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# CONTINUOUS TWO STEP HYBRID BLOCK PREDICTOR-HYBRID BLOCK CORRECTOR METHOD FOR THE SOLUTION OF SECOND ORDER INITIAL VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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**Abstract.** This paper considers the implementation of two steps, two hybrid points implemented in block predictor-block corrector method with constant step size. The method is developed using collocation and interpolation of power series approximate solution to give a continuous linear multistep method in which the predictor and corrector are implemented in block method. The basic properties of the corrector was investigated and found to be zero stable, consistent and convergent. The efficiency of the derived method was tested on some numerical examples and found to give better approximation than the existing methods.

**Keywords**: collocation, interpolation, zero stable, consistent, convergent, efficiency

**2010 AMS Subject Classification:** 65L05, 65L06, 65D30

## 1. Introduction

This paper considers the numerical solution to general second order intial value problem in the form

$$y'' = f(x, y, y') \;\; y(x_0) = y_0, y'(x_0) = y_0' \tag{1}$$

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where $x_0$ is the initial point, $y(x_0)$ and $y'(x_0)$ are the solutions at $x_0$ and $f$ is continuous within the interval of integration. Method of reduction of (1) to systems of first order ordinary differential equation has been reported to increase the dimension of the resulting differential equation by the order of the differential equations. It was concluded that the method of reduction is not efficient and unstable for general purpose. Among the authors that discussed the setbacks of the method of reduction are Awoyemi *et al.* [1], Vigor-Aguiar and Ramos [2], Bun and Vasil'Yev [3], Awoyemi [4], Adesanya *et al.* [5]. Direct method for the solution of higher order ordinary differential equations are later proposed to cater for the setbacks of reduction method. The direct method includes the development of linear multistep method which is implemented in predictor corrector method and block method which gives independent solution without overlapping and possess the characteristics of Runge Kutta method for being self starting and does not need starting values. Among the authors that discussed these methods are Jator [6], Adesanya *et al.*[7],

Adesanya *et al.*[11] reported that the major setback of block method is that the interpolation points considered in developing the method cannot exceed the order of the differential equation, hence methods with lower order are developed. Adesanya *et al.*[11] later proposed a method which combined the properties of the predictor corrector method and block method. The method is in such a way that the predictors are implemented in block method; hence it provides a constant order predictor to implement the corrector. Despite the success of this method over block method in term of accuracy, it was discovered that the results are generated at an overlapping interval, hence this affects the accuracy of the method. in this paper, we develop a method in which the predictor and the corrector are implemented in block method. This method gives result at a non overlapping interval and combines the properties of the predictor corrector method and block method.

## 2. Methodology

### 2.1. **Development of block corrector.**

We considered a power series approximate solution in the form

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \tag{2}$$

where $r$ and $s$ are the number of interpolation and collocation points respectively. Substituting the second derivative of (2) into (1) gives

$$y'' = \sum_{j=2}^{r+s-1} j(j-1) a_j x^{j-2}. \tag{3}$$

Interpolating (2) at $x_{n+r}, r = 0, \frac{1}{2}, 1$ and collocating (3) at $x_{n+s}, s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ gives a system of non linear equation

$$AX = U \tag{4}$$

where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{bmatrix}^T$$

$$U = \begin{bmatrix} y_n & y_{n+\frac{1}{2}} & y_{n+1} & f_n & f_{n+\frac{1}{2}} & f_{n+1} & f_{n+\frac{3}{2}} & f_{n+2} \end{bmatrix}^T$$

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+\frac{1}{2}} & x_{n+\frac{1}{2}}^2 & x_{n+\frac{1}{2}}^3 & x_{n+\frac{1}{2}}^4 & x_{n+\frac{1}{2}}^5 & x_{n+\frac{1}{2}}^6 & x_{n+\frac{1}{2}}^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x_{n+\frac{1}{2}}^2 & 20x_{n+\frac{1}{2}}^3 & 30x_{n+\frac{1}{2}}^4 & 42x_{n+\frac{1}{2}}^5 \\ 0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & 20x_{n+1}^3 & 30x_{n+1}^4 & 42x_{n+1}^5 \\ 0 & 0 & 2 & 6x_{n+\frac{3}{2}} & 12x_{n+\frac{3}{2}}^2 & 20x_{n+\frac{3}{2}}^3 & 30x_{n+\frac{3}{2}}^4 & 42x_{n+\frac{3}{2}}^5 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 & 20x_{n+2}^3 & 30x_{n+2}^4 & 42x_{n+2}^5 \end{bmatrix}$$

Solving (4) for the unknown constant $a_j's$ using Guassian elimination method and substituting back into (2) gives a continuous hybrid linear multistep method in the form

$$y(x) = \sum_{j=0}^{1} \alpha_j y_{n+j} + \alpha_{\frac{1}{2}} y_{n+\frac{1}{2}} + h^2 \left[ \sum_{j=0}^{2} \beta_j f_{n+j} + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} \right] \tag{5}$$

where

$$\alpha_0 = \frac{1}{21}\left(128t^7 - 896t^6 + 2352t^5 - 2800t^4 + 1344t^3 + 149t + 21\right)$$

$$\alpha_{\frac{1}{2}} = -\frac{1}{21}\left(256t^7 - 1792t^6 + 4704t^5 - 5600t^4 + 2688t^3 + 256t\right)$$

$$\alpha_1 = \frac{1}{21}\left(128t^7 - 896t^6 + 2352t^5 - 2800t^4 + 1344t^3 - 107t\right)$$

$$\beta_{\frac{1}{2}} = -\frac{1}{315}\left(408t^7 - 2828t^6 + 7308t^5 - 8470t^4 + 3864t^3 - 282t\right)$$

$$\beta_1 = -\frac{1}{180}\left(16t^7 - 136t^6 + 438t^5 - 635t^4 + 348t^3 - 31t\right)$$

$$\beta_{\frac{3}{2}} = -\frac{1}{315}\left(8t^7 - 28t^6 + 70t^4 - 56t^3 + 6t\right)$$

$$\beta_2 = \frac{1}{2520}\left(16t^7 - 56t^6 + 42t^5 + 34t^4 - 42t^3 + 5t\right)$$

$t = \frac{x - x_n}{h}$.

Evaluating (5) at $t = \frac{3}{2}, 2$ and the first derivative of (5) at $t = 0, \frac{1}{2}$ and writing in block gives

$$A^{(0)}Y_m = A^{(i)}Y_{m-1} + A^{(k)}Y_{m-2} + h^2\left[B^{(0)}F_{m-1} + B^{(i)}F_m\right] \tag{6}$$

where

$A^{(0)} = 4 \times 4$ identity matrix $Y_{m-1} = \begin{bmatrix} y_{n-1} & y_{n-2} & y_{n-3} & y_n \end{bmatrix}^T$

$Y_{m-2} = \begin{bmatrix} y'_{n-1} & y'_{n-2} & y'_n & y'_{n+\frac{1}{2}} \end{bmatrix}^T, F_{m-1} = \begin{bmatrix} f_{n-1} & f_{n-2} & f_{n-3} & f_n \end{bmatrix}^T$

$Y_m = \begin{bmatrix} y_{n+\frac{1}{2}} & y_{n+1} & y_{n+\frac{3}{2}} & y_{n+2} \end{bmatrix}^T, F_m = \begin{bmatrix} f_{n+\frac{1}{2}} & f_{n+1} & f_{n+\frac{3}{2}} & f_{n+2} \end{bmatrix}^T$

$$A^{(i)} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad A^{(k)} = \begin{bmatrix} 0 & 0 & \frac{41}{189} & \frac{107}{378} \\ 0 & 0 & \frac{61}{189} & \frac{128}{189} \\ 0 & 0 & \frac{3}{7} & \frac{15}{14} \\ 0 & 0 & \frac{122}{189} & \frac{256}{189} \end{bmatrix}$$

$$F_{m-1}=\begin{bmatrix} 0 & 0 & 0 & \frac{15649}{1088640} \\ 0 & 0 & 0 & \frac{397}{13608} \\ 0 & 0 & 0 & \frac{577}{13440} \\ 0 & 0 & 0 & \frac{638}{8505} \end{bmatrix},\ F_m=\begin{bmatrix} \frac{-4523}{136080} & \frac{533}{181440} & \frac{-19}{27216} & \frac{97}{1088640} \\ \frac{818}{8505} & \frac{463}{11340} & \frac{-46}{8505} & \frac{41}{68040} \\ \frac{421}{1680} & \frac{629}{2240} & \frac{5}{336} & \frac{1}{13440} \\ \frac{3904}{8505} & \frac{292}{567} & \frac{2176}{8505} & \frac{152}{8505} \end{bmatrix}$$

2.2. **Development of the block predictor.** Adesanya *et al.* [11] proposed a block predictor using power series approximate solution in the form

$$A^{(0)}\mathbf{Y}_m = \mathbf{e}y_n + h^2\mathbf{d}f\left(y_n\right) + h^2\mathbf{bF}\left(\mathbf{Y}_m\right) \tag{7}$$

where

$A^{(0)} = 8 \times 8$ identity matrix

$$\mathbf{b}=\begin{bmatrix} \frac{3}{32} & \frac{2}{5} & \frac{117}{160} & \frac{16}{15} & \frac{323}{720} & \frac{31}{45} & \frac{51}{80} & \frac{32}{45} \\ \frac{-47}{960} & \frac{-1}{12} & \frac{27}{320} & \frac{4}{15} & \frac{-11}{60} & \frac{2}{15} & \frac{9}{20} & \frac{4}{15} \\ \frac{29}{1440} & \frac{2}{45} & \frac{3}{32} & \frac{16}{45} & \frac{53}{720} & \frac{1}{45} & \frac{21}{80} & \frac{32}{45} \\ \frac{-7}{1920} & \frac{-1}{120} & \frac{-9}{640} & 0 & \frac{-19}{1440} & \frac{-1}{180} & \frac{-3}{160} & \frac{7}{45} \end{bmatrix}^T$$

$$\mathbf{d}=\begin{bmatrix} \frac{367}{5760} & \frac{53}{360} & \frac{147}{640} & \frac{14}{45} & \frac{251}{1440} & \frac{29}{180} & \frac{27}{160} & \frac{7}{45} \end{bmatrix}^T$$

$$\mathbf{e}=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Y_m=\begin{bmatrix} y_{n+\frac{1}{2}} & y_{n+1} & y_{n+\frac{3}{2}} & y_{n+2} & y'_{n+\frac{1}{2}} & y'_{n+1} & y'_{n+\frac{3}{2}} & y'_{n+2} \end{bmatrix}^T$$

$$y_n=\begin{bmatrix} y_{n-1} & y_{n-2} & y_{n-3} & y_n & y'_{n-1} & y'_{n-2} & y'_{n-3} & y'_n \end{bmatrix}^T$$

$$F(Y_m) = \left[ \begin{array}{cccccccc} f_{n+\frac{1}{2}} & f_{n+1} & f_{n+\frac{3}{2}} & f_{n+2} & {f^*}_{n+\frac{1}{2}} & f^*_{n+1} & f^*_{n+\frac{3}{2}} & f^*_{n+2} \end{array} \right]^T$$

$$f(y_n) = \left[ \begin{array}{cccccccc} f_{n-1} & f_{n-2} & f_{n-3} & f_n & {f^*}_{n-1} & f^*_{n-2} & f^*_{n-3} & f^*_n \end{array} \right]^T$$

where $f^*_n = f$ when considering the first derivative at $n$

Readers are kindly referred to Adesanya *et al.*[11] for the development of the method

## 3. **Analysis of the basic properties of the corrector**

3.1. **Order of the method.** Let the linear operator $\mathcal{L}\left\{y(x):h\right\}$ associated with the block method be defined as

(8) $$\mathcal{L}\left\{y(x):h\right\} = A^{(0)}Y_m - A^{(i)}Y_{m-1} - A^{(k)}Y_{m-2} - h^2\left[B^{(0)}F_{m-1} + B^{(i)}F_m\right]$$

Expanding (8) in Taylors series gives

$\mathcal{L}\left\{y(x):h\right\} = C_0y(x) + C_1hy(x) + ... + C_ph^py(x) + C_{p+1}h^{p+1}y^{p+1}(x)$

**Definition 1.** *Order*

The linear operator and associated block method (8) are said to be of order $p$ if

$C_0 = C_1 = ...C_{p+1} = 0$ and $C_{p+2} \neq 0$, $C_{p+2}$ is called the error constant and implies that the local truncation error is given by

$$t_{n+k} = C_{p+2}h^{p+2}y^{p+2}(x) + 0(h^{p+3})$$

the order of our (6) is 6 with error constant

$\left[ \begin{array}{cccc} \frac{-3373}{58522528640} & \frac{-139}{45722880} & \frac{-253}{72253440} & \frac{-41}{2857680} \end{array} \right]^T$

3.2. **Zero stability.** A block method is said to be zero stable if as h→0 the root $r_j, j =$

$1(1)k$ of the first characteristics polynomial $\rho(r) = 0$ that is $\rho(r) = \det\left[\sum A^0R^{k-1}\right]$=0 satisfying $\mid R \mid \leq 1$, for those root with $\mid R \mid \leq 1$ must be simple.

For our method

$$\rho(r) = \left[ R \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right] = 0$$

$R = 0, 0, 0, 1$

hence our method is zero stable

3.3. **Consistency.** A block method is said to be consistent if it has order greater than one.

3.4. **Convergence:** A block method is said to be convergent if and only if it is consistent and zero stable. From above, it shows clearly that our method is convergent.

## 4. Numerical Examples

### Problem 1

consider the non linear initial value problem (I.V.P)

$$y'' - x\,(y') = 0,\ y(0) = 1,\ y'\,(0) = \frac{1}{2}, h = 0.05$$

Exact solution: $y(x) = 1 + \frac{1}{2} In \left(\frac{2+x}{2-x}\right)$

### Problem II

We consider the non linear initial value problem

$$y'' = \left(\frac{y'}{2y}\right) - 2y,\ y\left(\frac{\pi}{6}\right) = \frac{1}{4},\ y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}, h = 0.05$$

Exact solution: $y\,(x) = (\sin x)^2$

These problems were solved by Adesanya *et al.*[11]using a constant order predictor corrector method of order eight. We solve these problems using thesame step-size $(h) = 0.05$. We compared our result with these results as shown in Tables I and II

Table I Showing result of problem I

| $x$ | $Exact\ result$ | $Computed\ result$ | $Error$ | $ERA$ |
|---|---|---|---|---|
| 0.1 | 1.05004172927849 | 1.050041729276734 | 1.7574(−12) | 7.5028(−13) |
| 0.2 | 1.10033534773107 | 1.100335347727339 | 3.7361(−12) | 9.7410(−12) |
| 0.3 | 1.51140435936467 | 1.151140435930267 | 6.1999(−12) | 3.7638(−11) |
| 0.4 | 1.20273255405408 | 1.202732554044594 | 9.4875(−12) | 9.7765(−11) |
| 0.5 | 1.25541281188299 | 1.255412811868792 | 1.4203(−11) | 2.0825(−10) |
| 0.6 | 1.30951960420311 | 1.309519604181470 | 2.1641(−11) | 3.9604(−10) |
| 0.7 | 1.36544375427139 | 1.365443754236747 | 3.4648(−11) | 7.0460(−10) |
| 0.8 | 1.42364893019360 | 1.423648930133906 | 5.9692(−11) | 1.2095(−09) |
| 0.9 | 1.48470027859405 | 1.484700278482146 | 1.1190(−10) | 2.0511(−09) |
| 1.0 | 1.54930614433405 | 1.549306144105986 | 2.2806(−10) | 3.5066(−09) |

Note:

ERA→ Error in Adesanya *et al.*[11]

ERROR→|Exact result - Computed result|

Table II Showing result of problem II

| $x$ | $Exact\ result$ | $Computed\ result$ | $Error$ | $ERA$ |
|---|---|---|---|---|
| 1.11 | 0.80135065164745 | 0.803150651635315 | 1.2141(−11) | 1.8811(−10) |
| 1.21 | 0.87610222589166 | 0.876102225876513 | 1.5156(−11) | 2.4539(−10) |
| 1.31 | 0.93405979124878 | 0.934059791230213 | 1.8574(−11) | 3.0306(−10) |
| 1.41 | 0.97471276248369 | 0.974712762461345 | 2.2350 (−11) | 3.5819(−10) |
| 1.51 | 0.99644043392112 | 0.996440433894826 | 2.6302(−11) | 4.0838(−10) |
| 0.61 | 0.99837659186650 | 0.998376591836405 | 3.0102(−11) | 4.5128(−10) |
| 0.71 | 0.98044404781244 | 0.980444047779133 | 3.3316(−11) | 4.8473(−10) |
| 0.81 | 0.94335771570097 | 0.943357715665520 | 3.6040(−11) | 5.0696(−10) |
| 0.91 | 0.88859611056068 | 0.888596110524639 | 3.6040(−11) | 5.1697(−10) |
| 2.01 | 0.81834240477827 | 0.818342404743569 | 3.4701(−11) | 5.1381(−10) |

4.1. **Discussion of result.** The efficiency of our method was tested on two numerical examples and the result are shown in Tables I and II. The existing method we compared our result with was implemented in constant order predictor corrector method, the corrector was of order eight and the constant order predictor is the same as the predictor used in this paper. It has been shown clearly that our is an improvement on constant order predictor corrector method. The reason for the better approximation is due to the fact that the method presented in this paper gives result at non overlapping interval.

## 5. Conclusion

We have proposed a two steps block predictor-block corrector of order six in this paper. This method gives result at a non overlapping interval, this property gives the new method ability to give better approximation than the existing method. This method enables us to understand the behaviour of the model at the selected grid points

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