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L-APPROXIMATIONS AND JOIN PRESERVING OPERATORS

YONG CHAN KIM

Department of Mathematics, Gangneung-Wonju National University, Gangneung, Gangwondo 210-702, Korea

Abstract. In this paper, we show that join (resp. meet, meet join, join meet) preserving maps and upper (resp. lower, meet join, join meet) approximation maps are equivalent in complete residuated lattices. We investigate relations between their maps and fuzzy connections.

Keywords:complete residuated lattices; join (meet) preserving maps, lower and upper approximation operators, meet-join (join-meet) operators; residuated (Galois) connections

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1. Introduction

Pawlak [7,8] introduced rough set theory as a formal tool to deal with imprecision and uncertainty in data analysis. Hájek [4] introduced a complete residuated lattice which is an algebraic structure for many valued logic. By using the concepts of lower and upper approximation operators, information systems and decision rules are investigated in complete residuated lattices [1-3, 9.10]. Bělohlávek [1,2] developed the notion of fuzzy contexts using Galois connections with $R \in L^{X\times Y}$ on a complete residuated lattice. Zhang [11,12] introduced the fuzzy complete lattice which is defined by join and meet on fuzzy

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posets. It is an important mathematical tool for algebraic structure of fuzzy contexts [1-3,5-7].

In this paper, we show that join (resp. meet, meet join, join meet) preserving maps and upper (resp. lower, meet join, join meet) approximation maps are equivalent in complete residuated lattices. We investigate relations between their maps and fuzzy connections.

Definition 1.1. [1,2,4] An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a *complete residuated lattice* if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume $(L, \wedge, \vee, \odot, \rightarrow,^* 0, 1)$ is a complete residuated lattice with the law of double negation;i.e. $x^{**} = x$. We denote $1_x \in L^X$ as $1_x(x) = 1, 1_x(y) = 0$ for otherwise.

Lemma 1.2.[1,4] For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.

(2) $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$.

(3) $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$.

(4) $\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ and $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.

(5) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

(6) $x \odot y = (x \rightarrow y^*)^*$.

(7) $x \odot (x \rightarrow y) \leq y$.

(8) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.

Definition 1.3. [11,12] Let X be a set. A function $e_X : X \times X \rightarrow L$ is called:

(E1) *reflexive* if $e_X(x, x) = 1$ for all $x \in X$,

(E2) *transitive* if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$,

(E3) if $e_X(x, y) = e_X(y, x) = 1$, then $x = y$.

If e satisfies (E1) and (E2), (X, e_X) is a *fuzzy preorder set.* If e satisfies (E1), (E2) and (E3), (X, e_X) is a *fuzzy partially order set* (simply, fuzzy poset).

Example 1.4.(1) We define a function $e_L : L \times L \to L$ as $e_L(x, y) = x \to y$. Then (L, e_L) is a fuzzy poset.

(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(A, B) = \bigwedge_{x \in X}(A(x) \to B(x))$. Then (L^X, e_{L^X}) is a fuzzy poset from Lemma 1.2 (8).

Definition 1.5. [11,12] Let (X, e_X) be a fuzzy poset and $A \in L^X$.

(1) A point x_0 is called a *join* of A, denoted by $x_0 = \sqcup A$, if it satisfies

(J1) $A(x) \leq e_X(x, x_0)$,

(J2) $\bigwedge_{x \in X}(A(x) \to e_X(x, y)) \leq e_X(x_0, y)$.

A point x_1 is called a *meet* of A, denoted by $x_1 = \sqcap A$, if it satisfies

(M1) $A(x) \leq e_X(x_1, x)$,

(M2) $\bigwedge_{x \in X}(A(x) \to e_X(y, x)) \leq e_X(y, x_1)$.

Remark 1.6. Let (X, e_X) be a fuzzy poset and $A \in L^X$.

(1) If x_0 is a join of A, then it is unique because $e_X(x_0, y) = e_X(y_0, y)$ for all $y \in X$, put $y = x_0$ or $y = y_0$, then $e_X(x_0, y_0) = e_X(y_0, x_0) = \top$ implies $x_0 = y_0$. Similarly, if a meet of A exist, then it is unique.

(2) x_0 is a join of A iff $\bigwedge_{x \in X}(A(x) \to e_X(x, y)) = e_X(x_0, y)$.

(3) x_1 is a meet of A iff $\bigwedge_{x \in X}(A(x) \to e_X(y, x)) = e_X(y, x_1)$.

Remark 1.7. Let (L, e_L) be a fuzzy poset and $A \in L^L$.

(1) Since x_0 is a join of A iff $\bigwedge_{x \in L}(A(x) \to e_L(x, y)) = \bigwedge_{x \in L}(A(x) \to (x \Rightarrow y)) = \bigvee_{x \in L}(x \odot A(x)) \to y = e_L(x_0, y) = x_0 \to y$, then $x_0 = \sqcup A = \bigvee_{x \in L}(x \odot A(x))$.

(2) Since x_0 is a join of A iff $\bigwedge_{x \in L}(A(x) \to e_L(x, y) = \bigwedge_{x \in L}(A(x) \to (y \to x)) = \bigwedge_{x \in L}(y \to (A(x) \to x)) = y \to \bigwedge_{x \in L}(A(x) \to x) = y \to \sqcap A$, then $\sqcap A = \bigwedge_{x \in L}(A(x) \to x)$.

Remark 1.8. Let (L^X, e_{L^X}) be a fuzzy poset and $\Phi \in L^{L^X}$.

(1) Since $\bigwedge_{A \in L^X}(\Phi(A) \to e_{L^X}(A, B)) = e_{L^X}(\bigvee_{A \in L^X}(\Phi(A) \odot A), B) = e_{L^X}(\sqcup\Phi, B)$, then $\sqcup\Phi = \bigvee_{A \in L^X}(\Phi(A) \odot A)$.

(2) Since $\bigwedge_{A \in L^X}(\Phi(A) \to e_{L^X}(B, A) = \bigwedge_{A \in L^X} e_{L^X}(B, (\Phi(A) \to A)) = e_{L^X}(B, \bigwedge_{A \in L^X}(\Phi(A) \to A))$, then $\sqcap\Phi = \bigwedge_{A \in L^X}(\Phi(A) \to A)$.

Definition 1.9. [1,3,5,10] Let X and Y be two sets. Let $\mathcal{H}, \mathcal{K}_1, \mathcal{M}_1 : L^X \to L^Y$ and $\mathcal{J}, \mathcal{K}_2, \mathcal{M}_2 : L^Y \to L^X$ be operators.

(1) $(e_{L^X}, \mathcal{H}, \mathcal{J}, e_{L^Y})$ is called a *residuated connection* if for $A \in L^X$ and $B \in L^Y$, $e_{L^Y}(\mathcal{H}(A), B) = e_{L^X}(A, \mathcal{J}(B))$.

(2) $(e_{L^X}, \mathcal{K}_1, \mathcal{K}_2, e_{L^X})$ is called a *Galois connection* if for $A \in L^X$ and $B \in L^Y$, $e_{L^Y}(B, \mathcal{K}_1(A)) = e_{L^X}(A, \mathcal{K}_2(B))$.

(3) $(e_{L^X}, \mathcal{M}_1, \mathcal{M}_2, e_{L^Y})$ is called a *dual Galois connection* if for $A \in L^X$ and $B \in L^Y$, $e_{L^Y}(\mathcal{M}_1(A), B) = e_{L^X}(\mathcal{M}_2(B), A)$.

Definition 1.10. [11,12] Let (L^X, e_{L^X}) and (L^Y, e_{L^Y}) be fuzzy posets and $\mathcal{F} : L^X \to L^Y$ a map. For each $\Phi \in L^{L^X}$, we define $\mathcal{F}^{\to}(\Phi)(B) = \bigvee_{\mathcal{F}(A)=B} \Phi(A)$.

(1) $\mathcal{F}$ is a *join preserving map* if $\mathcal{F}(\sqcup\Phi) = \sqcup\mathcal{F}^{\to}(\Phi)$.

(2) $\mathcal{F}$ is a *meet preserving map* if $\mathcal{F}(\sqcap\Phi) = \sqcap\mathcal{F}^{\to}(\Phi)$ for all $\Phi \in L^{L^X}$.

(3) $\mathcal{F}$ is a *meet-join preserving map* if $\mathcal{F}(\sqcap\Phi) = \sqcup\mathcal{F}^{\to}(\Phi)$ for all $\Phi \in L^{L^X}$.

(4) $\mathcal{F}$ is a *join-meet preserving map* if $\mathcal{F}(\sqcup\Phi) = \sqcap\mathcal{F}^{\to}(\Phi)$ for all for all $\Phi \in L^{L^X}$.

2. L-approximation and join preserving operators

Definition 2.1.[8,9] (1) A map $\mathcal{H} : L^X \to L^Y$ is called an *L-upper approximation operator* iff it satisfies the following conditions

(H1) $\mathcal{H}(\alpha \odot A) = \alpha \odot \mathcal{H}(A)$,

(H2) $\mathcal{H}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{H}(A_i)$.

(2) A map $\mathcal{J} : L^X \to L^Y$ is called an *L-lower approximation operator* iff it satisfies the following conditions

(J1) $\mathcal{J}(\alpha \to A) = \alpha \to \mathcal{J}(A)$,

(J2) $\mathcal{J}(\bigwedge_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{J}(A_i)$.

(3) A map $\mathcal{K} : L^X \to L^Y$ is called an *L-join meet approximation operator* iff it satisfies the following conditions

(K1) $\mathcal{K}(\alpha \odot A) = \alpha \to \mathcal{K}(A)$,

(K2) $\mathcal{K}(\bigvee_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{K}(A_i)$.

(4) A map $\mathcal{M} : L^X \to L^Y$ is called an *L-meet join approximation operator* iff it satisfies the following conditions

(M1) $\mathcal{M}(\alpha \to A) = \alpha \odot \mathcal{M}(A)$,

(M2) $\mathcal{M}(\bigwedge_{i\in I} A_i) = \bigvee_{i\in I} \mathcal{M}(A_i)$.

Theorem 2.2. *Let X and Y be two sets. Let $\mathcal{H} : L^X \to L^Y$ be an operator. Then the following statements are equivalent:*

(1) *$\mathcal{H}$ is a join preserving map.*

(2) *$\mathcal{H}$ is an L-upper approximation operator.*

(3) *There exists $R \in L^{X\times Y}$ such that*

$$\mathcal{H}(A)(y) = \bigvee_{x\in X} (A(x) \odot R(x, y)).$$

Proof. (1) $\Rightarrow$ (2) Since $\mathcal{H}$ is a join preserving map, we have $\mathcal{H}(\sqcup\Phi) = \sqcup\mathcal{H}^{\to}(\Phi)$ for all $\Phi \in L^{L^X}$.

(H1) Define $\Phi_1 : L^X \to L$ as $\Phi_1(A) = \alpha$ and $\Phi_1(B) = 0$, otherwise. By Remark 1.8(1),

$$(\sqcup\Phi_1)(x) = \bigvee_{A\in L^X} (\Phi_1(A) \odot A(x)) = \alpha \odot A(x).$$

Since $\mathcal{H}^{\to}(\Phi_1)(B) = \bigvee_{B=\mathcal{H}(A)} \Phi_1(A)$ and $\mathcal{H}(\sqcup\Phi_1) = \sqcup\mathcal{H}^{\to}(\Phi_1)$ for all $\Phi_1 \in L^{L^X}$, we have

$$\begin{aligned} \sqcup\mathcal{H}^{\to}(\Phi_1)(y) &= \textstyle\bigvee_{B\in L^Y} (\mathcal{H}^{\to}(\Phi_1)(B) \odot B(y)) \\ &= \Phi_1(A) \odot \mathcal{H}(A)(y) = \alpha \odot \mathcal{H}(A)(y) \\ &= \mathcal{H}(\sqcup\Phi_1)(y) = \mathcal{H}(\alpha \odot A)(y). \end{aligned}$$

Hence $\mathcal{H}(\alpha \odot A) = \alpha \odot \mathcal{H}(A)$.

(H2) Let $\{A_i \in L^X \mid i \in \Gamma\}$ be given. Define $\Phi : L^X \to L$ as $\Phi(A_i) = 1$ for $i \in \Gamma$ and $\Phi(B) = 0$, otherwise. By Remark 1.8(1),

$$(\sqcup\Phi)(x) = \bigvee_{A\in L^X} (\Phi(A) \odot A(x)) = \bigvee_{i\in\Gamma} A_i(x).$$

Since $\mathcal{H}^{\rightarrow}(\Phi)(B) = \bigvee_{B=\mathcal{H}(A)} \Phi(A)$ and $\mathcal{H}(\sqcup\Phi) = \sqcup\mathcal{H}^{\rightarrow}(\Phi)$ for all $\Phi \in L^{L^X}$, we have

$$\begin{array}{ll} \mathcal{H}(\sqcup\Phi)(y) & = \mathcal{H}(\bigvee_{i\in\Gamma} A_i)(y), \\ \sqcup\mathcal{H}^{\rightarrow}(\Phi)(y) & = \bigvee_{B\in L^Y}(\mathcal{H}^{\rightarrow}(\Phi)(B) \odot B(y)) \\ & = \bigvee_{B\in L^Y}((\bigvee_{B=\mathcal{H}(A)} \Phi(A)) \odot B(y)) \\ & = \bigvee_{A\in L^X}(\Phi(A) \odot \mathcal{H}(A)(y)) \\ & = \bigvee_{i\in\Gamma} \mathcal{H}(A_i)(y). \end{array}$$

Hence $\mathcal{H}(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} \mathcal{H}(A_i)$. Thus, $\mathcal{H}$ is an L-upper approximation operator.

(2) $\Rightarrow$ (3) Define $R(x,y) = \mathcal{H}(1_x)(y)$. Since $A = \bigvee_{x\in X}(A(x) \odot 1_x)$, we have

$$\begin{array}{ll} \mathcal{H}(A)(y) & = \mathcal{H}(\bigvee_{x\in X} A(x) \odot 1_x)(y) \\ & = \bigvee_{x\in X}(A(x) \odot \mathcal{H}(1_x)(y)) \\ & = \bigvee_{x\in X}(A(x) \odot R(x,y)). \end{array}$$

(3) $\Rightarrow$ (1) Put $B_0 = \sqcup\mathcal{H}^{\rightarrow}(\Phi)$. Then

$$\begin{array}{ll} e_{L^Y}(B_0, B) & = \bigwedge_{C\in L^Y}(\mathcal{H}^{\rightarrow}(\Phi)(C) \rightarrow e_{L^Y}(C,B)) \\ & = \bigwedge_{C\in L^Y}((\bigvee_{\mathcal{H}(A)=C} \Phi(A) \rightarrow e_{L^Y}(\mathcal{H}(A), B)) \\ & = \bigwedge_{A\in L^X}(\Phi(A) \rightarrow e_{L^Y}(\mathcal{H}(A), B)) \\ & = \bigwedge_{A\in L^X} e_{L^Y}(\Phi(A) \odot \mathcal{H}(A), B) \\ & = e_{L^Y}(\bigvee_{A\in L^X} \Phi(A) \odot \mathcal{H}(A), B). \end{array}$$

Hence $\mathcal{H}(\sqcup\Phi) = \sqcup\mathcal{H}^{\rightarrow}(\Phi)$ from:

$$\begin{array}{l} \sqcup\mathcal{H}^{\rightarrow}(\Phi)(y) = B_0(y) = \bigvee_{A\in L^X} \Phi(A) \odot \mathcal{H}(A)(y) \\ = \bigvee_{A\in L^X}(\Phi(A) \odot \bigvee_{x\in X}(A(x) \odot R(x,y))) \\ = \bigvee_{x\in X}(\bigvee_{A\in L^X}(\Phi(A) \odot A(x)) \odot R(x,y)) \\ = \bigvee_{x\in X}(\sqcup\Phi(x) \odot R(x,y)) = \mathcal{H}(\sqcup\Phi)(y). \end{array}$$

Theorem 2.3. *Let X be a set. Let $\mathcal{H} : L^X \rightarrow L^X$ be an operator. Then the following statements are equivalent:*

(1) *$\mathcal{H}$ is a join preserving map with $1_x \leq \mathcal{H}(1_x)$ and $\mathcal{H}(\mathcal{H}(1_x)) \leq \mathcal{H}(1_x)$ for all $x \in X$.*

(2) *$\mathcal{H}$ is an L-upper approximation operator with $A \leq \mathcal{H}(A)$ and $\mathcal{H}(\mathcal{H}(A)) \leq \mathcal{H}(A)$ for all $A \in L^X$.*

(3) *There exists a preorder $e_X \in L^{X\times X}$ such that*

$$\mathcal{H}(A)(y) = \bigvee_{x\in X} (A(x) \odot e_X(x, y)).$$

Proof. (1) $\Rightarrow$ (2) Since $A = \bigvee_{x\in X}(A(x) \odot 1_x)$, $\mathcal{H}(A)(y) = \mathcal{H}(\bigvee_{x\in X}(A(x) \odot 1_x))(y) = \bigvee_{x\in X}(A(x) \odot \mathcal{H}(1_x)(y)) \geq \bigvee_{x\in X}(A(x) \odot 1_x)(y) = A(y)$ and

$$\begin{aligned} \mathcal{H}(\mathcal{H}(A))(y) &= \mathcal{H}(\mathcal{H}(\textstyle\bigvee_{x\in X} A(x) \odot 1_x))(y) \\ &= \textstyle\bigvee_{x\in X}(A(x) \odot \mathcal{H}(\mathcal{H}(1_x))(y)) \\ &\leq \textstyle\bigvee_{x\in X}(A(x) \odot \mathcal{H}(1_x)(y)) = \mathcal{H}(A)(y). \end{aligned}$$

(2) $\Rightarrow$ (1) Put $A = 1_x$. It is trivial.

(2) $\Rightarrow$ (3) Define $e_X(x, y) = \mathcal{H}(1_x)(y)$. Since $A = \bigvee_{x\in X}(A(x) \odot 1_x)$, we have

$$\begin{aligned} \mathcal{H}(A)(y) &= \mathcal{H}(\textstyle\bigvee_{x\in X} A(x) \odot 1_x)(y) \\ &= \textstyle\bigvee_{x\in X}(A(x) \odot \mathcal{H}(1_x)(y)) \\ &= \textstyle\bigvee_{x\in X}(A(x) \odot e_X(x, y)). \end{aligned}$$

Since $1 = 1_x(x) \leq \mathcal{H}(1_x)(x) = e_X(x, x)$, then e_X is reflexive. Since $\mathcal{H}(1_x) = \bigvee_{y\in X}(\mathcal{H}(1_x)(y) \odot 1_y)$, then

$$\mathcal{H}(\mathcal{H}(1_x))(z) = \bigvee_{y\in X} (\mathcal{H}(1_x)(y) \odot \mathcal{H}(1_y)(z)) \leq \mathcal{H}(1_x)(z).$$

Hence $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$;i.e. e_X is transitive. Thus e_X is a preorder.

(3) $\Rightarrow$ (1) Since $\mathcal{H}(1_x)(y) = e_X(x, y) \geq 1_x(y)$, then $1_x \leq \mathcal{H}(1_x)$. Since

$$\begin{aligned} \mathcal{H}(\mathcal{H}(1_x))(z) &= \textstyle\bigvee_{y\in X}(\mathcal{H}(1_x)(y) \odot \mathcal{H}(1_y)(z)) \\ &= \textstyle\bigvee_{y\in X}(e_X(x, y) \odot e_X(y, z)) \leq e_X(x, z) = \mathcal{H}(1_x)(z). \end{aligned}$$

Theorem 2.4. *Let X and Y be two sets. Let $\mathcal{J} : L^X \to L^Y$ be an operator. Then the following statements are equivalent:*

(1) *$\mathcal{J}$ is a meet preserving map.*

(2) *$\mathcal{J}$ is an L-lower approximation operator.*

(3) *There exists* $R \in L^{X\times Y}$ *such that*

$$\mathcal{J}(A)(y) = \bigwedge_{x\in X} (R(x,y) \to A(x)).$$

Proof. (1) ⇒ (2) Since $\mathcal{J}$ is a meet preserving map, then $\mathcal{J}(\sqcap\Phi) = \sqcap\mathcal{J}^{\to}(\Phi)$ for all $\Phi \in L^{L^X}$.

(J1) Define $\Phi_1 : L^X \to L$ as $\Phi_1(A) = \alpha$ and $\Phi_1(B) = 0$, otherwise. By Remark 1.8(2),

$$(\sqcap\Phi_1)(x) = \bigwedge_{A\in L^X} (\Phi_1(A) \to A(x)) = \alpha \to A(x).$$

Since $\mathcal{J}^{\to}(\Phi_1)(B) = \bigvee_{B=\mathcal{J}(A)} \Phi_1(A)$ and $\mathcal{J}(\sqcap\Phi_1) = \sqcap\mathcal{J}^{\to}(\Phi_1)$ for all $\Phi_1 \in L^{L^X}$, we have

$$\begin{aligned} \sqcap\mathcal{J}^{\to}(\Phi_1)(y) &= \textstyle\bigwedge_{B\in L^X}(\mathcal{J}^{\to}(\Phi_1)(B) \to B(y)) \\ &= \textstyle\bigwedge_{B\in L^X}(\Phi_1(A) \to \mathcal{J}(A)(y)) \\ &= \alpha \to \mathcal{J}(A)(y) \\ &= \mathcal{J}(\sqcap\Phi_1)(y) = \mathcal{J}(\alpha \to A)(y). \end{aligned}$$

Hence $\mathcal{J}(\alpha \to A) = \alpha \to \mathcal{J}(A)$.

(J2) Let $\{A_i \in L^X \mid i \in \Gamma\}$ be given. Define $\Phi : L^X \to L$ as $\Phi(A_i) = 1$ for $i \in \Gamma$ and $\Phi(B) = 0$ otherwise. By Remark 1.8(2),

$$\sqcap\Phi(x) = \bigwedge_{A\in L^X} (\Phi(A) \to A(x)) = \bigwedge_{i\in\Gamma} A_i(x).$$

Since $\mathcal{J}^{\to}(\Phi)(B) = \bigvee_{B=\mathcal{J}(A)} \Phi(A)$ and $\mathcal{J}(\sqcap\Phi) = \sqcap\mathcal{J}^{\to}(\Phi)$ for all $\Phi \in L^{L^X}$, we have

$$\begin{aligned} \sqcap\mathcal{J}^{\to}(\Phi)(y) &= \textstyle\bigwedge_{B\in L^Y}(\mathcal{J}^{\to}(\Phi)(B) \to B(y)) = \bigwedge_{B\in L^Y}(\bigvee_{B=\mathcal{J}(A)} \Phi(A) \to B(y)) \\ &= \textstyle\bigwedge_{A\in L^X}(\Phi(A) \to \mathcal{J}(A)(y)) = \bigwedge_{i\in\Gamma} \mathcal{J}(A_i)(y) \\ &= \mathcal{J}(\sqcap\Phi)(y) = \mathcal{J}(\textstyle\bigwedge_{i\in\Gamma} A_i)(y). \end{aligned}$$

Hence $\mathcal{J}(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} \mathcal{J}(A_i)$. Thus $\mathcal{J}$ is an L-lower approximation operator.

(2) ⇒ (3) Define $R(x,y) = \mathcal{J}(1_x^*)^*(y)$. Since $A = \bigwedge_{x\in X}(A^*(x) \to 1_x^*)$, we have

$$\begin{aligned} \mathcal{J}(A)(y) &= \mathcal{J}(\textstyle\bigwedge_{x\in X}(A^*(x) \to 1_x^*))(y) \\ &= \textstyle\bigwedge_{x\in X}(A^*(x) \to \mathcal{J}(1_x^*)(y)) \\ &= \textstyle\bigwedge_{x\in X}(\mathcal{J}(1_x^*)^*(y) \to A(x)) \\ &= \textstyle\bigwedge_{x\in X}(R(x,y) \to A(x)). \end{aligned}$$

(3) $\Rightarrow$ (1) Put $B_1 = \sqcap \mathcal{J}^{\rightarrow}(\Phi)$. Then

$$\begin{aligned} e_{L^Y}(B, B_1) &= \bigwedge_{C\in L^Y}(\mathcal{J}^{\rightarrow}(\Phi)(C) \rightarrow e_{L^Y}(B, C)) \\ &= \bigwedge_{C\in L^Y}((\bigvee_{\mathcal{J}(A)=C} \Phi(A) \rightarrow e_{L^Y}(B, C))) \\ &= \bigwedge_{A\in L^X}(\Phi(A) \rightarrow e_{L^Y}(B, \mathcal{J}(A)) \\ &= \bigwedge_{A\in L^X} e_{L^Y}(B, \Phi(A) \rightarrow \mathcal{J}(A)) \\ &= e_{L^Y}(B, \bigwedge_{A\in L^X} \Phi(A) \rightarrow \mathcal{J}(A)). \end{aligned}$$

Hence $\mathcal{J}(\sqcap\Phi) = \sqcap\mathcal{J}^{\rightarrow}(\Phi)$ from:

$$\begin{aligned} &\sqcap\mathcal{J}^{\rightarrow}(\Phi)(y) = B_1(y) = \bigwedge_{A\in L^X}(\Phi(A) \rightarrow \mathcal{J}(A)(y)) \\ &= \bigwedge_{A\in L^X}(\Phi(A) \rightarrow \bigwedge_{x\in X}(R(x, y) \rightarrow A(x))) \\ &= \bigwedge_{A\in L^X} \bigwedge_{x\in X}(\Phi(A) \rightarrow (R(x, y) \rightarrow A(x))) \\ &= \bigwedge_{A\in L^X} \bigwedge_{x\in X}(\Phi(A) \odot R(x, y) \rightarrow A(x)) \\ &= \bigwedge_{x\in X}(R(x, y) \rightarrow \bigwedge_{A\in L^X}(\Phi(A) \rightarrow A(x))) \\ &= \bigwedge_{x\in X}(R(x, y) \rightarrow \sqcap\Phi(x)) \quad \text{(by Remark 1.8(2))} \\ &= \mathcal{J}(\sqcap\Phi)(y). \end{aligned}$$

Theorem 2.5. *Let X be a set. Let $\mathcal{J} : L^X \rightarrow L^X$ be an operator. Then the following statements are equivalent:*

(1) *$\mathcal{J}$ is a meet preserving map with $\mathcal{J}(1^*_x) \leq 1^*_x$ and $\mathcal{J}(\mathcal{J}(1^*_x)) \geq \mathcal{J}(1^*_x)$ for all $x \in X$.*

(2) *$\mathcal{J}$ is an L-lower approximation operator with $\mathcal{J}(A) \leq A$ and $\mathcal{J}(\mathcal{J}(A)) \geq \mathcal{J}(A)$ for all $A \in L^X$.*

(3) *There exists a preorder $e_X \in L^{X\times X}$ such that*

$$\mathcal{J}(A)(y) = \bigvee_{x\in X}(e_X(x, y) \rightarrow A(x)).$$

Proof. (1) $\Leftrightarrow$ (2) Since $A = \bigwedge_{x\in X}(A^*(x) \rightarrow 1^*_x)$, $\mathcal{J}(A)(y) = \mathcal{J}(\bigwedge_{x\in X}(A^*(x) \rightarrow 1^*_x))(y) = \bigwedge_{x\in X}(A^*(x) \rightarrow \mathcal{J}(1^*_x)(y)) \leq \bigwedge_{x\in X}(A^*(x) \rightarrow 1^*_x)(y) = A(y)$ and

$$\begin{aligned} \mathcal{J}(\mathcal{J}(A))(y) &= \mathcal{J}(\mathcal{J}(\bigwedge_{x\in X}(A^*(x) \rightarrow 1^*_x))(y) \\ &= \mathcal{J}(\bigwedge_{x\in X}(A^*(x) \rightarrow \mathcal{J}(1^*_x)(y))) \\ &= \bigwedge_{x\in X}(A^*(x) \rightarrow \mathcal{J}(\mathcal{J}(1^*_x)(y))) \\ &\geq \bigwedge_{x\in X}(A^*(x) \rightarrow \mathcal{J}(1^*_x)(y)) = \mathcal{J}(A)(y). \end{aligned}$$

(2) ⇒ (3) Define $e_X(x,y) = \mathcal{J}(1_x^*)^*(y)$. Since $A = \bigwedge_{x\in X}(A^*(x) \to 1_x^*)$, we have

$$\begin{aligned} \mathcal{J}(A)(y) &= \mathcal{J}(\textstyle\bigwedge_{x\in X}(A^*(x) \to 1_x^*))(y) \\ &= \textstyle\bigwedge_{x\in X}(A^*(x) \to \mathcal{J}(1_x^*)(y)) \\ &= \textstyle\bigwedge_{x\in X}(\mathcal{J}(1_x^*)^*(y) \to A(x)) \\ &= \textstyle\bigwedge_{x\in X}(e_X(x,y) \to A(x)). \end{aligned}$$

Since $1 = 1_x(x) \le \mathcal{J}(1_x^*)^*(x) = e_X(x,x)$, then e_X is reflexive.

$$\begin{aligned} \mathcal{J}(\mathcal{J}(1_x^*))(z) &= \mathcal{J}(\textstyle\bigwedge_{y\in X}(\mathcal{J}(1_x^*)^*(y) \to 1_y^*))(z) \\ &= \textstyle\bigwedge_{x\in X}(\mathcal{J}(1_x^*)^*(y) \to \mathcal{J}(1_y^*)(z)) \ge \mathcal{J}(1_x^*)(z) \\ \Leftrightarrow \quad & \mathcal{J}(1_x^*)^*(y) \odot \mathcal{J}(1_x^*)(z) \le \mathcal{J}(1_y^*)(z) \\ \Leftrightarrow \quad & \mathcal{J}(1_x^*)^*(y) \to \mathcal{J}(1_x^*)^*(z) \ge \mathcal{J}(1_y^*)^*(z) \\ \Leftrightarrow \quad & \mathcal{J}(1_x^*)^*(y) \odot \mathcal{J}(1_y^*)^*(z)) \le \mathcal{J}(1_x^*)^*(z) \\ \Leftrightarrow \quad & e_X(x,y) \odot e_X(y,z) \le e_X(x,z). \end{aligned}$$

Hence $e_X(x,y) \odot e_X(y,z) \le e_X(x,z)$;i.e. e_X is transitive. Thus e_X is a preorder.

(3) ⇒ (1) Since $\mathcal{J}(1_x^*)^*(y) = e_X(x,y) \ge 1_x(y)$, then $\mathcal{J}(1_x^*) \le 1_x^*$. Since $e_X(x,y) \odot e_X(y,z) \le e_X(x,z)$, then $\mathcal{J}(\mathcal{J}(1_x^*))(z) \ge \mathcal{J}(1_x^*)(z)$.

Theorem 2.6. *Let X and Y be two sets. Let $\mathcal{K} : L^X \to L^Y$ be an operator. Then the following statements are equivalent:*

(1) *$\mathcal{K}$ is an L-join meet preserving map.*

(2) *$\mathcal{K}$ is an L-join meet approximation operator.*

(3) *There exists $R \in L^{X\times Y}$ such that*

$$\mathcal{K}(A)(y) = \bigwedge_{x\in X}(A(x) \to R(x,y)).$$

Proof. (1) ⇒ (2) Since $\mathcal{K}$ is an L-join meet preserving map, then $\mathcal{K}(\sqcup\Phi) = \sqcap\mathcal{K}^{\to}(\Phi)$ for all $\Phi \in L^{L^X}$.

(K1) Define $\Phi_1 : L^X \to L$ as $\Phi_1(A) = \alpha$ and $\Phi_1(B) = 0$, otherwise. Then

$$(\sqcup\Phi_1)(x) = \bigvee_{A\in L^X}(\Phi_1(A) \odot A(x)) = \alpha \odot A(x)$$

Since $\mathcal{K}^{\rightarrow}(\Phi_1)(B) = \bigvee_{B=\mathcal{K}(A)} \Phi_1(A)$ and $\mathcal{K}(\sqcup\Phi_1) = \sqcup\mathcal{K}^{\rightarrow}(\Phi_1)$ for all $\Phi_1 \in L^{L^X}$, we have

$$\begin{aligned} \sqcap\mathcal{K}^{\rightarrow}(\Phi_1)(y) &= \textstyle\bigwedge_{B\in L^X}(\mathcal{K}^{\rightarrow}(\Phi_1)(B) \rightarrow B(y)) = \Phi_1(A) \rightarrow \mathcal{K}(A)(y) \\ &= \alpha \rightarrow \mathcal{K}(A)(y) \\ &= \mathcal{K}(\sqcup\Phi_1)(y) = \mathcal{K}(\alpha \odot A)(y). \end{aligned}$$

Hence $\mathcal{K}(\alpha \odot A) = \alpha \rightarrow \mathcal{K}(A)$.

(K2) Let $\{A_i \in L^X \mid i \in \Gamma\}$ be given. Define $\Phi : L^X \rightarrow L$ as $\Phi(A_i) = 1$ for $i \in \Gamma$ and $\Phi(B) = 0$ otherwise. Then

$$(\sqcup\Phi)(x) = \bigvee_{A\in L^X} (\Phi(A) \odot A(x)) = \bigvee_{i\in\Gamma} A_i(x).$$

Since $\mathcal{K}^{\rightarrow}(\Phi)(B) = \bigvee_{B=\mathcal{K}(A)} \Phi(A)$ and $\mathcal{K}(\sqcup\Phi) = \sqcap\mathcal{K}^{\rightarrow}(\Phi)$ for all $\Phi \in L^{L^X}$, we have

$$\begin{aligned} \sqcap\mathcal{K}^{\rightarrow}(\Phi)(y) &= \textstyle\bigwedge_{B\in L^Y}(\mathcal{K}^{\rightarrow}(\Phi)(B) \rightarrow B(y)) = \bigwedge_{B\in L^Y}(\bigvee_{B=\mathcal{K}(A)} \Phi(A) \rightarrow B(y)) \\ &= \textstyle\bigwedge_{A\in L^X}(\Phi(A) \rightarrow \mathcal{K}(A)(y)) = \bigwedge_{i\in\Gamma} \mathcal{K}(A_i)(y) \\ &= \textstyle\mathcal{K}(\sqcup\Phi)(y) = \mathcal{K}(\bigvee_{i\in\Gamma} A_i)(y). \end{aligned}$$

Hence $\mathcal{K}(\bigvee_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} \mathcal{K}(A_i)$.

(2) $\Rightarrow$ (3) Define $R(x, y) = \mathcal{K}(1_x)(y)$. Since $A = \bigvee_{x\in X}(A(x) \odot 1_x)$, we have

$$\begin{aligned} \mathcal{K}(A)(y) &= \textstyle\mathcal{K}(\bigvee_{x\in X}(A(x) \odot 1_x))(y) \\ &= \textstyle\bigwedge_{x\in X}(A(x) \rightarrow \mathcal{K}(1_x)(y)) \\ &= \textstyle\bigwedge_{x\in X}(A(x) \rightarrow R(x, y)). \end{aligned}$$

(3) $\Rightarrow$ (1) Put $B_1 = \sqcap\mathcal{K}^{\rightarrow}(\Phi)$. Then

$$\begin{aligned} e_{L^Y}(B, B_1) &= \textstyle\bigwedge_{C\in L^Y}(\mathcal{K}^{\rightarrow}(\Phi)(C) \rightarrow e_{L^Y}(B, C)) \\ &= \textstyle\bigwedge_{C\in L^Y}((\bigvee_{\mathcal{K}(A)=C} \Phi(A) \rightarrow e_{L^Y}(B, \mathcal{K}(A))) \\ &= \textstyle\bigwedge_{A\in L^X}(\Phi(A) \rightarrow e_{L^Y}(B, \mathcal{K}(A)) \\ &= \textstyle\bigwedge_{A\in L^X} e_{L^Y}(B, \Phi(A) \rightarrow \mathcal{K}(A)) \text{ (by Lemma 1.2(5))} \\ &= \textstyle e_{L^Y}(B, \bigwedge_{A\in L^X} \Phi(A) \rightarrow \mathcal{K}(A)). \end{aligned}$$

Hence $\mathcal{K}(\sqcup\Phi) = \sqcap\mathcal{K}^{\rightarrow}(\Phi)$ from

$$\begin{aligned}
&\sqcap\mathcal{K}^{\rightarrow}(\Phi)(y) = B_1(y) = \textstyle\bigwedge_{A\in L^X}(\Phi(A) \to \mathcal{K}(A)(y)) \\
&= \textstyle\bigwedge_{A\in L^X}(\Phi(A) \to \bigwedge_{x\in X}(A(x) \to R(x,y))) \\
&= \textstyle\bigwedge_{A\in L^X}\bigwedge_{x\in X}(\Phi(A) \to (A(x) \to R(x,y))) \\
&= \textstyle\bigwedge_{A\in L^X}\bigwedge_{x\in X}(\Phi(A) \odot A(x) \to R(x,y)) \text{ (by Lemma 1.2(5))} \\
&= \textstyle\bigwedge_{x\in X}(\bigvee_{A\in L^X}(\Phi(A) \odot A(x)) \to R(x,y)) \\
&= \textstyle\bigwedge_{x\in X}(\sqcup\Phi(x) \to R(x,y)) \\
&= \mathcal{K}(\sqcup\Phi)(y).
\end{aligned}$$

Theorem 2.7. *Let X and Y be two sets. Let $\mathcal{M} : L^X \to L^Y$ be an operator. Then the following statements are equivalent:*

(1) *$\mathcal{M}$ is an L-meet join preserving map.*

(2) *$\mathcal{M}$ is an L-meet join approximation operator.*

(3) *There exists $R \in L^{X\times Y}$ such that*

$$\mathcal{M}(A)(y) = \bigvee_{x\in X}(A^*(x) \odot R(x,y)).$$

Proof. (1) $\Rightarrow$ (2) Since $\mathcal{M}$ is an L-meet join operator, then $\mathcal{M}(\sqcap\Phi) = \sqcup\mathcal{M}^{\rightarrow}(\Phi)$ for all $\Phi \in L^{L^X}$.

(M1) Define $\Phi_1 : L^X \to L$ as $\Phi_1(A) = \alpha$ and $\Phi_1(B) = 0$ otherwise. Then

$$(\sqcap\Phi_1)(x) = \bigwedge_{A\in L^X}(\Phi_1(A) \to A(x)) = \alpha \to A(x).$$

Since $\mathcal{M}^{\rightarrow}(\Phi_1)(B) = \bigvee_{B=\mathcal{M}(A)}\Phi_1(A)$ and $\mathcal{M}(\sqcap\Phi_1) = \sqcup\mathcal{M}(\Phi_1)$ for all $\Phi_1 \in L^{L^X}$, we have

$$\begin{aligned}
\sqcup\mathcal{M}^{\rightarrow}(\Phi_1)(y) \quad &= \textstyle\bigvee_{B\in L^X}(\mathcal{M}^{\rightarrow}(\Phi_1)(B) \odot B(y)) \\
&= \Phi_1(A) \odot \mathcal{M}(A)(y) = \alpha \odot \mathcal{M}(A)(y) \\
&= \mathcal{M}(\sqcap\Phi_1)(y) = \mathcal{M}(\alpha \to A)(y).
\end{aligned}$$

Hence $\mathcal{M}(\alpha \to A) = \alpha \odot \mathcal{M}(A)$.

(M2) Let $\{A_i \in L^X \mid i \in \Gamma\}$ be given. Define $\Phi : L^X \to L$ as $\Phi(A_i) = 1$ for $i \in \Gamma$ and $\Phi(B) = 0$ otherwise. Then

$$(\sqcap\Phi)(x) = \bigwedge_{A \in L^X} (\Phi(A) \to A(x)) = \bigwedge_{i \in \Gamma} A_i(x).$$

Since $\mathcal{M}^{\rightarrow}(\Phi)(B) = \bigvee_{B=\mathcal{M}(A)} \Phi(A)$ and $\mathcal{M}(\sqcap\Phi) = \sqcup\mathcal{M}^{\rightarrow}(\Phi)$ for all $\Phi \in L^{L^X}$, we have

$$\begin{aligned}\sqcup\mathcal{M}^{\rightarrow}(\Phi)(y) &= \bigvee_{A \in L^X} (\mathcal{M}^{\rightarrow}(\Phi)(B) \odot B(y)) \\ &= \bigvee_{B=\mathcal{M}(A)} (\Phi(A) \odot \mathcal{M}(A)(y)) = \bigvee_{i \in \Gamma} \mathcal{M}(A_i)(y) \\ &= \mathcal{M}(\sqcap\Phi)(y) = \mathcal{M}(\bigwedge_{i \in \Gamma} A_i)(y).\end{aligned}$$

Hence $\mathcal{M}(\bigwedge_{i \in \Gamma} A_i) = \bigvee_{i \in \Gamma} \mathcal{M}(A_i)$. Thus $\mathcal{M}$ is an L-meet join approximation operator.

(2) $\Rightarrow$ (3) Define $R(x, y) = \mathcal{M}(1_x^*)(y)$. Since $A = \bigwedge_{x \in X}(A^*(x) \to 1_x^*)$, we have

$$\begin{aligned}\mathcal{M}(A)(y) &= \mathcal{M}(\bigwedge_{x \in X} (A(x)^* \to 1_x^*)(y) \\ &= \bigvee_{x \in X} (A(x)^* \odot \mathcal{M}(1_x^*)(y)) \\ &= \bigvee_{x \in X} (A(x)^* \odot R(x, y)).\end{aligned}$$

(3) $\Rightarrow$ (1) Put $B_0 = \sqcup\mathcal{M}^{\rightarrow}(\Phi)$. Then

$$\begin{aligned}e_{L^Y}(B_0, B) &= \bigwedge_{C \in L^Y} (\mathcal{M}^{\rightarrow}(\Phi)(C) \to e_{L^Y}(C, B)) \\ &= \bigwedge_{C \in L^Y} ((\bigvee_{\mathcal{M}(A)=C} \Phi(A) \to e_{L^Y}(\mathcal{M}(A), B)) \\ &= \bigwedge_{A \in L^X} (\Phi(A) \to e_{L^Y}(\mathcal{M}(A), B)) \\ &= e_{L^Y}(\bigvee_{A \in L^X} (\Phi(A) \odot \mathcal{M}(A)), B).\end{aligned}$$

Hence $\mathcal{M}(\sqcap\Phi) = \sqcup\mathcal{M}^{\rightarrow}(\Phi)$ from:

$$\begin{aligned}&\sqcup\mathcal{M}^{\rightarrow}(\Phi)(y) = B_0(y) = \bigvee_{A \in L^X} (\Phi(A) \odot \mathcal{M}(A)(y)) \\ &= \bigvee_{A \in L^X} (\Phi(A) \odot \bigvee_{x \in X} (A^*(x) \odot R(x, y))) \\ &= \bigvee_{x \in X} (\bigvee_{A \in L^X} (\Phi(A) \odot A^*(x)) \odot R(x, y)) \\ &= \bigvee_{x \in X} ((\bigwedge_{A \in L^X} (\Phi(A) \to A(x)))^* \odot R(x, y)) \text{ (by Lemma 1.2(6))} \\ &= \bigvee_{x \in X} ((\sqcap\Phi)^*(x) \odot R(x, y)) \\ &= \mathcal{M}(\sqcap\Phi)(y).\end{aligned}$$

Theorem 2.8. *Let $\mathcal{H} : L^X \to L^Y$ and $\mathcal{J} : L^Y \to L^X$ be maps. Then the following statements are equivalent:*

(1) $(e_{L^X}, \mathcal{H}, \mathcal{J}, e_{L^Y})$ *is a residuated connection.*

(2) $\mathcal{H}(\sqcup\Phi) = \sqcup\mathcal{H}^{\rightarrow}(\Phi)$ *and* $\mathcal{J}(\sqcap\Phi) = \sqcap\mathcal{J}^{\rightarrow}(\Phi)$ *with* $\mathcal{H}(1_x)(y) = \mathcal{J}(1_y^*)^*(x)$ *for all* $\Phi \in L^{L^X}$.

(3) $\mathcal{H}$ *is an* L*-upper approximation operator and* $\mathcal{J}$ *is an* L*-lower approximation operator with* $\mathcal{H}(1_x)(y) = \mathcal{J}(1_y^*)^*(x)$

(4) *There exists* $R \in L^{X\times Y}$ *such that*

$$\mathcal{H}(A)(y) = \bigvee_{x\in X}(A(x) \odot R(x,y)),$$

$$\mathcal{J}(B)(x) = \bigvee_{y\in Y}(R(x,y) \rightarrow B(y)).$$

Proof. (1)⇒ (2). Put $B_0 = \sqcup\mathcal{H}^{\rightarrow}(\Phi)$. Then

$$\begin{aligned} e_{L^Y}(B_0, B) &= \bigwedge_{C\in L^Y}(\mathcal{H}^{\rightarrow}(\Phi)(C) \rightarrow e_{L^Y}(C,B)) \\ &= \bigwedge_{C\in L^Y}((\bigvee_{\mathcal{H}(A)=C}\Phi(A) \rightarrow e_{L^Y}(C,B)) \\ &= \bigwedge_{A\in L^X}(\Phi(A) \rightarrow e_{L^Y}(\mathcal{H}(A),B)) \\ &= \bigwedge_{A\in L^X}(\Phi(A) \rightarrow e_{L^X}(A,\mathcal{J}(B))) \\ &= e_{L^X}(\bigvee_{A\in L^X}(\Phi(A) \odot \phi(A)), \mathcal{J}(B)) \\ &= e_{L^X}(\sqcup\Phi, \mathcal{J}(B)) = e_{L^Y}(\mathcal{H}(\sqcup\Phi), B) \end{aligned}$$

Hence $\mathcal{H}(\sqcup\Phi) = \sqcup\mathcal{H}^{\rightarrow}(\Phi)$.

Put $B_1 = \sqcap\mathcal{J}^{\rightarrow}(\Phi)$. Then

$$\begin{aligned} e_{L^X}(B, B_1) &= \bigwedge_{C\in L^X}(\mathcal{J}^{\rightarrow}(\Phi)(C) \rightarrow e_{L^X}(B,C)) \\ &= \bigwedge_{C\in L^X}(\bigvee_{\mathcal{J}(A)=C}\Phi(A) \rightarrow e_{L^X}(B,C)) \\ &= \bigwedge_{A\in L^Y}(\Phi(A) \rightarrow e_{L^X}(B,\mathcal{J}(A))) \\ &= \bigwedge_{A\in L^X}(\Phi(A) \rightarrow e_{L^Y}(\mathcal{H}(B),A)) \\ &= e_{L^Y}(\mathcal{H}(B), \bigwedge_{A\in L^X}(\Phi(A) \rightarrow (A))) \\ &= e_{L^Y}(\mathcal{H}(B), \sqcap\Phi)) \\ &= e_{L^X}(B, \mathcal{J}(\sqcap\Phi)). \end{aligned}$$

Hence $\mathcal{J}(\sqcap\Phi) = \sqcap\mathcal{J}^{\rightarrow}(\Phi)$.

$$\begin{aligned}\mathcal{J}(1_y^*)(x) &= e_{L^X}(1_x, \mathcal{J}(1_y^*)) = e_{L^Y}(\mathcal{H}(1_x), 1_y^*)\\ &= e_{L^Y}(1_y, \mathcal{H}(1_x)^*) = \mathcal{H}(1_x)^*(y).\end{aligned}$$

(2)⇒ (3) and (3)⇒ (4) are follows from Theorems 2.2 and 2.4.

(4)⇒ (1). For each $A \in L^X, B \in L^Y$,

$$\begin{aligned}e_{L^Y}(\mathcal{H}(A), B) &= \bigwedge_{y\in Y}(\mathcal{H}(A)(y) \to B(y))\\ &= \bigwedge_{y\in Y}\left(\bigvee_{x\in X}(A(x)\odot R(x,y)) \to B(y)\right)\\ &= \bigwedge_{y\in Y}\bigwedge_{x\in X}\left(A(x) \to (R(x,y) \to B(y))\right)\\ &= \bigwedge_{x\in X}\left(A(x) \to \bigwedge_{y\in Y}(R(x,y) \to B(y))\right)\\ &= \bigwedge_{x\in X}\left(A(x) \to \mathcal{J}(B)(x)\right) = e_{L^X}(A, \mathcal{J}(B)).\end{aligned}$$

Theorem 2.9. *Let $\mathcal{K}_1 : L^X \to L^Y$ and $\mathcal{K}_2 : L^Y \to L^X$ be maps. Then the following statements are equivalent:*

(1) *$(e_{L^X}, \mathcal{K}_1, \mathcal{K}_2, e_{L^Y})$ is a Galois connection.*

(2) *$\mathcal{K}_i(\sqcup\Phi) = \sqcap\mathcal{K}_i^{\rightarrow}(\Phi)$ with $\mathcal{K}_1(1_x)(y) = \mathcal{K}_2(1_y)(x)$ for all $i \in \{1,2\}, \Phi \in L^{L^X}$.*

(3) *For all $i \in \{1,2\}$, $\mathcal{K}_i$ is an L-join meet approximation operator with $\mathcal{K}_1(1_x)(y) = \mathcal{K}_2(1_y)(x)$ for $x \in X, y \in Y$.*

(4) *There exists $R \in L^{X\times Y}$ such that*

$$\mathcal{K}_1(A)(y) = \bigvee_{x\in X}(A(x) \to R(x,y)),$$

$$\mathcal{K}_2(B)(x) = \bigvee_{y\in Y}(B(y) \to R(x,y)).$$

Proof. (1)⇒ (2). Put $B_1 = \sqcap\mathcal{K}_1^{\rightarrow}(\Phi)$. Then

$$\begin{aligned}e_{L^Y}(B, B_1) &= \bigwedge_{C\in L^Y}(\mathcal{K}_1^{\rightarrow}(\Phi)(C) \to e_{L^Y}(B, C))\\ &= \bigwedge_{C\in L^Y}(\bigvee_{\mathcal{K}_1(A)=C}\Phi(A) \to e_{L^Y}(B, C))\\ &= \bigwedge_{A\in L^X}(\Phi(A) \to e_{L^Y}(B, \mathcal{K}_1(A)))\\ &= \bigwedge_{A\in L^X}(\Phi(A) \to e_{L^X}(A, \mathcal{K}_2(B))\\ &= e_{L^X}(\bigvee_{A\in L^X}(\Phi(A)\odot A), \mathcal{K}_2(B))\\ &= e_{L^Y}(B, \mathcal{K}_1(\bigvee_{A\in L^X}(\Phi(A)\odot A)))\\ &= e_{L^Y}(B, \mathcal{K}_1(\sqcup\Phi)).\end{aligned}$$

Hence $\mathcal{K}_1(\sqcup\Phi) = \sqcap\mathcal{K}_1^{\rightarrow}(\Phi)$. Similarly, $\mathcal{K}_2(\sqcup\Phi) = \sqcap\mathcal{K}_2^{\rightarrow}(\Phi)$. Moreover,

$$K_1(1_x)(y) = e_{L^Y}(1_y, K_1(1_x)) = e_{L^X}(1_x, K_2(1_y)) = K_2(1_y)(x).$$

(2)⇒ (3) and (3)⇒ (4) are follows from Theorem 2.6.

(4)⇒ (1) For each $A \in L^X, B \in L^Y$,

$$\begin{aligned} e_{L^X}(A, \mathcal{K}_2(B)) &= \textstyle\bigwedge_{x\in X}(A(x) \to \mathcal{K}_2(B)(x)) \\ &= \textstyle\bigwedge_{x\in X}\Big(A(x) \to \bigwedge_{y\in Y}(B(y) \to R(x,y))\Big) \\ &= \textstyle\bigwedge_{x\in X}\Big(A(x) \to \bigwedge_{y\in Y}(B(y) \to R(x,y))\Big) \\ &= \textstyle\bigwedge_{x\in X}\bigwedge_{y\in Y}\Big(A(x) \to (B(y) \to R(x,y))\Big) \\ &= \textstyle\bigwedge_{x\in X}\bigwedge_{y\in Y}\Big(B(y) \to (A(x) \to R(x,y))\Big) \\ &= \textstyle\bigwedge_{x\in X}\bigwedge_{y\in Y}\Big(B(y) \to \bigwedge_{x\in X}(A(x) \to R(x,y))\Big) \\ &= e_{L^Y}(B, \mathcal{K}_1(A)) \end{aligned}$$

Theorem 2.10. *Let $\mathcal{M}_1 : L^X \to L^Y$ and $\mathcal{M}_2 : L^Y \to L^X$ be maps. Then the following statements are equivalent:*

(1) *$(e_{L^X}, \mathcal{M}_1, \mathcal{M}_2, e_{L^Y})$ is a dual Galois connection.*

(2) *$\mathcal{M}_i(\sqcap\Phi) = \sqcup\mathcal{M}_i^{\rightarrow}(\Phi)$ with $\mathcal{M}_1(1_x^*)(y) = \mathcal{M}_2(1_y^*)(x)$ for all $i \in \{1,2\}, \Phi \in L^{L^X}$.*

(3) *For all $i \in \{1,2\}$, $\mathcal{M}_i$ is an L-meet join operator with $\mathcal{M}_1(1_x^*)(y) = \mathcal{M}_2(1_y^*)(x)$ for $x \in X, y \in Y$.*

(4) *There exists $R \in L^{X\times Y}$ such that*

$$\mathcal{M}_1(A)(y) = \bigvee_{x\in X}(A^*(x) \odot R(x,y)),$$

$$\mathcal{M}_2(B)(x) = \bigvee_{y\in Y}(B^*(y) \odot R(x,y)).$$

Proof. (1)⇒ (2). Put $B_0 = \sqcup \mathcal{M}_1^{\rightarrow}(\Phi)$. Then

$$\begin{aligned} e_{L^Y}(B_0, B) &= \bigwedge_{C\in L^Y}(\mathcal{M}_1^{\rightarrow}(\Phi)(C) \rightarrow e_{L^Y}(C,B)) \\ &= \bigwedge_{C\in L^Y}((\bigvee_{\mathcal{M}_1(A)=C} \Phi(A) \rightarrow e_{L^Y}(C,B)) \\ &= \bigwedge_{A\in L^X}(\Phi(A) \rightarrow e_{L^Y}(\mathcal{M}_1(A), B)) \\ &= \bigwedge_{A\in L^X}(\Phi(A) \rightarrow e_{L^X}(\mathcal{M}_2(B), A)) \\ &= \bigwedge_{x\in X}(\mathcal{M}_2(B)\Phi(A) \rightarrow \bigwedge_{A\in L^X}(\Phi(A), A)) \\ &= e_{L^X}(\mathcal{M}_2(B), \bigwedge_{A\in L^X}(\Phi(A) \rightarrow A)) \\ &= e_{L^Y}(\mathcal{M}_1(\sqcap\Phi), B). \end{aligned}$$

Hence $\mathcal{M}_1(\sqcap\Phi) = \sqcup\mathcal{M}_1^{\rightarrow}(\Phi)$. Similarly, $\mathcal{M}_2(\sqcap\Phi) = \sqcup\mathcal{M}_2^{\rightarrow}(\Phi)$. Moreover,

$$\mathcal{M}_1(1_x^*)^*(y) = e_{L^Y}(\mathcal{M}_1(1_x^*), 1_y^*) = e_{L^X}(\mathcal{M}_2(1_y^*), 1_x^*) = \mathcal{M}_1(1_x^*)^*(y).$$

(2)⇒ (3) and (3)⇒ (4) are follows from Theorem 2.7.

(4)⇒ (1). For each $A \in L^X, B \in L^Y$,

$$\begin{aligned} e_{L^X}(\mathcal{M}_2(B), A) &= \bigwedge_{x\in X}(\mathcal{M}_2(B)(x) \rightarrow A(x)) \\ &= \bigwedge_{x\in X}\Big(\bigvee_{y\in Y} B^*(y) \odot R(x,y) \rightarrow A(x)\Big) \\ &= \bigwedge_{x\in X}\bigwedge_{y\in Y}\Big(B^*(y) \rightarrow (R(x,y) \rightarrow A(x))\Big) \\ &= \bigwedge_{x\in X}\bigwedge_{y\in Y}\Big((R(x,y) \rightarrow A(x))^* \rightarrow (B(y)^*)^*\Big) \\ &= \bigwedge_{y\in Y}\Big(\bigvee_{x\in X}(R(x,y) \rightarrow A(x))^* \rightarrow B(y)\Big) \\ &= \bigwedge_{y\in Y}\Big(\bigvee_{x\in X}(A(x)^* \odot R(x,y)) \rightarrow B(y)\Big) \\ &= e_{L^Y}(\mathcal{M}_1(A), B). \end{aligned}$$

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