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A NEW CLOSURE OPERATOR IN BITOPOLOGICAL SPACES

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Abstract: In this paper we introduce a concept of ωα-closure in bitopological spaces and derive some basic properties of ωα-closure in a bitopological spaces.

Key words: Bitopological Spaces, ωα-closed sets, ωα-closure.

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1. Introduction

In 2007, Benchalli et al.[1] introduced the notion of ωα - closed sets using ω-open sets[8] and showed that this class properly contains the class of α-sets. Recently the present author extended the notion of ωα-closed sets to bitopological spaces [7]. Dunham [3] introduced the concept of generalized closure operator c* using generalized closed sets of Levine [6]. Then Fukutake [4] introduced and studied the concept of pairwise generalized closure operator (τ_i, τ_j) - cl* in bitopological spaces.

In this paper, we introduce the notion of ωα-closure operator in bitopological spaces by using ωα-closed sets in bitopological spaces [7]. Also it is proved that ωα- closure satisfies Kuratowaski closure operator type properties in bitopological spaces.

We recall some definitions and concepts which are useful in the following sections.

2. *Preliminaries:*

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If A is a subset of X with a topology τ, then the closure, interior and α-closure of A is denoted by cl(A), int(A) and αcl(A) respectively and the complement of A is denoted by A^c or X – A

Definition 2.2: A subset A of a topological space X is called ω-closed [8] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is semi-open in X. The compliment of ω-closed set is ω-open.

Definition 2.3: Let (X, τ) be a topological space and let A $\subset$ X. Then A is called $\omega\alpha$-closed set [1] if αcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is ω-open in (X,τ) and its compliment A^c(or X – A) is called $\omega\alpha$- open.

Definition 2.4: A topological space (X,τ) is said to be $T_{\omega\alpha}$-space [1] if every $\omega\alpha$-closed set is closed.

Throughout this paper the spaces X and Y always represent nonempty bitopological spaces (X, τ_1, τ_2) and (Y, μ_1, μ_2) on which no separation axioms are assumed unless explicitly mentioned and the integers i, j $\in$\{1, 2\}. For a A $\subset$ X, τ_i- cl(A), τ_i - int(A) and τ_i- αcl(A) denote the closure of A, interior of A, and α-closure of A with respect of the topology τ_i respectively.

We denote the family of all (τ_i, τ_j) - $\omega\alpha$ - closed sets in (X, τ_1, τ_2) by B (τ_i, τ_j).

Definition 2.5: Let i, j$\in$ \{1, 2\} be fixed integers. In a bitopological space (X, τ_1, τ_2), a subset A of (X, τ_1, τ_2) is called (τ_i, τ_j) - $\omega\alpha$ - closed set [7] if τ_j-cl(A)) $\subseteq$ U whenever A $\subseteq$ U and U is ω-open set in τ_i..

Definition 2.6: A bitopological space (X, τ_1, τ_2) is called (τ_i, τ_j) - $T_{\omega\alpha}$ - space [7] if every (τ_i, τ_j) - $\omega\alpha$ - closed set is τ_j - closed.

3. $\omega\alpha$ -Closure in Bitopological Spaces

In this section we define (τ_i, τ_j) - $\omega\alpha$ closure and study some characterizations.

Definition 3.1: Let (X, τ_1, τ_2) be a bitopological space and E be a subset of X. Then $\omega\alpha$ -closure of E denoted by (τ_i, τ_j) - $\omega\alpha$cl*(E) is defined as (τ_i, τ_j) - $\omega\alpha$cl*(E) = $\cap$\{A$\subseteq$X / E $\subseteq$ A $\in$ B(τ_i, τ_j)\}.

Theorem 3.2: If E and F are subsets of a bitopological space (X, τ_1, τ_2), then the following properties hold good:

(i) (τ_i, τ_j) - $\omega\alpha cl^*(X) = X$.

(ii) (τ_i, τ_j) - $\omega\alpha cl^*(\phi) = \phi$.

(iii) $A \subseteq (\tau_i, \tau_j)$ - $\omega\alpha cl^*(A)$.

(iv) If B is any (τ_i, τ_j) - $\omega\alpha$-closed set containing A, then (τ_i, τ_j) - $\omega\alpha cl^*(A) \subseteq B$

Proof: Follows from the Definition 3.1.

Theorem 3.3: Let E be a subset of (X, τ_1, τ_2). Then, we have the following results:

(i) $E \subseteq (\tau_i, \tau_j)$ - $\omega\alpha cl^*(E) \subseteq \tau_j$- $\omega\alpha cl(E)$.

(ii) If E is (τ_i, τ_j) - $\omega\alpha$-closed then (τ_i, τ_j) - $\omega\alpha cl^*(E) = E$.

Proof: (i) $E \subseteq (\tau_i, \tau_j)$ - $\omega\alpha cl^*(E)$ follows from the Definition 3.1 Suppose that B is τ_j-closed set. So B is (τ_i, τ_j)- $\omega\alpha$-closed. Then $\{\tau_j$ - closed set$\} \subseteq \{(\tau_i, \tau_j)$ - $\omega\alpha$-closed set$\} \cap \{(\tau_i, \tau_j)$ - $\omega\alpha$-closed set containing E$\} \subseteq \cap \{\tau_j$ - closed set containing E$\}$. That is (τ_i, τ_j) - $\omega\alpha cl^*(E) \subseteq \tau_j$-$\omega\alpha cl(E)$.

(ii) Follows from Definition 3.1 and Theorem 3.3(i).

Remark 3.4: The containment relations in the Theorem 3.3(i) may be proper and the converse of the Theorem 3.3(ii) is not true in general as seen from the following examples.

Example 3.5: Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\phi,\{a\},\{b, c\}, X\}$. Then the subset $A = \{b\}$ of X, (τ_1, τ_2) - $\omega\alpha cl^*(\{b\}) = \{b, c\}$ and τ_2 - $\omega\alpha cl(\{b\}) = \{b, c\}$. So $E \subseteq (\tau_i, \tau_j)$ - $\omega\alpha cl^*(E) \subseteq \tau_j$- $\omega\alpha cl(E)$.

Example 3.6: Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, \{a\}, X\}$ and $\tau_2 = \{\phi, \{a\}, \{a, b\}, X\}$.Then for a subset $A = \{a\}$ of (X, τ_1, τ_2), (τ_1, τ_2) - $\omega\alpha cl^*(\{a\}) = \{a\}$ but A is not (τ_1, τ_2) - $\omega\alpha$-closed set.

Theorem 3.7: Let E and F be two subsets of (X, τ_1, τ_2).

(i) If $E \subseteq F$, then (τ_i, τ_j) - $\omega\alpha cl^*(E) \subseteq (\tau_i, \tau_j)$ - $\omega\alpha cl^*(F)$

(ii) If $\tau_1 \subseteq \tau_2$, then (τ_1, τ_2) - $\omega\alpha cl^*(E) \subseteq (\tau_2, \tau_1)$ - $\omega\alpha cl^*(E)$.

Proof: (i) Let $E \subseteq F$. By Definition 3.1,(τ_i, τ_j) - $\omega\alpha cl^*(F) = \cap\{A \subseteq X / F \subseteq A \in B(\tau_i, \tau_j)\}$. If $F \subseteq A \in B(\tau_i, \tau_j)$, then $E \subseteq F \subseteq A \in B(\tau_i, \tau_j)$. We have (τ_i, τ_j) - $\omega\alpha cl^*(E) \subseteq A$. Then (τ_i, τ_j) -

$\omega\alpha cl^*(E) \subseteq \cap\{A:F \subseteq A \in B(\tau_i, \tau_j)\} = (\tau_i, \tau_j) - \omega\alpha cl^*(F)$. That is $(\tau_i, \tau_j) - \omega\alpha cl^*(E) \subseteq (\tau_i, \tau_j) - \omega\alpha cl^*(F)$

(ii) $\tau_1 \subseteq \tau_2$ implies $B(\tau_2, \tau_1) \subseteq B(\tau_1, \tau_2)$, which implies

$\{C \in X/ E \subseteq C \in B(\tau_2, \tau_1)\} \subseteq \{A \in X/ E \subseteq A \in B(\tau_1, \tau_2)\}$

$\cap\{A \in X/ E \subseteq A \in B(\tau_1, \tau_2)\} \subseteq \cap\{C \in X/E \subseteq C \in B(\tau_2, \tau_1)\}$

Thus $(\tau_1, \tau_2) - \omega\alpha cl^*(E) \subseteq (\tau_2, \tau_1) - \omega\alpha cl^*(E)$.

Theorem 3.8: The operator $(\tau_i,\tau_j) - \omega\alpha cl^*$ is the same as the Kuratowski closure operator.

Proof:

(i) It follows from Theorem 3.2(ii) that $(\tau_i,\tau_j) - \omega\alpha cl^*(\phi) = \phi$.

(ii) $E \subseteq (\tau_i, \tau_j) - \omega\alpha cl^*(E)$ follows from Theorem 3.3(i).

(iii) Suppose E and F are two sets of (X, τ_1, τ_2). It follows from Theorem 3.7(i) , $(\tau_i, \tau_j) - \omega\alpha cl^*(E) \subseteq (\tau_i, \tau_j) - \omega\alpha cl^*(E \cup F)$ and $(\tau_i, \tau_j) - \omega\alpha cl^*(F) \subseteq (\tau_i, \tau_j) - \omega\alpha cl^*(E \cup F)$. Hence we have $(\tau_i, \tau_j) - \omega\alpha cl^*(E) \cup (\tau_i, \tau_j) - \omega\alpha cl^*(F) \subseteq (\tau_i, \tau_j) - \omega\alpha cl^*(E \cup F)$.

Now if $x \notin (\tau_i, \tau_j) - \omega\alpha cl^*(E) \cup (\tau_i, \tau_j) - \omega\alpha cl^*(F)$ then there exist $A, B \in B(\tau_i,\tau_j)$ such that $E \subseteq A$, $x \notin A$ and $F \subseteq B$, $x \notin B$. Hence $E \cup F \subseteq A \cup B$ and $x \notin A \cup B$. Since $A \cup B$ is (τ_i, τ_j) - $\omega\alpha$-closed by [1], $x \notin (\tau_i, \tau_j)-\omega\alpha cl^*(E \cup F)$. Then we have $(\tau_i, \tau_j) - \omega\alpha cl^*(E \cup F) \subseteq (\tau_i, \tau_j) - \omega\alpha cl^*(E) \cup (\tau_i, \tau_j) - \omega\alpha cl^*(F)$. Therefore we have $(\tau_i, \tau_j) - \omega\alpha cl^*(E \cup F) = (\tau_i, \tau_j)-\omega\alpha cl^*(E) \cup (\tau_i, \tau_j) - \omega\alpha cl^*(F)$.

(iv) Let E be a subset of (X, τ_1, τ_2) and A be a (τ_i, τ_j) - $\omega\alpha$-closed set containing E. Since $(\tau_i, \tau_j) - \omega\alpha cl^*(E) \subseteq A$, we have $(\tau_i, \tau_j) - \omega\alpha cl^*(E) \supset (\tau_i,\tau_j) - \omega\alpha cl^*((\tau_i,\tau_j) - \omega\alpha cl^*(E))$. Conversely $(\tau_i, \tau_j) - \omega\alpha cl^*(E) \subseteq (\tau_i, \tau_j)-\omega\alpha cl^*((\tau_i, \tau_j) - \omega\alpha cl^*(E))$ is true by Theorem 3.3(i). Then we have $(\tau_i, \tau_j)-\omega\alpha cl^*(E) = (\tau_i, \tau_j) - \omega\alpha cl^*((\tau_i, \tau_j) - \omega\alpha cl^*(E))$. Hence the proof.

From the above Theorem 3.8, $(\tau_i, \tau_j) - \omega\alpha cl^*$ defines a new topology on X.

Definition 3.9: Let $i, j \in \{1, 2\}$ be two fixed integers. Let $\tau_{\omega\alpha}{}^*(\tau_i, \tau_j)$ be the topology on X generated by $(\tau_i,\tau_j) - \omega\alpha cl^*$ in the usual manner. That is $\tau_{\omega\alpha}{}^*(\tau_i,\tau_j) = \{E \subseteq X; (\tau_i,\tau_j) - \omega\alpha cl^*(E^c) = E^c\}$.

Theorem 3.10: Let i, j $\in$ {1, 2} be two fixed integers. Let (X, τ_1, τ_2) be a bitopological space, then $\tau_j \subseteq \tau_{\omega\alpha}*(\tau_i,\tau_j)$.

Proof: Let G be any τ_j- open set. It follows that G^c is τ_j- closed. By [1], G^c is (τ_i,τ_j) - $\omega\alpha$-closed. Therefore (τ_i,τ_j) - $\omega\alpha cl*(G^c) = G^c$, by Theorem 3.3 (ii). That is $G\in \tau_{\omega\alpha}*(\tau_i,\tau_j)$ and hence $\tau_j \subseteq \tau_{\omega\alpha}*(\tau_i, \tau_j)$.

Remark 3.11: Containment relation in the above Theorem 3.10 may be proper as seen from the following example.

Example 3.12: In Example 3.5, the (τ_1,τ_2) - $\omega\alpha$-closed sets are ϕ, {a}, {c}, {a, c},{b, c}, X and $\tau_{\omega\alpha}*(\tau_1,\tau_2)$ = {ϕ, {a}, {b}, {a, b},{b, c}, X}. Clearly $\tau_2 \subseteq \tau_{\omega\alpha}*(\tau_1,\tau_2)$ but $\tau_2 \neq \tau_{\omega\alpha}*(\tau_1,\tau_2)$.

Theorem 3.13: Let i, j $\in$ {1,2} be two fixed integers. Let (X, τ_1, τ_2) be a bitopological space. If a subset E of X is (τ_i,τ_j) - $\omega\alpha$-closed, then E is $\tau_{\omega\alpha}*(\tau_1,\tau_2)$ - closed.

Proof: Let a subset E of X be (τ_i,τ_j) - $\omega\alpha$-closed. By Theorem 3.3(ii), (τ_i,τ_j) - $\omega\alpha cl*(E) = E$. That is (τ_i,τ_j) - $\omega\alpha cl*\{(E^c)^c\} = (E^c)^c$. It follows that $E^c \subseteq \tau_{\omega\alpha}*(\tau_i,\tau_j)$. Therefore E is $\tau_{\omega\alpha}*(\tau_i,\tau_j)$-closed.

However the converse of the above Theorem 3.13 need not be true as seen from the following example.

Example 3.14: In Example 3.5, the set A = {b} is $\tau_{\omega\alpha}*(\tau_1,\tau_2)$ - closed but not (τ_1,τ_2) -$\omega\alpha$- closed in (X, τ_1, τ_2).

Theorem 3.15: For any point x of (X, τ_1, τ_2), {x} is τ_i - ω-closed or $\{x\}^c$ is $\tau*(\tau_i,\tau_j)$ -closed.

Proof: Suppose {x} is not τ_i - ω-closed. Then $\{x\}^c$ is (τ_i,τ_j) - $\omega\alpha$-closed by[1] .Then by Theorem 3.10 (ii), $\{x\}^c$ is $\tau*(\tau_i,\tau_j)$-closed.

Corollary 3.16: If $\tau_1 \subseteq \tau_2$ in (X, τ_1, τ_2) then $\tau*(\tau_2,\tau_1) \subseteq \tau*(\tau_1,\tau_2)$.

Proof: Let $E\in\tau*(\tau_2,\tau_1)$.Then $E \in \tau*(\tau_1,\tau_2)$ by Theorems 3.5(ii), 3.3(ii) and by assumption. Hence $\tau*(\tau_2,\tau_1) \subseteq \tau*(\tau_1,\tau_2)$.

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