

Theorem 2.4. *Let $T \in B(H)$ and $r \geq 1$. If $B + iC$ is the Cartesian decomposition of T, then*

$$w^{2r}(T) \leq 2^{r-1}\left[\left(\frac{\|B^2 + C^2\| + \|B^2 - C^2\|}{2}\right)^r + w^r(CB)\right]. \tag{2.2}$$

Proof. Let $x \in H$ be a unit vector. By Lemma 2.1 and the arguments used in the proof of Theorem 2.3, we deduce that

$$\begin{aligned}
|\langle Tx, x\rangle|^{2r} &\leq \left(\max\{\|Bx\|^2, \|Cx\|^2\} + |\langle CBx, x\rangle|\right)^r \\
&= \left(\frac{\langle (B^2+C^2)\,x, x\rangle + |\langle (B^2-C^2)\,x, x\rangle|}{2} + |\langle CBx, x\rangle|\right)^r \\
&\leq 2^{r-1}\left(\left(\frac{\langle (B^2+C^2)\,x, x\rangle + |\langle (B^2-C^2)\,x, x\rangle|}{2}\right)^r + |\langle CBx, x\rangle|^r\right).
\end{aligned}$$

Take the supremum over all unit vectors $x \in H$, we reach our theorem. This completes the proof.

It is clear that the equalities for the inequalities (2.1)-(2.2) are satisfied when $r = 1$ and $T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

A straightforward technique plus Lemma 2.1 and Lemma 2.2, we derive the following two theorems.

Theorem 2.5. *Let $T \in B(H)$ and $r \geq 1$. Suppose that $B + iC$ is the Cartesian decomposition of T. Then for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$w^{2r}(T) \leq 2^{r+\frac{1}{q}-1}\left\|B^{2rp} + C^{2rq}\right\|^{\frac{1}{p}}.$$

Proof. For any unit vector $x \in H$, by Lemmas 2.1-2.2 plus Holder's inequality, we get that

$$\begin{aligned}
|\langle Tx, x\rangle|^{2r} &\leq \left(\|Bx\|^2 + \|Cx\|^2\right)^r \\
&\leq 2^{r-1}\left(\left\langle B^2x, x\right\rangle^r + \left\langle C^2x, x\right\rangle^r\right) \\
&\leq 2^{r+\frac{1}{q}-1}\left(\left\langle B^{2r}x, x\right\rangle^p + \left\langle C^{2r}x, x\right\rangle^p\right)^{\frac{1}{p}} \\
&\leq 2^{r+\frac{1}{q}-1}\left(\left\langle B^{2rp}x, x\right\rangle + \left\langle C^{2rp}x, x\right\rangle\right)^{\frac{1}{p}} \\
&\leq 2^{r+\frac{1}{q}-1}\left\langle\left(B^{2rp} + C^{2rp}\right)x, x\right\rangle^{\frac{1}{p}}.
\end{aligned}$$

By taking the supremum over all unit vectors $x \in H$, we complete the proof.

Theorem 2.6. *Suppose that $T \in B(H)$, and that $B + iC$ is the Cartesian decomposition of T. Then for $r \geq 1$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$w^{2r}(T) \leq \left\|B^{2rp} + I\right\|^{\frac{1}{p}} \left\|C^{2rq} + I\right\|^{\frac{1}{q}},$$

where I is the identity operator.

Proof. Assume that $x \in H$ with $\|x\| = 1$. Applying Holders inequality, Lemma 2.1 and Lemma 2.2 give that

$$\begin{aligned}
|\langle Tx, x\rangle|^{2r} &\leq \left(\|Bx\|^2 + \|Cx\|^2\right)^r \\
&\leq 2^{r-1}\left(\left\langle B^2x, x\right\rangle^r + \left\langle C^2x, x\right\rangle^r\right) \\
&\leq 2^{r-1}\left[\left(\left\langle B^2x, x\right\rangle^{rp} + \langle Ix, x\rangle\right)^{\frac{1}{p}}\left(\left\langle C^2x, x\right\rangle^{rq} + \langle Ix, x\rangle\right)^{\frac{1}{q}}\right] \\
&\leq 2^{r-1}\left(\left\langle \left(B^{2rp} + I\right)x, x\right\rangle^{\frac{1}{p}}\left\langle \left(C^{2rq} + I\right)x, x\right\rangle^{\frac{1}{q}}\right).
\end{aligned}$$

We attain our theorem by taking the supremum over all x. This finishes the proof.

Similar procedure to what used in Theorem 2.6, Clarkson's inequality, (see [10]), provides new upper bounds for $w^{2r}(T)$ with $r \geq 2$. In particular, we establish the following.

Theorem 2.7. *Let $T \in B(H)$, and let $B+iC$ be the Cartesian decomposition of T. Then for any $r \geq 2$,*

$$w^{2r}(T) \leq 2^{r-2}\left(\left\|B^2 + C^2\right\|^r + \left\|B^2 - C^2\right\|^r\right).$$

Proof. For any unit vector $x \in H$, applying Lemma 2.1 and Clarkson's inequality, we get that

$$\begin{aligned}
|\langle Tx, x\rangle|^{2r} &\leq \left(\|Bx\|^2 + \|Cx\|^2\right)^r \\
&\leq 2^{r-1}\left(\left\langle B^2x, x\right\rangle^r + \left\langle C^2x, x\right\rangle^r\right) \\
&\leq 2^{r-2}\left(\left|\left\langle B^2x, x\right\rangle + \left\langle C^2x, x\right\rangle\right|^r + \left|\left\langle B^2x, x\right\rangle - \left\langle C^2x, x\right\rangle\right|^r\right) \\
&= 2^{r-2}\left(\left|\left\langle (B^2 + C^2)x, x\right\rangle\right|^r + \left|\left\langle (B^2 - C^2)x, x\right\rangle\right|^r\right).
\end{aligned}$$

By this, we satisfy the desired inequality by taking the supremum over all x. This completes the proof.

As a direct application of Lemma 2.2, we deduce for any $0 < r \le 1$ and any $T \in B(H)$ with $T = B + iC$ is the Cartesian decomposition of T, that

$$w^2(T) \le \left\| B^{2r} \right\|^{\frac{1}{r}} + \left\| C^{2r} \right\|^{\frac{1}{r}}. \tag{2.3}$$

We end our sequels with the following theorem.

Theorem 2.8. *Assume that $T \in B(H)$ and that $A + iB$ is the Cartesian decomposition of T. then for any $r \ge 1$,*

$$w(T) \le 2 \max \left\{ \left(\frac{\|A\|^r + \||A|^r\|}{2} \right)^{\frac{1}{r}}, \left(\frac{\|B\|^r + \||B|^r\|}{2} \right)^{\frac{1}{r}} \right\}.$$

Proof. Let $A + iB$ be the Cartesian decomposition of a given $T \in B(H)$. Then, [[1], Lemma 2.5] implies that

$$\begin{aligned} w(T) &= w\left(\begin{bmatrix} A & 0 \\ 0 & iB \end{bmatrix} + \begin{bmatrix} iB & 0 \\ 0 & A \end{bmatrix} \right) \\ &\le 2w\left(\begin{bmatrix} A & 0 \\ 0 & iB \end{bmatrix} \right) \\ &\le \alpha + \beta + \sqrt{(\alpha - \beta)^2}, \end{aligned}$$

where $\alpha = \left(\frac{\|A\|^r + \||A|^r\|}{2} \right)^{\frac{1}{r}}$ and $\beta = \left(\frac{\|B\|^r + \||B|^r\|}{2} \right)^{\frac{1}{r}}$. Therefore, $w(T) \le \alpha + \beta + |\alpha - \beta| = 2\max\{\alpha, \beta\}$.

References

[1] R. Bhatia, Matrix Analysis, Springer-Verlag, Berlin, 1997.

[2] S. Dragomir, Inequalities for the norm and the numerical radius of linear operator in Hilbert spaces, Demonstratio Math. 40 (2007), no. 2, 411-417.

[3] S. Dragomir, Norm and numerical radius inequalities for sums of bounded linear operators in Hilbert spaces, Ser. Math. Inform. 22 (2007), no. 1, 61-75.

[4] S. Dragomir, Power inequalities for the numarical radius of a product of two operators in Hilbert spaces, Seraj. J. math. 5 (2009), no. 18, 269-278.

[5] M. El-Haddad and F. Kittaneh, Numerical radius inequalities for Hilbert space operators. II. Studia Math. 182 (2007), 133-140.

[6] K. Gustafson and D. Rao, Numerical range, Springer-Verlage, New York, 1997.

[7] P. Halmos, A Hilbert Space Problem Book, 2nd ed., Springer-Verlage, New York, 1982.

[8] O. Hirzallah, F. Kittaneh and K. Shebrawi, Numerical Radius Inequalities for Commutators of Hilbert space Operators. Numerical Functional Analysis and Optimization 32 (2011) no. 7, 739-749.

[9] F. Kittaneh, A numerical radius inequality and an estimate for the numerical radius of the Frobenius companion matrix, Studia Math. 158 (2003) no.1, 11-17.

[10] F. Kittaneh, Notes on some inequalities for Hilbert space operators, Pub1. Res. Inst. Math. Sci. 24 (1988), 283-293.

[11] F. Kittaneh, Numerical radius inequalities for Hillbert space operators, Studia Math. 168 (2005), no. 1, 73-80.