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INTRA-REGULAR AND COMPLETELY REGULAR FUZZY ORDERED SEMIGROUPS

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Abstract. We characterize the intra-regular, the left (right) regular and the completely regular ordered semigroups in terms of fuzzy sets. Using computer we apply our results to finite ordered semigroups.

Keywords: ordered semigroup; intra-regular; left (right) regular; regular; completely regular; left (right) ideal; bi-ideal; interior ideal; fuzzy left (right) ideal; fuzzy bi-ideal; fuzzy interior ideal; semiprime; fuzzy semiprime.

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1. Introduction and prerequisites

Let $(S, ., \leq)$ be an ordered semigroup (: an ordered set with a multiplication which is compatible with the ordering). For a subset H of S, $(H]$ denotes the subset of S defined by $(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}$. For $H = \{a, b, c, ...\}$ we write $(a, b, c, ...]$ instead of $(\{a, b, c, ...\}]$. Clearly $S = (S]$, and for any subsets A, B of S, we have $A \subseteq (A]$, if $A \subseteq B$ then $(A] \subseteq (B]$, $(A](B] \subseteq (AB]$, and $((A]] = (A]$. A nonempty subset A of S is called a *left* (resp. *right*) *ideal* of S if (1) $SA \subseteq A$ (resp. $AS \subseteq A$) and (2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$, that is $(A] = A$. A is called an *ideal* of S if it is both a left and a right ideal of S. A nonempty subset A of S is called a *bi-ideal* of S if (1)

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$ASA \subseteq A$ and 2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$. For an element a of S, $L(a)$, $R(a)$, $I(a)$, $B(a)$ denote the left ideal, right ideal, the ideal and the bi-ideal of S, respectively, generated by a, and we have $L(a) = (a \cup Sa]$, $R(a) = (a \cup aS]$, $I(a) = (a \cup aS \cup Sa \cup aSa]$, $B(a) = (a \cup aSa]$. A left (resp. right) ideal A of S is clearly a subsemigroup of S i.e. $A^2 \subseteq A$. A nonempty subset A of S is called an *interior ideal* of S if (1) $SAS \subseteq A$ and 2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$. If A an ideal of S, then A is an interior ideal of S. Indeed, $S(AS) \subseteq SA \subseteq A$ and $(A] = A$. If A is a left (resp. right) ideal of S, then A is a bi-ideal of S. An ordered semigroup $(S, ., \leq)$ is called *left regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$, that is $a \in (Sa^2]$ for all $a \in S$ or $A \subseteq (SA^2]$ for all $A \subseteq S$. It is called *right regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq a^2x$, that is $a \in (a^2S]$ for all $a \in S$ or $A \subseteq (A^2S]$ for all $A \subseteq S$. An ordered semigroup $(S, ., \leq)$ is called *regular* if for any $a \in S$ there exists $x \in S$ such that $a \leq axa$ i.e. $a \in (aSa]$ for every $a \in S$ or $A \subseteq (ASA]$ for every $A \subseteq S$. It is called *intra-regular* if for each $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$, that is $a \in (Sa^2S]$ for all $a \in S$ or $A \subseteq (SA^2S]$ for all $A \subseteq S$. An ordered semigroup $(S, ., \leq)$ is called *completely regular* if it is at the same time left regular, right regular and regular. In regular and in intra-regular ordered semigroups the ideals and the interior ideals are the same. A subset T of S is called *semiprime* if for any $a \in S$ such that $a^2 \in T$, we have $a \in T$, equivalently, if $A \subseteq S$ such that $A^2 \subseteq T$ implies $A \subseteq T$.

If S is an ordered semigroup, a *fuzzy subset of* S (or *fuzzy set in* S) is a mapping f of S into the real closed interval $[0, 1]$ of real numbers. For a subset A of S, denote by f_A the characteristic function on A, that is the fuzzy subset of S defined by

$$f_A : S \to [0,1] \mid f_A(x) := \begin{cases} 1 & \text{if } \ x \in A \\ 0 & \text{if } \ x \notin A. \end{cases}$$

A fuzzy subset f of S is called a *fuzzy subsemigroup* of S if

(1) $f(xy) \geq \min\{f(x), f(y)\}$ for every $x, y \in S$ and

(2) if $x \leq y$, then $f(x) \geq f(y)$.

A fuzzy subset f of S is called a *fuzzy left ideal* (resp. *fuzzy right ideal*) of S if

(1) $f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$) for every $x, y \in S$ and

(2) if $x \le y$, then $f(x) \ge f(y)$.

For a fuzzy subset f of S, we say that f is a *fuzzy ideal* of S if it is both a fuzzy left and a fuzzy right ideal of S. A fuzzy subset f of S is called a *fuzzy bi-ideal* of S if

(1) $f(xyz) \ge \min\{f(x), f(z)\}$ for all $x, y, z \in S$ and

(2) if $x \le y$, then $f(x) \ge f(y)$.

It is called a *fuzzy interior ideal* of S if

(1) $f(xay) \ge f(a)$ for all $x, a, y \in S$ and

(2) if $x \le y$, then $f(x) \ge f(y)$.

If f a fuzzy ideal of S, then f is a fuzzy interior ideal of S. If f is a fuzzy left (resp. fuzzy right) ideal of S, then f is a fuzzy bi-ideal of S. In regular and in intra-regular ordered semigroups the concepts of fuzzy ideals and fuzzy interior ideals coincide [9].

By a fuzzy ordered semigroup we mean an ordered semigroup having a fuzzy set. Fuzzy ordered semigroups have been first considered by Kehayopulu and Tsingelis in [6]. Fuzzy interior ideals in ordered semigroups have been studied in [9]. For fuzzy bi-ideals of ordered semigroups we refer to [8]. Intra-regular ordered semigroups play an important role is studying the structure of ordered semigroups. An ordered semigroup S is intra-regular if and only if it is a semilattice of simple semigroups, equivalently, if S is a union of simple subsemigroups of S [2], which means that intra-regular ordered semigroups are decomposable into simple components. Moreover, an ordered semigroup S is both regular and intra-regular if and only if it is a semilattice of simple and regular semigroups [10]. Recall that every completely regular ordered semigroup is, by definition, left (resp. right) regular. Every left (resp. right) regular ordered semigroup is intra-regular. It is known, by the same author, that the intra-regular, left regular, and completely regular ordered semigroups can be also defined as the ordered semigroups in which the ideals, the left ideals, and the bi-ideals, respectively, are semiprime (cf., for example [4,5,7]). The present paper adds some additional information on the same type of ordered semigroups using fuzzy sets as well. Fuzzy semigroups (without order) have been systematically studied by N. Kuroki. Characterizations of intra-regular semigroups (without order) in terms of fuzzy sets (of semigroups) have been given by N. Kuroki in [11].

2. Main results

Lemma 1. (cf. also [6; Proposition 2]) *Let S be an ordered semigroup. If A is a left (resp. right) ideal of S, then the characteristic function f_A is a fuzzy left (resp. fuzzy right) ideal of S. "Conversely", if A is a nonempty subset of S and f_A a fuzzy left (resp. fuzzy right) ideal of S, then A is a left (resp. right) ideal of S.*

Lemma 2. (cf. also [8; Theorem 1]) *Let S be an ordered semigroup. If A is a bi-ideal of S, then the characteristic function f_A is a fuzzy bi-ideal of S. "Conversely", if A is a nonempty subset of S and f_A a fuzzy bi-ideal of S, then A is a bi-ideal of S.*

Lemma 3. *Let S be an ordered semigroup. If A is a subsemigroup of S, then the characteristic function f_A is a fuzzy subsemigroup of S. "Conversely", if A is a nonempty subset of S and f_A a fuzzy subsemigroup of S, then A is subsemigroup of S.*

Lemma 4. (cf. also [9; Proposition 2.3]) *Let S is an ordered semigroup. If A is an interior ideal of S, then the characteristic function f_A is a fuzzy interior ideal of S. "Conversely", if A is a nonempty subset of S and f_A a fuzzy interior ideal of S, then A is an interior ideal of S.*

Definition 5. Let S be an ordered semigroup. A fuzzy subset f of S is called *fuzzy semiprime* if

$$f(a) \geq f(a^2) \text{ for every } a \in S.$$

Theorem 6. *Let S be an ordered semigroup. The following are equivalent:*

(1) *S is intra-regular.*

(2) *Every interior ideal of S is semiprime.*

(3) *Every fuzzy interior ideal of S is fuzzy semiprime.*

(4) *If f is a fuzzy interior ideal and at the same time a fuzzy subsemigroup of S, then $f(a) = f(a^2)$ for every $a \in S$.*

(5) *$a \in I(a^2)$ for every $a \in S$.*

(6) *$I(a) = I(a^2)$ for every $a \in S$.*

(7) *Every ideal of S is semiprime.*

(8) *Every fuzzy ideal of S is fuzzy semiprime.*

Proof. (1) $\Longrightarrow$ (2). Let A be an interior ideal of S and $a \in S$, $a^2 \in A$. Since S is intra-regular, we have $a \in (Sa^2S] \subseteq (SAS] \subseteq (A] = A$, and A is semiprime.

(2) $\Longrightarrow$ (3). Let f be a fuzzy interior ideal of S and $a \in S$. The set $(Sa^2S]$ is an interior ideal of S. This is because it is a nonempty subset of S,

$$S(Sa^2S]S = (S](Sa^2S](S] \subseteq (S(Sa^2S)S] \subseteq (Sa^2S],$$

and $((Sa^2S)] = (Sa^2S]$. Then, by (2), $(Sa^2S]$ is semiprime. Since $a^4 \in (Sa^2S]$, we have $a^2 \in (Sa^2S]$, and $a \in (Sa^2S]$. Then $a \leq xa^2y$ for some $x, y \in S$. Since f is a fuzzy interior ideal of S, we have $f(a) \geq f(xa^2y) \geq f(a^2)$, so f is fuzzy semiprime.

(3) $\Longrightarrow$ (4). Let f be a fuzzy interior ideal at the same time a fuzzy subsemigroup of S and $a \in S$. By (3), f is a fuzzy semiprime fuzzy subsemigroup of S, so we have

$$f(a) \geq f(a^2) \geq \min\{f(a), f(a)\} = f(a),$$

then $f(a) = f(a^2)$.

(4) $\Longrightarrow$ (5). Let $a \in S$. Since $I(a^2)$ is an ideal of S, $I(a^2)$ is an interior ideal at the same time a subsemigroup of S. By Lemmas 3 and 4, the characteristic function $f_{I(a^2)}$ is a fuzzy interior ideal and at the same time a fuzzy subsemigroup of S. By (4), we have $f_{I(a^2)}(a) = f_{I(a^2)}(a^2) = 1$. Then $a \in I(a^2)$.

(5) $\Longrightarrow$ (6). If $a \in S$ then, since $a \in I(a^2)$, we have

$$I(a) \subseteq I(a^2) = (a^2 \cup Sa^2 \cup a^2S \cup Sa^2S] \subseteq (Sa \cup aS \cup SaS] \subseteq I(a),$$

thus we have $I(a) = I(a^2)$.

(6) $\Longrightarrow$ (7). If A is an ideal of S and $a \in S$ such that $a^2 \in A$ then, by (6), we have $a \in I(a) = I(a^2) \subseteq A$, so A is semiprime.

(7) $\Longrightarrow$ (8). Let f be a fuzzy ideal of S and $a \in S$. As we have already seen the set $(Sa^2S]$ is a nonempty subset of S and $((Sa^2S]] = (Sa^2S]$, moreover $S(Sa^2S] = (S](Sa^2S] \subseteq (S^2a^2S] \subseteq (SaS]$, so $(Sa^2S]$ is a left ideal of S, similarly it is a right ideal, and so an ideal of S. By hypothesis, $(Sa^2S]$ is semiprime. Since $a^4 \in (Sa^2S]$, we have $a^2 \in (Sa^2S]$, and

$a \in (Sa^2S]$. Then $a \le xa^2y$ for some $x, y \in S$. Then we get $f(a) \ge f(xa^2y) \ge f(a^2)$, so f is fuzzy semiprime.

(8) $\Longrightarrow$ (7). Let A be an ideal of S and $a \in S$, $a^2 \in A$. Since f_A is a fuzzy ideal of S, by hypothesis, f_A is fuzzy semiprime, so we have $f_A(a) \ge f_A(a^2) = 1$. On the other hand, since f_A is a fuzzy subset of S, we have $f_A(a) \le 1$. Thus we have $f_A(a) = 1$, and $a \in A$, so A is semiprime.

(7) $\Longrightarrow$ (1). Let $a \in S$. Since $(Sa^2S]$ is an ideal of S, by hypothesis, it is semiprime. Since $a^4 \in (Sa^2S]$, we have $a \in (Sa^2S]$, so S is intra-regular.

One can also prove directly the implication (8) $\Rightarrow$ (1). In this case the proof is more technical. □

Theorem 7. *Let S be an ordered semigroup. The following are equivalent:*

(1) *S is left regular.*
(2) *The left ideals of S are semiprime.*
(3) *The fuzzy left ideals of S are fuzzy semiprime.*
(4) *If f is a fuzzy left ideal and at the same time a fuzzy subsemigroup of S, then $f(a) = f(a^2)$ for every $a \in S$.*
(5) *$a \in L(a^2)$ for every $a \in S$.*
(6) *$L(a) = L(a^2)$ for every $a \in S$.*

Proof. (1) $\Longrightarrow$ (2). Let A be a left ideal of S and $a \in S$ such that $a^2 \in A$. Then $a \in (Sa^2] \subseteq (SA] \subseteq (A] = A$, so A is semiprime.

(2) $\Longrightarrow$ (3). Let f be a fuzzy left ideal of S and $a \in S$. The set $(Sa^2]$ is a left ideal of S. Indeed, $\emptyset \ne (Sa^2] \subseteq S$, $S(Sa^2] = (S](Sa^2] \subseteq (S(Sa^2)] \subseteq (Sa^2]$, and $((Sa^2]] = (Sa^2]$. By (2), $(Sa^2]$ is semiprime. Since $a^4 \in (Sa^2]$, we have $a^2 \in (Sa^2]$, and $a \in (Sa^2]$. Then $a \le xa^2$ for some $x \in S$ from which $f(a) \ge f(xa^2) \ge f(a^2)$, and f is fuzzy semiprime.

(3) $\Longrightarrow$ (4). Let f be a fuzzy left ideal at the same time a fuzzy subsemigroup of S and $a \in S$. Since f is a fuzzy left ideal of S, by (3), f is fuzzy semiprime which means $f(a) \ge f(a^2)$. Since f is a fuzzy subsemigroup of S, we have $f(a^2) \ge \min\{f(a), f(a)\} = f(a)$. Then we obtain $f(a) = f(a^2)$.

(4) $\Longrightarrow$ (5). Let $a \in S$. As $L(a^2)$ is a left ideal and a subsemigroup of S, by Lemmas 1 and 3, $f_{L(a^2)}$ is a fuzzy left ideal and a fuzzy subsemigroup of S. By (4), we have $f_{L(a^2)}(a) = f_{L(a^2)}(a^2) = 1$. Then $a \in L(a^2)$ and condition (5) is satisfied.

(5) $\Longrightarrow$ (6). If $a \in S$, then $a \in L(a^2) = (a^2 \cup Sa^2] \subseteq (Sa] \subseteq L(a)$. Then we have $L(a) \subseteq L(a^2) \subseteq L(a)$, and $L(a) = L(a^2)$.

(6) $\Longrightarrow$ (1). Let $a \in S$. We have $a \in L(a) = L(a^2) = (a^2 \cup Sa^2]$. Then

$$a^2 \in (a^2 \cup Sa^2](a] \subseteq (a^3 \cup Sa^3] \subseteq (Sa^2].$$

Thus we have $a \in ((Sa^2] \cup Sa^2] = ((Sa^2]] = (Sa^2]$, and S is left regular. □

The right analogue of Theorem 7 also holds and we have the following theorem.

Theorem 8. *Let S be an ordered semigroup. The following are equivalent:*

(1) *S is right regular.*
(2) *The right ideals of S are semiprime.*
(3) *The fuzzy right ideals of S are fuzzy semiprime.*
(4) *If f is a fuzzy right ideal and at the same time a fuzzy subsemigroup of S, then $f(a) = f(a^2)$ for every $a \in S$.*
(5) *$a \in R(a^2)$ for every $a \in S$.*
(6) *$R(a) = R(a^2)$ for every $a \in S$.*

Lemma 9. (cf. also [4]) *An ordered semigroup S is completely regular if and only if, for every $a \in S$, we have $a \in (a^2Sa^2]$.*

Proof. $\Longrightarrow$. Since S is completely regular, we have

$$a \in (aSa] \subseteq ((a^2S]S(Sa^2]] = ((a^2S](S](Sa^2]] \subseteq ((a^2S)S(Sa^2)] \subseteq (a^2Sa^2],$$

thus $a \in (a^2Sa^2]$.

$\Longleftarrow$. Let $a \in S$. Since $a \in (a^2Sa^2] \subseteq (aSa]$, $(Sa^2]$, $(a^2S]$, S is regular, left regular and right regular. □

Theorem 10. *Let S be an ordered semigroup. The following are equivalent:*

(1) *S is completely regular.*

(2) *Every bi-ideal of S is semiprime.*

(3) *Every fuzzy bi-ideal of S is fuzzy semiprime.*

(4) $a \in B(a^2)$ *for every* $a \in S$.

(5) $B(a) = B(a^2)$ *for every* $a \in S$.

Proof. (1) $\Longrightarrow$ (2). Let A be a bi-ideal of S and $a \in S$ such that $a^2 \in A$. Since S is completely regular, by Lemma 9, we have $a \in (a^2Sa^2] \subseteq (ASA] \subseteq (A] = A$, and $a \in A$.

(2) $\Longrightarrow$ (3). Let f be a fuzzy bi-ideal of S and $a \in S$. The set $(a^2Sa^2]$ is a bi-ideal of S. This is because $(a^2Sa^2]$ is a nonempty subset of S,

$$(a^2Sa^2]S(a^2Sa^2] = (a^2Sa^2](S](a^2Sa^2] \subseteq ((a^2Sa^2)S(a^2Sa^2)] \subseteq (a^2Sa^2],$$

and $((a^2Sa^2]] = (a^2Sa^2]$. By (2), $(a^2Sa^2]$ is semiprime. Since $(a^4)^2 = a^8 \in (a^2Sa^2]$, we have $(a^2)^2 = a^4 \in (a^2Sa^2]$, $a^2 \in (a^2Sa^2]$, and $a \in (a^2Sa^2]$. Then $a \leq a^2xa^2$ for some $x \in S$. Since f is a fuzzy bi-ideal of S, we have

$$f(a) \geq f(a^2xa^2) \geq \min\{f(a^2), f(a^2)\} = f(a^2),$$

and f is fuzzy semiprime.

(3) $\Longrightarrow$ (4). Let $a \in S$. We consider the bi-ideal $B(a^2)$ of S generated by a. This is the set $(a^2 \cup a^2Sa^2]$. Since $B(a^2)$ is a bi-ideal of S, by Lemma 2, $f_{B(a^2)}$ is a fuzzy bi-ideal of S. By (3), we have $f_{B(a^2)}(a) \geq f_{B(a^2)}(a^2) = 1$. Since $f_{B(a^2)}$ is a fuzzy set in S, we have $f_{B(a^2)}(a) \leq 1$. Then $f_{B(a^2)}(a) = 1$, and $a \in B(a^2)$.

(4) $\Longrightarrow$ (5). Let $a \in S$. Since $B(a)$ is the bi-ideal of S generated by a, by (4), we have $a \in B(a) \subseteq B(a^2) = (a^2 \cup a^2Sa^2]$. Then

$$\begin{aligned} a^2 \in (a^2 \cup a^2Sa^2](a] &\subseteq ((a^2 \cup a^2Sa^2)a] = (a^3 \cup a^2Sa^3] \subseteq (aSa] \\ &\subseteq (a \cup aSa] = B(a), \end{aligned}$$

and $B(a^2) \subseteq B(a)$. Thus we obtain $B(a) = B(a^2)$.

(5) $\Longrightarrow$ (1). Let $a \in S$. By (5), we have $a \in B(a) = B(a^2) = (a^2 \cup a^2Sa^2]$. Then $a \leq a^2$ or $a \leq a^2ya^2$ for some $y \in S$. If $a \leq a^2$, then $a \leq a^2a^2 = aaa^2 \leq a^2aa^2$. For $x := a$, we get $a \leq a^2xa^2$, and the proof is complete. □

A left (or right) regular ordered semigroup S is intra-regular. Indeed, if S is left regular, then for each $a \in S$, we have

$$a \in (Sa^2] \subseteq (S(Sa^2]a] \subseteq ((S](Sa^2](a]] \subseteq (S(Sa^2)a] \subseteq (Sa^2S],$$

so S is intra-regular. The converse does not hold in general.

As an application of Theorems 6 and 8, we give the following example which is at the same time an example of an intra-regular ordered semigroup which is not left regular.

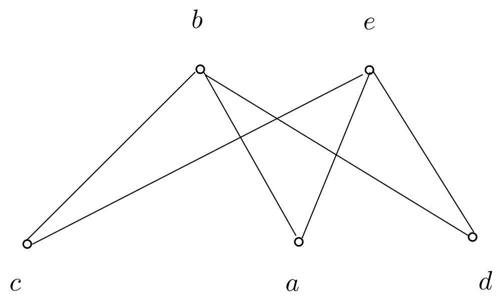
Example 10. (cf. also [3]) We consider the ordered semigroup $S = \{a, b, c, d, e\}$ with the multiplication and the order below:

.	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	a	a	a
d	a	b	a	a	a
e	a	b	a	a	e

$$\leq := \{(a, a), (a, b), (a, e), (b, b), (c, b), (c, c), (c, e), (d, b), (d, d), (d, e), (e, e)\}.$$

We give the covering relation and the figure of S.

$$\prec = \{(a, b), (a, e), (c, b), (c, e), (d, b), (d, e)\}.$$



For an easy way to check that this is an ordered semigroup we refer to [1]. This is intra-regular, not left regular (: c and a are incomparable), and right regular ordered semigroup. The right ideals of S are the sets: $\{a, b, c, d\}$ and S.

The left ideal of S are the sets: $\{a\}$, $\{a,c\}$, $\{a,d\}$, $\{a,c,d\}$, $\{a,b,c,d\}$, $\{a,b,c,e\}$, S.
The ideals of S are the sets: $\{a,b,c,d\}$ and S.
According to Theorem 6 (or Theorem 8), the set $\{a,b,c,d\}$ is semiprime, the characteristic function

$$f_{\{a,b,c,d\}} : S \to [0,1] \mid x \to f_{\{a,b,c,d\}}(x) = \begin{cases} 1 & if\ x \in \{a,b,c,d\} \\ 0 & if\ x \notin \{a,b,c,d\} \end{cases}$$

is a fuzzy semiprime fuzzy ideal of S and $x \in I(x^2)$ for every $x \in S$.

Independently, for the characteristic function $f_{\{a,b,c,d\}}$, by easy calculation, we have

$$f_{\{a,b,c,d\}}(xy) \geq f_{\{a,b,c,d\}}(x) \text{ and } f_{\{a,b,c,d\}}(y) \geq f_{\{a,b,c,d\}}(y)$$

for every $x,y \in S$ which means that $f_{\{a,b,c,d\}}$ is a fuzzy ideal of S. The set $A := \{a,b,c,d\}$ is semiprime since $a^2 \in A$ and $a \in A$; $b^2 \in A$ and $b \in A$; $c^2 \in A$ and $c \in A$; $d^2 \in A$ and $d \in A$ (recall that $e^2 \notin A$). The fuzzy set $f_{\{a,b,c,d\}}$ is fuzzy semiprime. Indeed, we have $f_{\{a,b,c,d\}}(a) = f_{\{a,b,c,d\}}(b) = f_{\{a,b,c,d\}}(c) = f_{\{a,b,c,d\}}(d) = 1$, $f_{\{a,b,c,d\}}(e) = 0$, so $f_{\{a,b,c,d\}}(x) \geq f_{\{a,b,c,d\}}(x^2)$ for every $x \in S$. Moreover we observe that $a \in I(a) = I(a^2)$, $b \in I(b) = I(b^2)$, $c \in \{a,b,c,d\} = I(c^2)$, $d \in \{a,b,c,d\} = I(d^2)$ and $e \in I(e^2)$.

As an application of Theorem 10, we give the following example.

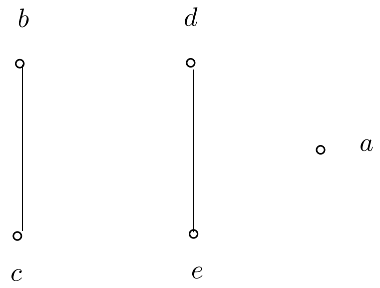
Example 11. (cf. also [2]) We consider the ordered semigroup $S = \{a,b,c,d,e\}$ with the multiplication and the order below:

.	a	b	c	d	e
a	b	a	a	a	a
b	a	b	b	b	b
c	a	b	b	b	b
d	a	b	b	d	d
e	a	b	c	d	e

$$\leq := \{(a,a),(b,b),(c,b),(c,c),(d,d),(e,d),(e,e)\}.$$

We give the covering relation and the figure of S.

$$\prec = \{(c,b),(e,d)\}.$$



This is regular, left regular and right regular, that is a completely regular ordered semigroup. The sets $\{a,b,c\}$ and S are the only bi-ideals of S. According to Theorem 10, the set $\{a,b,c\}$ is semiprime, the characteristic function $f_{\{a,b,c\}}$ is a fuzzy semiprime fuzzy bi-ideal of S and $x \in B(x^2)$ for every $x \in S$.

Independently, one can check that for the characteristic function $f_{\{a,b,c\}}$, we have

$$f_{\{a,b,c\}}(xyz) \geq \min\{f_{\{a,b,c\}}(x), f_{\{a,b,c\}}(z)\}$$

for every $x,y,z \in S$ which means that $f_{\{a,b,c\}}$ is a fuzzy bi-ideal of S. The set $A = \{a,b,c\}$ is semiprime since $a^2 \in A$ and $a \in A$; $b^2 \in A$ and $b \in A$; $c^2 \in A$ and $c \in A$ (recall that $d^2 \notin A$ and $e^2 \notin A$). We have $f_{\{a,b,c\}}(a) = f_{\{a,b,c\}}(b) = f_{\{a,b,c\}}(c) = 1$, $f_{\{a,b,c\}}(d) = f_{\{a,b,c\}}(e) = 0$, so $f_{\{a,b,c\}}(x) \geq f_{\{a,b,c\}}(x^2)$ for every $x \in S$ which means that the fuzzy bi-ideal $f_{\{a,b,c\}}$ is fuzzy semiprime. Finally we have $a \in \{a,b,c\} = B(a^2)$, $b \in \{a,b,c\} = B(b^2)$, $c \in \{a,b,c\} = B(c^2)$, $d \in B(d^2)$, $e \in B(e^2)$.

For the examples of the paper we used computer programs.

References

[1] N. Kehayopulu, On regular duo ordered semigroups, Math. Japon. 37, no. 3 (1992), 535–540.

[2] N. Kehayopulu, On intra-regular ordered semigroups, Semigroup Forum 46, no. 3 (1993), 271–278.

[3] N. Kehayopulu, On regular, intra-regular ordered semigroups, Pure Math. Appl. 4, no. 4 (1993), 447–461.

[4] N. Kehayopulu, On completely regular ordered semigroups, Scient. Math. 1, no. 1 (1998), 27–32.

[5] N. Kehayopulu, Ordered semigroups whose elements are separated by prime ideals, Math. Slovaca 62, no. 3 (2012), 417–424.

[6] N. Kehayopulu, M. Tsingelis, Fuzzy sets in ordered groupoids, Semigroup Forum 65, no. 1 (2002), 128–132.

[7] N. Kehayopulu, M. Tsingelis, On left regular ordered semigroups, Southeast Asian Bull. Math. 25 (2002), 609–615.

[8] N. Kehayopulu, M. Tsingelis, Fuzzy bi-ideals in ordered semigroups, Inform. Sci. 171 (2005), 13–28.

[9] N. Kehayopulu, M. Tsingelis, Fuzzy interior ideals in ordered semigroups, Lobachevskii J. Math. 21 (2006), 65–71.

[10] N. Kehayopulu, M. Tsingelis, On ordered semigroups which are semilattice of simple and regular semigroups, Comm. Algebra 41 (2013), 3252–3260.

[11] N. Kuroki, Fuzzy semiprime ideals in semigroups, Fuzzy sets and Systems 8 (1982), 71–79.