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GALOIS CONNECTIONS AND RIGHT CLOSURE OPERATORS

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Abstract. In this paper, we investigate the relations between right (left) closure operators and residuated (Galois) connections on generalized residuated lattices.

Keywords: generalized residuated lattices; isotone (antitone) maps; residuated (Galois) connections; right (left) closure (interior) operators

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1. Introduction

Bělohlávek [1-3] developed the notion of fuzzy contexts using Galois connections with $R \in L^{X \times Y}$ on a complete residuated lattice. Georgescu and Popescue [4.5] introduced non-commutative fuzzy Galois connection in a generalized residuated lattice which is induced by two implications. Kim [7] investigated the properties of right and left closure on a generalized residuated lattice.

In this paper, we investigate the relations between right (left) closure operators and residuated (Galois) connections on generalized residuated lattices. We give their examples.

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Definition 1.1. [4,5] A structure $(L, \vee, \wedge, \odot, \rightarrow, \Rightarrow, \bot, \top)$ is called a *generalized residuated lattice* if it satisfies the following conditions:

(GR1) $(L, \vee, \wedge, \top, \bot)$ is a bounded where $\top$ is the universal upper bound and $\bot$ denotes the universal lower bound;

(GR2) $(L, \odot, \top)$ is a monoid;

(GR3) it satisfies a residuation , i.e.

$$a \odot b \leq c \text{ iff } a \leq b \rightarrow c \text{ iff } b \leq a \Rightarrow c.$$

We call that a generalized residuated lattice has the law of double negation if $a = (a^*)^0 = (a^0)^*$ where $a^0 = a \rightarrow \bot$ and $a^* = a \Rightarrow \bot$.

Remark 1.2.[4-8] (1) A generalized residuated lattice is a residuated lattice ($\rightarrow = \Rightarrow$) iff $\odot$ is commutative.

(2) A left-continuous t-norm $([0,1], \leq, \odot)$ defined by $a \rightarrow b = \bigvee\{c \mid a \odot c \leq b\}$ is a residuated lattice

(3) Let $(L, \leq, \odot, \bot, \top)$ be a quantale. For each $x, y \in L$, we define

$$x \rightarrow y = \bigvee\{z \in L \mid z \odot x \leq y\}, \;\; x \Rightarrow y = \bigvee\{z \in L \mid x \odot z \leq y\}.$$

Then it satisfies Galois correspondence, that is,

$(x \odot y) \leq z$ iff $x \leq (y \rightarrow z)$ iff $y \leq (x \Rightarrow z)$.

Hence $(L, \vee, \wedge, \odot, \rightarrow, \Rightarrow, \bot, \top)$ is a generalized residuated lattice.

(4) A pseudo MV-algebra is a generalized residuated lattice with the law of double negation.

In this paper, we assume $(L, \wedge, \vee, \odot, \rightarrow, \Rightarrow, \bot, \top)$ is a generalized residuated lattice with the law of double negation and if the family supremum or infumum exists, we denote $\bigvee$ and $\bigwedge$.

Lemma 1.3.[4-8] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$ for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(2) $x \odot y \leq x \wedge y \leq x \vee y$.

(3) $x \rightarrow (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma}(x \rightarrow y_i)$ and $(\bigvee_{i \in \Gamma} x_i) \rightarrow y = \bigwedge_{i \in \Gamma}(x_i \rightarrow y)$ for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(4) $x \to (\bigvee_{i\in\Gamma} y_i) \geq \bigvee_{i\in\Gamma}(x \to y_i)$, for $\to \in \{\to, \Rightarrow\}$.

(5) $(\bigwedge_{i\in\Gamma} x_i) \to y \geq \bigvee_{i\in\Gamma}(x_i \to y)$, for $\to \in \{\to, \Rightarrow\}$.

(6) $(x \odot y) \to z = x \to (y \to z)$ and $(x \odot y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)$.

(7) $x \to (y \Rightarrow z) = y \Rightarrow (x \to z)$ and $x \Rightarrow (y \to z) = y \to (x \Rightarrow z)$.

(8) $x \odot (x \Rightarrow y) \leq y$ and $(x \to y) \odot x \leq y$.

(9) $(x \Rightarrow y) \odot (y \Rightarrow z) \leq x \Rightarrow z$ and $(y \to z) \odot (x \to y) \leq x \to z$.

(10) $(x \Rightarrow z) \leq (y \odot x) \Rightarrow (y \odot z)$ and $(x \to z) \leq (x \odot y) \to (z \odot y)$.

(11) $(x \Rightarrow y) \leq (y \Rightarrow z) \to (x \Rightarrow z)$ and $(y \Rightarrow z) \leq (x \Rightarrow y) \Rightarrow (x \Rightarrow z)$.

(12) $x_i \to y_i \leq (\bigwedge_{i\in\Gamma} x_i) \to (\bigwedge_{i\in\Gamma} y_i)$ for $\to \in \{\to, \Rightarrow\}$.

(13) $x_i \to y_i \leq (\bigvee_{i\in\Gamma} x_i) \to (\bigvee_{i\in\Gamma} y_i)$ for $\to \in \{\to, \Rightarrow\}$.

(14) $x \to y = \top$ iff $x \leq y$.

(15) $x \to y = y^0 \Rightarrow x^0$ and $x \Rightarrow y = y^* \to x^*$.

(16) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(17) $\bigwedge_{i\in\Gamma} x_i^0 = (\bigvee_{i\in\Gamma} x_i)^0$ and $\bigvee_{i\in\Gamma} x_i^0 = (\bigwedge_{i\in\Gamma} x_i)^0$.

Definition 1.4.[7] Let X be a set. A function $e_X : X \times X \to L$ is called a *right preorder* if it satisfies:

(E1) $e_X(x, x) = \top$ for all $x \in X$,

(R) $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$.

A function e_X is called a *left preorder* if it satisfies (E1) and

(L) $e_X(y, z) \odot e_X(x, y) \leq e_X(x, z)$, for all $x, y, z \in X$.

The pair (X, e_X) is a right preorder (resp. left-preorder) set.

Remark 1.5.(1) We define two functions $e_L^{\Uparrow}, e_L^{\uparrow} : L \times L \to L$ as $e_L^{\Uparrow}(x, y) = x \Rightarrow y$ and $e_L^{\uparrow}(x, y) = x \to y$. Then $e_L^{\Uparrow}$ is a right preorder and $e_L^{\uparrow}$ is a left preorder from Lemma 1.3 (9).

(2) We define two functions $e_{L^X}^{\Uparrow}, e_{L^X}^{\uparrow} : L^X \times L^X \to L$ as

$$e_{L^X}^{\Uparrow}(A, B) = \bigwedge_{x\in X}(A(x) \Rightarrow B(x)),\ e_{L^X}^{\uparrow}(A, B) = \bigwedge_{x\in X}(A(x) \to B(x)).$$

Then $e_{L^X}^{\Uparrow}$ is a right preorder and $e_{L^X}^{\uparrow}$ is a left preorder from Lemma 1.3 (9).

Definition 1.6.[7,10] Let X and Y be two sets. Let $F, H : L^X \to L^Y$ and $G, K : L^Y \to L^X$ be operators.

(1) The pair (F, G) is called a *residuated connection* between X and Y if for $A \in L^X$ and $B \in L^Y$, $F(A) \leq B$ iff $A \leq G(B)$.

(2) The pair (H, K) is called a *Galois connection* between X and Y if for $A \in L^X$ and $B \in L^Y$, $B \leq H(A)$ iff $A \leq K(B)$.

Definition 1.7.[7] (1) A map $G : L^X \to L^Y$ is a *right isotone map* if for all $A, B \in L^X$, $e^{\Uparrow}_{L^X}(A, B) \leq e^{\Uparrow}_{L^Y}(G(A), G(B))$.

(2) A map $G : L^X \to L^Y$ is a *left isotone map* if for all $A, B \in L^X$, $e^{\uparrow}_{L^X}(A, B) \leq e^{\uparrow}_{L^Y}(G(A), G(B))$.

(3) A map $G : L^X \to L^Y$ is a *right antitone map* if for all $A, B \in L^X$, $e^{\uparrow}_{L^X}(A, B) \leq e^{\Uparrow}_{L^Y}(G(B), G(A))$.

(4) A map $G : L^X \to L^Y$ is a *left antitone map* if for all $A, B \in L^X$, $e^{\Uparrow}_{L^X}(A, B) \leq e^{\uparrow}_{L^Y}(G(B), G(A))$.

Definition 1.8.[7] A map $C : L^X \to L^X$ is called a *right (resp. left) closure operator* if it satisfies the following conditions:

(C1) $A \leq C(A)$, for all $A \in L^X$.

(C2) $C(C(A)) = C(A)$, for all $A \in L^X$.

(C3) C is a right (resp. left) isotone map.

A map $I : L^X \to L^X$ is called a *right (resp. left) interior operator* if it satisfies the following conditions:

(I1) $I(A) \leq A$, for all $A \in L^X$.

(I2) $I(I(A)) = I(A)$, for all $A \in L^X$.

(I3) I is a right (resp. left) isotone map.

Theorem 1.9.[7]*Let $G : L^X \to L^Y$ and $H : L^Y \to L^X$ be two maps.*

(1) *A pair (G, H) is a residuated connection with two right isotone maps G and H iff for all $A \in L^X$ and $B \in L^Y$, $e^{\Uparrow}_{L^Y}(G(A), B) = e^{\Uparrow}_{L^X}(A, H(B))$.*

(2) *A pair* (G,H) *is a residuated connection with two left isotone maps* G *and* H *iff for all* $A\in L^X$ *and* $B\in L^Y$, $e^{\uparrow}_{L^Y}(G(A),B)=e^{\uparrow}_{L^X}(A,H(B))$.

(3) *A pair* (G,H) *is a Galois connection with right antitone map* G *and left antitone map* H *iff for all* $A\in L^X$ *and* $B\in L^Y$, $e^{\uparrow}_{L^X}(A,H(B))=e^{\Uparrow}_{L^Y}(B,G(A))$.

(4) *A pair* (G,H) *is a Galois connection with left antitone map* G *and right antitone map* H *iff for all* $A\in L^X$ *and* $B\in L^Y$, $e^{\Uparrow}_{L^X}(A,H(B))=e^{\uparrow}_{L^Y}(B,G(A))$.

Theorem 1.10.[7]*Let* $G:L^X\to L^Y$ *and* $H:L^Y\to L^X$ *be right isotone maps with a residuated connection* (G,H). *Then the following statements hold:*

(1) $H\circ G$ *is a right closure operator.*

(2) $G\circ H$ *is a right interior operator.*

Corollary 1.11.[7]*Let* $G:L^X\to L^Y$ *and* $H:L^Y\to L^X$ *be left isotone maps with a residuated connection* (G,H). *Then the following statements hold:*

(1) $H\circ G$ *is a left closure operator.*

(2) $G\circ H$ *is a left interior operator.*

Theorem 1.12.[7]*Let* $G:L^X\to L^Y$ *be a right antitone map and* $H:L^Y\to L^X$ *be a left antitone map with a Galois connection* (G,H). *Then*

(1) $H\circ G$ *is a left closure operator.*

(2) $G\circ H$ *is a right closure operator.*

Corollary 1.13.[7]*Let* $G:L^X\to L^Y$ *be a left antitone map and* $H:L^Y\to L^X$ *be a right antitone map with a Galois connection* (G,H). *Then*

(1) $H\circ G$ *is a right closure operator.*

(2) $G\circ H$ *is a left closure operator.*

2. Galois connections and right closure operators

Theorem 2.1._Let_ (X,e_X) *be a left preordered set. Let* $(e_X)^{\Uparrow},(e_X)^{\odot},:L^X\to L^X$ *be maps as follows:*

$$(e_X)^{\Uparrow}(A)(x)=\bigwedge_{y\in X}(e_X(x,y)\Rightarrow A(y)),\quad (e_X)^{\odot}(A)(x)=\bigvee_{y\in X}(e_X(y,x)\odot A(y)).$$

Then the following statements hold.

(1) $(e_X)^{\Uparrow}$ *is a right interior operator.*

(2) $(e_X)^{\odot}$ *is a right closure operator.*

(3) $e^{\Uparrow}_{L^X}((e_X)^{\odot}(A), B) = e^{\Uparrow}_{L^X}(A, (e_X)^{\Uparrow}(B))$.

(4) $(e_X)^{\Uparrow} \circ (e_X)^{\odot} = (e_X)^{\odot}$.

(5) $(e_X)^{\odot} \circ (e_X)^{\Uparrow} = (e_X)^{\Uparrow}$.

Proof. (1) (I1) $(e_X)^{\Uparrow}(A)(x) \leq (e_X(x,x) \Rightarrow A(x)) = A(x)$.

(I2) Since e_X is a left preorder, $\bigvee_{y\in X}((e_X(y,z) \odot e_X(x,y)) = e_X(x,z)$. Thus

$$\begin{aligned}
&(e_X)^{\Uparrow}((e_X)^{\Uparrow}(B))(x)\\
&= \bigwedge_{y\in X}(e_X(x,y) \Rightarrow (e_X)^{\Uparrow}(B)(y))\\
&= \bigwedge_{y\in X}\Big(e_X(x,y) \Rightarrow \bigwedge_{z\in X}(e_X(y,z) \Rightarrow B(z))\Big)\\
&= \bigwedge_{y\in X}\bigwedge_{z\in X}\Big((e_X(y,z) \odot e_X(x,y)) \Rightarrow B(z)\Big) \text{ (by Lemma 1.3(6))}\\
&= \bigwedge_{z\in X}\Big((\bigvee_{y\in X}((e_X(y,z) \odot e_X(x,y))) \Rightarrow B(z)\Big)\\
&= \bigwedge_{z\in X}\Big(e_X(x,z) \Rightarrow B(z)\Big)\\
&= (e_X)^{\Uparrow}(B)(x).
\end{aligned}$$

(I3) Since $(e_X(x,y) \Rightarrow A(y)) \odot (A(y) \Rightarrow B(y)) \leq e_X(x,y) \Rightarrow B(y)$, then

$$e^{\Uparrow}_{L^X}(A,B) \leq e^{\Uparrow}_{L^X}((e_X)^{\Uparrow}(A), (e_X)^{\Uparrow}(B)).$$

Thus $(e_X)^{\Uparrow}$ is a right interior operator.

(2) (C1) $A \leq (e_X)^{\odot}(A)$.

(C2) Since e_X is a left preorder, $\bigvee_{y\in X}((e_X(y,z) \odot e_X(x,y)) = e_X(x,z)$.

$$\begin{aligned}
(e_X)^{\odot}((e_X)^{\odot}(A))(y) &= \bigvee_{z\in X}(e_X(z,y) \odot (e_X)^{\odot}(A)(z))\\
&= \bigvee_{z\in X}((e_X(z,y) \odot \bigvee_{x\in X}(e_X(x,z) \odot A(x)))\\
&= \bigvee_{z\in X}(\bigvee_{x\in X}((e_X(z,y) \odot e_X(x,z)) \odot A(x)))\\
&= \bigvee_{x\in X}(\bigvee_{z\in X}((e_X(z,y) \odot e_X(x,z)) \odot A(x)))\\
&= \bigvee_{x\in X}((e_X(x,y) \odot A(x))\\
&= (e_X)^{\odot}(A)(y).
\end{aligned}$$

(C3) Since $e_X(x,y) \odot A(x) \odot (A(x) \Rightarrow B(x)) \le e_X(x,y) \odot B(y)$, then

$$(A(x) \Rightarrow B(x)) \le (e_X(x,y) \odot A(x)) \Rightarrow (e_X(x,y) \odot B(y)),$$

$$e^{\Uparrow}_{L^X}(A,B) \le e^{\Uparrow}_{L^X}((e_X)^{\odot}(A), (e_X)^{\odot}(B)).$$

Thus $(e_X)^{\odot}$ is a right closure operator.

(3)

$$\begin{aligned} e^{\Uparrow}_{L^X}((e_X)^{\odot}(A), B) &= \bigwedge_{y\in X}((e_X)^{\odot}(A)(y) \Rightarrow B(y)) \\ &= \bigwedge_{y\in X}\Big(\bigvee_{x\in X}(e_X(x,y)\odot A(x)) \Rightarrow B(y)\Big) \\ &= \bigwedge_{y\in X}\bigwedge_{x\in X}\Big((e_X(x,y)\odot A(x)) \Rightarrow B(y)\Big) \\ &= \bigwedge_{x\in X}\bigwedge_{y\in X}\Big(A(x) \Rightarrow (e_X(x,y) \Rightarrow B(y))\Big) \\ &= \bigwedge_{x\in X}\Big(A(x) \Rightarrow \bigwedge_{y\in Y}(e_X(x,y) \Rightarrow B(y))\Big) \\ &= \bigwedge_{x\in X}(A(x) \Rightarrow (e_X)^{\Uparrow}(B)(x)) \\ &= e^{\Uparrow}_{L^X}(A, (e_X)^{\Uparrow}(B)). \end{aligned}$$

(4) By (1), since $(e_X)^{\odot} \ge (e_X)^{\Uparrow} \circ (e_X)^{\odot}$, we only show $(e_X)^{\odot} \le (e_X)^{\Uparrow} \circ (e_X)^{\odot}$ from:

$$\begin{aligned} \top &= e^{\Uparrow}_{L^X}((e_X)^{\odot}(A), (e_X)^{\odot}(A)) = e^{\Uparrow}_{L^X}((e_X)^{\odot}((e_X)^{\odot}(A)), (e_X)^{\odot}(A)) \\ &= e^{\Uparrow}_{L^X}((e_X)^{\odot}(A), (e_X)^{\Uparrow}((e_X)^{\odot}(A))) \ \text{ (by (3))}. \end{aligned}$$

(5) By (2), since $(e_X)^{\odot} \circ (e_X)^{\Uparrow} \ge (e_X)^{\Uparrow}$, we only show $(e_X)^{\odot} \circ (e_X)^{\Uparrow} \le (e_X)^{\Uparrow}$ from:

$$\begin{aligned} \top &= e^{\Uparrow}_{L^X}((e_X)^{\Uparrow}(B), (e_X)^{\Uparrow}(B)) = e^{\Uparrow}_{L^X}((e_X)^{\Uparrow}(B), (e_X)^{\Uparrow}((e_X)^{\Uparrow}(B))) \\ &= e^{\Uparrow}_{L^X}((e_X)^{\odot}((e_X)^{\Uparrow}(B)), (e_X)^{\Uparrow}(B)) \ \text{ (by (3))}. \end{aligned}$$

Theorem 2.2. *Let (X, e_X) be a right preordered set. Let $(e_X)^{\uparrow}, ^{\odot}(e_X), : L^X \to L^X$ be maps as follows:*

$$(e_X)^{\uparrow}(A)(x) = \bigwedge_{y\in X}(e_X(x,y) \to A(y)), \quad ^{\odot}(e_X)(A)(x) = \bigvee_{y\in X}(A(y)\odot e_X(y,x)).$$

Then the following statements hold.

(1) $(e_X)^{\uparrow}$ *is a left interior operator.*

(2) $^{\odot}(e_X)$ *is a left closure operator.*

(3) $e^{\uparrow}_{L^X}(^{\odot}(e_X)(A), B) = e^{\uparrow}_{L^X}(A, (e_X)^{\uparrow}(B))$.

(4) $(e_X)^{\uparrow} \circ ^{\odot}(e_X) = ^{\odot}(e_X)$.

(5) ${}^{\odot}(e_X)\circ(e_X)^{\uparrow}=(e_X)^{\uparrow}$.

Proof. (1) (I1) $(e_X)^{\uparrow}(A)\leq A$. (I2) Since e_X is a right preorder, $\bigvee_{y\in X}((e_X(x,y)\odot e_X(y,z))=e_X(x,z)$. Thus

$$\begin{aligned}
&(e_X)^{\uparrow}((e_X)^{\uparrow}(B))(x)\\
&=\bigwedge_{y\in X}(e_X(x,y)\to(e_X)^{\uparrow}(B)(y))\\
&=\bigwedge_{y\in X}\Big(e_X(x,y)\to\bigwedge_{z\in X}(e_X(y,z)\to B(z))\Big)\\
&=\bigwedge_{y\in X}\bigwedge_{z\in X}\Big((e_X(x,y)\odot e_X(y,z))\to B(z)\Big)\text{ (by Lemma 1.3(6))}\\
&=\bigwedge_{z\in X}\Big((\bigvee_{y\in X}((e_X(x,y)\odot e_X(y,z)))\to B(z)\Big)\\
&=\bigwedge_{z\in X}\Big(e_X(x,z)\to B(z)\Big)\\
&=(e_X)^{\uparrow}(B)(x).
\end{aligned}$$

(I3) Since $(A(y)\to B(y))\odot(e_X(x,y)\to A(y))\leq e_X(x,y)\to B(y)$, then

$$e^{\uparrow}_{L^X}(A,B)\leq e^{\uparrow}_{L^X}((e_X)^{\uparrow}(A),(e_X)^{\uparrow}(B)).$$

Thus $(e_X)^{\uparrow}$ is a left interior operator.

(2) (C1) $A\leq(e_X)^{\odot}(A)$.

(C2) Since e_X is a right preorder, $\bigvee_{y\in X}((e_X(x,y)\odot e_X(y,z))=e_X(x,z)$. Thus

$$\begin{aligned}
{}^{\odot}(e_X)({}^{\odot}(e_X)(A))(y)&=\bigvee_{z\in X}({}^{\odot}(e_X)(A)(z)\odot e_X(z,y))\\
&=\bigvee_{z\in X}(\bigvee_{x\in X}(A(x)\odot e_X(x,z))\odot(e_X(z,y))\\
&=\bigvee_{x\in X}(A(x)\odot\bigvee_{z\in X}((e_X(x,z)\odot e_X(z,y)))\\
&=\bigvee_{x\in X}(A(x)\odot e_X(x,y))\\
&={}^{\odot}(e_X)(A)(y).
\end{aligned}$$

(C3) Since $(A(y)\to B(y))\odot A(y)\odot e_X(x,y)\leq B(y)\odot e_X(x,y)$,

$$e^{\uparrow}_{L^X}(A,B)\leq e^{\uparrow}_{L^X}({}^{\odot}(e_X)(A),{}^{\odot}(e_X)(B)).$$

Thus ${}^{\odot}(e_X)$ is a left closure operator.

(3)

$$\begin{aligned} e^{\uparrow}_{L^X}(^{\odot}(e_X)(A), B) &= \bigwedge_{y\in X}(^{\odot}(e_X)(A)(y) \to B(y)) \\ &= \bigwedge_{y\in X}\Big(\bigvee_{x\in X}(A(x)\odot e_X(x,y)) \to B(y)\Big) \\ &= \bigwedge_{y\in X}\bigwedge_{x\in X}\Big((A(x)\odot e_X(x,y)) \to B(y)\Big) \\ &= \bigwedge_{x\in X}\bigwedge_{y\in X}\Big(A(x) \to (e_X(x,y) \to B(y))\Big) \\ &= \bigwedge_{x\in X}\Big(A(x) \to \bigwedge_{y\in Y}(e_X(x,y) \to B(y))\Big) \\ &= \bigwedge_{x\in X}(A(x) \to (e_X)^{\uparrow}(B)(x)) \\ &= e^{\uparrow}_{L^X}(A, (e_X)^{\uparrow}(B)). \end{aligned}$$

(4) By (1), since $^{\odot}(e_X) \geq (e_X)^{\uparrow} \circ^{\odot} (e_X)$, we only show $^{\odot}(e_X) \leq (e_X)^{\uparrow} \circ^{\odot} (e_X)$ from:

$$\begin{aligned} \top &= e^{\uparrow}_{L^X}(^{\odot}(e_X)(A), ^{\odot}(e_X)(A)) = e^{\uparrow}_{L^X}(^{\odot}(e_X)(^{\odot}(e_X)(A)), ^{\odot}(e_X)(A)) \\ &= e^{\uparrow}_{L^X}(^{\odot}(e_X)(A), (e_X)^{\uparrow}(^{\odot}(e_X)(A))). \end{aligned}$$

(5) By (2), since $^{\odot}(e_X) \circ (e_X)^{\uparrow} \geq (e_X)^{\uparrow}$, we only show $^{\odot}(e_X) \circ (e_X)^{\uparrow} \leq (e_X)^{\uparrow}$ from:

$$\begin{aligned} \top &= e^{\uparrow}_{L^X}((e_X)^{\uparrow}(B), (e_X)^{\uparrow}(B)) = e^{\uparrow}_{L^X}((e_X)^{\uparrow}(B), (e_X)^{\uparrow}((e_X)^{\uparrow}(B))) \\ &= e^{\uparrow}_{L^X}(^{\odot}(e_X)((e_X)^{\uparrow}(B)), (e_X)^{\uparrow}(B)). \end{aligned}$$

Theorem 2.3. *For each $A \in L^X$ and $B \in L^Y$ and $R \in L^{X\times Y}$, we define: $R^{\rightarrow}, R^{\Rightarrow} : L^X \to L^Y$ is defined as:*

$$R^{\rightarrow}(A)(y) = \bigwedge_{x\in X}(A(x) \to R(x,y)), \quad R^{\Rightarrow}(A)(y) = \bigwedge_{x\in X}(A(x) \Rightarrow R(x,y))$$

and $R^{\Leftarrow}, R^{\leftarrow} : L^Y \to L^X$ is defined as:

$$R^{\leftarrow}(B)(x) = \bigwedge_{y\in Y}(B(y) \to R(x,y)), \quad R^{\Leftarrow}(B)(x) = \bigwedge_{y\in Y}(B(y) \Rightarrow R(x,y)).$$

(1) *$R^{\Rightarrow}$ is a left antitone map and $R^{\rightarrow}$ is a right antitone map.*

(2) *$R^{\Leftarrow}$ is a left antitone map and $R^{\leftarrow}$ is a right antitone map.*

(3) *Let $R^{\Rightarrow}$ be a left antitone map and $R^{\leftarrow}$ a right antitone map with a Galois connection $(R^{\Rightarrow}, R^{\leftarrow})$.*

(4) *$R^{\Rightarrow} \circ R^{\leftarrow} : L^Y \to L^Y$ is a left closure operator and $R^{\leftarrow} \circ R^{\Rightarrow} : L^X \to L^X$ is a right closure operator.*

(5) *$e^{\Uparrow}_{L^Y}(B, R^{\rightarrow}(A)) = e^{\uparrow}_{L^X}(A, R^{\Leftarrow}(B))$.*

(6) *$R^{\rightarrow} \circ R^{\Leftarrow} : L^Y \to L^Y$ is a right closure operator and $R^{\Leftarrow} \circ R^{\rightarrow} : L^X \to L^X$ is a left closure operator.*

Proof. (1) Since $(A(x) \Rightarrow B(x)) \odot (B(x) \Rightarrow R(x,y)) \le (A(x) \Rightarrow R(x,y))$,

$$e^{\Uparrow}_{L^X}(A,B) \le e^{\uparrow}_{L^Y}(R^{\Rightarrow}(B), R^{\Rightarrow}(A)).$$

(2) Since $(B(y) \to R(x,y)) \odot (A(y) \to B(y)) \le (A(y) \to R(x,y))$,

$$e^{\uparrow}_{L^Y}(A,B) \le e^{\Uparrow}_{L^X}(R^{\leftarrow}(B), R^{\leftarrow}(A)).$$

(3) From Theorem 1.9(4), we only show that $e^{\Uparrow}_{L^X}(A, R^{\leftarrow}(B)) = e^{\uparrow}_{L^Y}(B, R^{\Rightarrow}(A))$ from:

$$\begin{aligned}
& e^{\Uparrow}_{L^X}(A, R^{\leftarrow}(B)) \\
&= \bigwedge_{x\in X}(A(x) \Rightarrow R^{\leftarrow}(B)(x)) \\
&= \bigwedge_{x\in X}(A(x) \Rightarrow \bigwedge_{y\in X}(B(y) \to R(x,y)) \\
&= \bigwedge_{x\in X}\bigwedge_{y\in X}(A(x) \Rightarrow (B(y) \to R(x,y)) \\
&= \bigwedge_{y\in X}(B(y) \to \bigwedge_{x\in X}(A(x) \Rightarrow R(x,y)) \text{ (by Lemma 1.3(7))} \\
&= \bigwedge_{y\in X}(B(y) \to R^{\Rightarrow}(A)(y)) = e^{\uparrow}_{L^Y}(B, R^{\Rightarrow}(A)).
\end{aligned}$$

(5) From Theorem 1.9(3), $e^{\Uparrow}_{L^Y}(B, R^{\rightarrow}(A)) = e^{\uparrow}_{L^X}(A, R^{\Leftarrow}(B))$ from

$$\begin{aligned}
e^{\uparrow}_{L^X}(A, R^{\Leftarrow}(B)) &= \bigwedge_{x\in X}(A(x) \to R^{\Leftarrow}(B)(x)) \\
&= \bigwedge_{x\in X}(A(x) \to \bigwedge_{y\in X}(B(y) \Rightarrow R(x,y)) \\
&= \bigwedge_{x\in X}\bigwedge_{y\in X}(A(x) \to (B(y) \Rightarrow R(x,y)) \\
&= \bigwedge_{y\in X}(B(y) \Rightarrow \bigwedge_{x\in X}(A(x) \to R(x,y)) \\
&= \bigwedge_{y\in X}(B(y) \Rightarrow R^{\rightarrow}(A)(y)) = e^{\Uparrow}_{L^Y}(B, R^{\rightarrow}(B)).
\end{aligned}$$

(4) and (6) are proved from Corollary 1.12 and Theorem 1.13, respectively.

Theorem 2.4. *Let $F, G : L^X \to L^X$ be maps such that*

$$e^{\Uparrow}_{L^X}(F(A), B) = e^{\Uparrow}_{L^X}(A, G(B)).$$

Then the following statements are equivalent.

(1) *F is a right interior operator.*

(2) *G is a right closure operator.*

(3) *$F \circ G = F$.*

(4) $G \circ F = G$.

Proof. Since $e^{\Uparrow}_{L^X}(F(A), B) = e^{\Uparrow}_{L^X}(A, G(B))$, by Theorem 1.9(1), F and G are right isotone maps.

(1) $\Rightarrow$(2). Since $\top = e^{\Uparrow}_{L^X}(F(A), A) = e^{\Uparrow}_{L^X}(A, G(A))$, then $A \leq G(A)$.

$$\begin{aligned} &e^{\Uparrow}_{L^X}(G(G(A)), G(A)) = e^{\Uparrow}_{L^X}(F(G(G(A))), A) = e^{\Uparrow}_{L^X}(F(F(G(G(A)))), A) \\ &= e^{\Uparrow}_{L^X}(F(G(G(A))), G(A)) = e^{\Uparrow}_{L^X}(G(G(A)), G(G(A))) = \top. \end{aligned}$$

Thus G is a right closure operator.

(2) $\Rightarrow$(3). Since F is a right isotone map, $\top = e^{\Uparrow}_{L^X}(A, G(A)) \leq e^{\Uparrow}_{L^X}(F(A), F(G(A)))$. Then $F(A) \leq F(G(A))$. Moreover, $F(A) = F(G(A))$ from:

$$\begin{aligned} &e^{\Uparrow}_{L^X}(F(G(A)), F(A)) = e^{\Uparrow}_{L^X}(G(A), G(F(A))) = e^{\Uparrow}_{L^X}(G(A), G(G(F(A)))) \\ &\geq e^{\Uparrow}_{L^X}(A, G(F(A))) = \top. \ (\text{ } G \text{ is a right isotone map}) \end{aligned}$$

(3) $\Rightarrow$(4). Let $F \circ G = F$. Then $G \circ F \circ G = G \circ F$. Since $G \circ F \circ G \geq G$ and $F \circ G(A) \leq A$ implies $G \circ F \circ G(A) \leq G(A)$. So, $G \circ F = G \circ F \circ G = G$.

(4) $\Rightarrow$(3). It follows from $F \circ G \circ F = F$.

(3) and (4) $\Rightarrow$(1). $e^{\Uparrow}_{L^X}(F(A), A) \geq e^{\Uparrow}_{L^X}(F(A), F(G(A))) \odot e^{\Uparrow}_{L^X}(F(G(A)), A) = \top \odot \top = \top$. Moreover, $e^{\Uparrow}_{L^X}(F(A), F(F(A))) = e^{\Uparrow}_{L^X}(A, G(F(F(A)))) = e^{\Uparrow}_{L^X}(A, G(F(A))) = \top$.

The following corollary are similarly proved as Theorem 2.4.

Corollary 2.5. *Let $F, G : L^X \to L^X$ be maps such that*

$$e^{\uparrow}_{L^X}(F(A), B) = e^{\uparrow}_{L^X}(A, G(B)).$$

Then the following statements are equivalent.

(1) *F is a left interior operator.*

(2) *G is a left closure operator.*

(3) $F \circ G = F$.

(4) $G \circ F = G$.

Theorem 2.6. *Let $F, G : L^X \to L^X$ be maps such that*

$$e^{\uparrow}_{L^X}(F(A), B) = e^{\uparrow}_{L^X}(A, G(B)).$$

Then the following statements are equivalent.

(1) *F is a left closure operator.*

(2) *G is a left interior operator.*

(3) $G \circ F = F$.

(4) $F \circ G = G$.

Proof. Since $e^{\uparrow}_{L^X}(F(A), B) = e^{\uparrow}_{L^X}(A, G(B))$, by Theorem 1.9(2), F and G are left isotone maps.

(1) $\Rightarrow$(3). Since $F(A) = F(G(F(A)))$, we have

$$e^{\uparrow}_{L^X}(G(F(A)), F(A)) = e^{\uparrow}_{L^X}(G(F(A)), F(G(F(A)))) = \top.$$

Then $G(F(A)) \leq F(A)$. Moreover,

$$e^{\uparrow}_{L^X}(F(A), G(F(A))) = e^{\uparrow}_{L^X}(F(F(A)), F(G(F(A)))) = e^{\uparrow}_{L^X}(F(F(A)), F(A)) = \top.$$

Then $G(F(A)) \geq F(A)$.

(3) $\Rightarrow$(1). Since F is a left isotone map and $A \leq G(F(A))$,

$$e^{\uparrow}_{L^X}(A, F(A)) = e^{\uparrow}_{L^X}(A, G(F(A))) = \top.$$

Then $A \leq F(A)$.

$$e^{\uparrow}_{L^X}(F(F(A)), F(A)) = e^{\uparrow}_{L^X}(F(A), G(F(A))) = e^{\Uparrow}_{L^X}(F(A), F(A)) = \top.$$

Thus $F(A) = F(F(A))$.

(3) $\Leftrightarrow$(4). It follows from $G \circ F \circ G = G$ and $F \circ G \circ F = F$.

(2) $\Leftrightarrow$(4). We prove a similar method as (1) $\Leftrightarrow$(3).

The following corollary are similarly proved as Theorem 2.5.

Corollary 2.7. *Let $F, G : L^X \to L^X$ be maps such that*

$$e^{\Uparrow}_{L^X}(F(A), B) = e^{\Uparrow}_{L^X}(A, G(B)).$$

Then the following statements are equivalent.

(1) *F is a right closure operator.*

(2) *G is a right interior operator.*

(3) $G \circ F = F$.

(4) $F \circ G = G$.

Example 2.8. Let $K = \{(x, y) \in R^2 \mid x > 0\}$ be a set and we define an operation $\otimes : K \times K \to K$ as follows:

$$(x_1, y_1) \otimes (x_2, y_2) = (x_1x_2, x_1y_2 + y_1).$$

Then $(K, \otimes)$ is a group with $e = (1, 0)$, $(x, y)^{-1} = (\frac{1}{x}, -\frac{y}{x})$.

We have a positive cone $P = \{(a, b) \in R^2 \mid a = 1, b \geq 0 \text{ , or } a > 1\}$ because $P \cap P^{-1} = \{(1, 0)\}$, $P \odot P \subset P$, $(a, b)^{-1} \odot P \odot (a, b) = P$ and $P \cup P^{-1} = K$. For $(x_1, y_1), (x_2, y_2) \in K$, we define

$$\begin{aligned} (x_1, y_1) \leq (x_2, y_2) \quad &\Leftrightarrow (x_1, y_1)^{-1} \odot (x_2, y_2) \in P,\ (x_2, y_2) \odot (x_1, y_1)^{-1} \in P \\ &\Leftrightarrow x_1 < x_2 \text{ or } x_1 = x_2, y_1 \leq y_2. \end{aligned}$$

Then $(K, \leq \otimes)$ is a lattice-group.

The structure $(L, \odot, \Rightarrow, \to, (\frac{1}{2}, 1), (1, 0))$ is a generalized residuated lattice with strong negation where $\perp = (\frac{1}{2}, 1)$ is the least element and $\top = (1, 0)$ is the greatest element from the following statements:

$$\begin{aligned} (x_1, y_1) \odot (x_2, y_2) \quad &= (x_1, y_1) \otimes (x_2, y_2) \vee (\tfrac{1}{2}, 1) = (x_1x_2, x_1y_2 + y_1) \vee (\tfrac{1}{2}, 1), \\ (x_1, y_1) \Rightarrow (x_2, y_2) \quad &= ((x_1, y_1)^{-1} \otimes (x_2, y_2)) \wedge (1, 0) = (\tfrac{x_2}{x_1}, \tfrac{y_2 - y_1}{x_1}) \wedge (1, 0), \\ (x_1, y_1) \to (x_2, y_2) \quad &= ((x_2, y_2) \otimes (x_1, y_1)^{-1}) \wedge (1, 0) = (\tfrac{x_2}{x_1}, -\tfrac{x_2y_1}{x_1} + y_2) \wedge (1, 0). \end{aligned}$$

Furthermore, we have $(x, y) = (x, y)^{*\circ} = (x, y)^{\circ *}$ from:

$$(x, y)^* = (x, y) \Rightarrow (\frac{1}{2}, 1) = (\frac{1}{2x}, \frac{1 - y}{x}),$$

$$(x, y)^{*\circ} = (\frac{1}{2x}, \frac{1 - y}{x}) \to (\frac{1}{2}, 1) = (x, y).$$

Let $X = \{a, b, c\}$ be a set. Define $(e^1_X(a, b)), (e^2_X(a, b)) \in L^{X \times X}$ as

$$e^1_X = \begin{pmatrix} (1, 0) & (\frac{5}{8}, \frac{5}{2}) & (\frac{5}{6}, \frac{5}{3}) \\ (\frac{5}{7}, \frac{30}{7}) & (1, 0) & (\frac{5}{8}, -\frac{5}{4}) \\ (1, -2) & (\frac{5}{7}, \frac{10}{3}) & (1, 0) \end{pmatrix} e^2_X = \begin{pmatrix} (1, 0) & (\frac{2}{3}, 5) & (\frac{5}{6}, 1) \\ (\frac{5}{7}, 3) & (1, 0) & (\frac{6}{7}, 4) \\ (\frac{5}{6}, -1) & (\frac{3}{4}, 2) & (1, 0) \end{pmatrix}$$

We easily show that e^1_X is a right partial order and e^2_X is a left partial order. But e^2_X is not a right partial order because

$$e^2_X(b,c)\odot e^2_X(c,a) = (\frac{6}{7},4)\odot(\frac{5}{6},-1) = (\frac{5}{7},\frac{22}{7}) \not\leq e^2_X(b,a) = (\frac{5}{7},3).$$

For $A = ((\frac{2}{3},1),(\frac{3}{5},-1),(1,-1))^t$,

$$^{\odot}(e^2_X)(A) = ((\frac{5}{6},-2),(\frac{3}{4},1),(1,-1))^t,$$

$$(e^2_X)^{\uparrow}(^{\odot}(e^2_X)(A)) =^{\odot} (e^2_X)(A),$$

$$(e^2_X)^{\odot}(A) = ((\frac{5}{6},-\frac{11}{6}),(\frac{3}{4},1),(1,-1))^t,$$

$$(e^2_X)^{\Uparrow}((e^2_X)^{\odot}(A)) = ((\frac{5}{6},-\frac{11}{6}),(\frac{3}{4},1),(1,-\frac{4}{3}))^t \neq (e^2_X)^{\odot}(A).$$

Since e^2_X is not a right partial order, by Theorem 2.1 (4), $(e^2_X)^{\Uparrow}((e^2_X)^{\odot}(A)) \neq (e^2_X)^{\odot}(A)$.

Let $X = \{a,b,c\}$ and $Y = \{u,v\}$ be sets. Define $R \in L^{X\times Y}$ as

$$R = \begin{pmatrix} (1,0) & (\frac{5}{8},\frac{5}{2}) \\ (\frac{5}{7},\frac{30}{7}) & (\frac{5}{8},-\frac{5}{4}) \\ (\frac{1}{2},2) & (\frac{5}{6},\frac{10}{3}) \end{pmatrix}$$

For $A = ((\frac{2}{3},1),(\frac{1}{2},2),(\frac{2}{3},-1))^t$,

$$R^{\rightarrow}(A) = ((\frac{3}{4},\frac{11}{4}),(\frac{15}{16},-\frac{25}{16}))^t, \quad R^{\Rightarrow}(A) = ((\frac{3}{4},\frac{9}{2}),(\frac{15}{16},\frac{9}{4}))^t$$

$$R^{\Leftarrow}(R^{\rightarrow}(A)) = ((\frac{2}{3},\frac{13}{3}),(\frac{2}{3},\frac{1}{3}),(\frac{2}{3},-1))^t,$$

$$R^{\leftarrow}(R^{\Rightarrow}(A)) = ((\frac{2}{3},\frac{85}{24}),(\frac{2}{3},-\frac{5}{24}),(\frac{2}{3},\frac{1}{6}))^t.$$

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