



Available online at <http://scik.org>

J. Math. Comput. Sci. 3 (2013), No. 5, 1211-1223

ISSN: 1927-5307

ON THE CONVERGENCE OF CERTAIN BASIC SETS OF POLYNOMIALS

A. EL-SAYED AHMED^{1,2}

¹Department of Mathematics, Faculty of Science, Taif University, 888 El-Hawiyah, Saudi Arabia

²Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt

Abstract. In this paper, various problems relating to the properties of the Hadamard set of simple basic sets of polynomials are treated with particular emphasis on distinction between the single and several complex variables cases. An important result is established for the relationship between the Cannon functions of simple sets of polynomials in several complex variables and those of the directly Hadamard sets. Possible results on the effectiveness of Hadamard set for several complex variables at the origin are also discussed.

Keywords: basic sets of polynomials; Hadamard product, hyperellipse.

2010 AMS Subject Classification: 32A17, 32A30

1. Introduction

Let \mathbb{C} represent the field of complex variables. In the space \mathbb{C}^2 of the two complex variables z and w , the successive monomial $1, z, w, z^2, zw, w^2, \dots$ are arranged so that the enumeration number of the monomial $z^j w^k$ in the above sequence is

$$\frac{1}{2}(j+k)(j+k) + k; \quad j, k \geq 0.$$

The enumeration number of the last monomial of a polynomial $P(z, w)$ in two complex variables is called the degree of the polynomial. A sequence $\{P_i(z; w)\}_0^\infty$ of polynomials in two complex variables in which the order of each polynomial is equal to its degree is called a simple set (see [9], [11] and [20]). Such a set is conveniently denoted $\{P_i(z; w)\}$, where the last monomial in $P_{m,n}(z, w)$ is $z^m w^n$.

If further, the coefficient of this last monomial is 1, the simple set is termed monic. Thus, in the simple monic set $\{P_{m,n}(z; w)\}$ the polynomial $P_{m,n}(z, w)$ is represented as follows.

$$P_{m,n}(z, w) = \sum_{k=0}^{m+n} \sum_{j=0}^k P_{k-j,j}^{m,n} z^{k-j} w^j \quad (P_{m,n}^{m,n} = 1; P_{m+n-j,j}^{m,n} = 0, j > n).$$

Let $\mathbf{z} = (z_1, z_2, \dots, z_n)$ be an element of \mathbb{C}^n ; the space of several complex variables. The following definition is introduced in [15] and [16].

Definition 1.1. A set of polynomials $\{P_{\mathbf{m}}[\mathbf{z}]\} = \{P_0, P_1, P_2, \dots, P_n, \dots\}$ is said to be basic when every polynomial in the complex variables z_s ; $s \in I = \{1, 2, 3, \dots, n\}$ can be uniquely expressed as a finite linear combination of the elements of the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$.

Thus according to [16], the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be basic if and only if there exists a unique row-finite matrix \bar{P} such that $\bar{P}P = P\bar{P} = \mathbf{I}$, where $P = [P_{\mathbf{m},\mathbf{h}}]$ is the matrix of coefficients, \bar{P} is the matrix of operators of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and \mathbf{I} is the infinite unit matrix.

Similar definition for a simple monic set can be extended to the case of several complex variables by replacing m, n by $(\mathbf{m}) = (m_1, m_2, m_3, \dots, m_n)$, j, k by $(\mathbf{h}) = (h_1, h_2, h_3, \dots, h_n)$ and z, w by \mathbf{z} , where each of (\mathbf{m}) and (\mathbf{h}) be multi-indices of non-negative integers.

The fact that the simple monic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of several complex variables is necessarily basic follows from the observation that the matrix $[P_{\mathbf{m},\mathbf{h}}]$ of coefficients of the polynomials of the set is a lower triangular matrix with non-zero diagonal elements. (These elements are each equal to 1 for monic sets).

Definition 1.2. The basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is said to be algebraic of degree ℓ when its matrix of coefficients P satisfies the usual identity

$$\alpha_0 P^\ell + \alpha_1 P^{\ell-1} + \dots + \alpha_\ell I = 0.$$

Thus, we have a relation of the form

$$\bar{P}_{\mathbf{m},\mathbf{h}} = \delta_{\mathbf{m},\mathbf{h}}\gamma_0 + \sum_{s_1=1}^{\ell-1} \gamma_{s_1} P_{\mathbf{m},\mathbf{h}}^{(s_1)},$$

where $P_{\mathbf{m},\mathbf{h}}^{(s_1)}$ are the elements of the power matrix P^{s_1} and $\gamma_{s_1}, s_1 = 0, 1, 2, \dots, \ell - 1$ are constant numbers. In the space of several complex variables \mathbb{C}^n , an open elliptical region $\sum_{s=1}^n \frac{|z_s|^2}{r_s^2} < 1$ is here denoted by \mathbf{E}_{r_s} and its closure $\sum_{s=1}^n \frac{|z_s|^2}{r_s^2} \leq 1$; is denoted by $\bar{\mathbf{E}}_{r_s}$, where $r_s; s \in I$ are positive numbers. In terms of the introduced notations these regions satisfy the following inequalities:

$$\mathbf{E}_{r_s} = \{\mathbf{w} : |\mathbf{w}| < 1\}$$

$$\bar{\mathbf{E}}_{r_s} = \{\mathbf{w} : |\mathbf{w}| \leq 1\},$$

where $\mathbf{w} = (w_1, w_2, w_3, \dots, w_n)$, $w_s = \frac{z_s}{r_s}; s \in I$. Suppose now that the function $f(\mathbf{z})$, is given by

$$f(\mathbf{z}) = \sum_{\mathbf{m}=0}^{\infty} a_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$$

is regular in $\bar{\mathbf{E}}_{r_s}$ and

$$M[f; r_s] = \sup_{\bar{\mathbf{E}}_{r_s}} |f(\mathbf{z})|.$$

Then it follows that $\{|z_s| \leq r_s t_s; |t_s| = 1\} \subset \bar{\mathbf{E}}_{r_s}$; hence

$$\begin{aligned} |a_{\mathbf{m}}| &\leq \frac{M[f; \rho_s]}{\rho_{\mathbf{m}} t^{\mathbf{m}}} = \frac{M[f; \rho_s]}{\prod_{s=1}^n \rho_s^{m_s} t_s^{m_s}} \leq \inf_{|t|=1} \frac{M[f; \rho_s]}{\prod_{s=1}^n (\rho_s t_s)^{m_s}} \\ &= \sigma_{\mathbf{m}} \frac{M[f; \rho_s]}{\prod_{s=1}^n \rho_s^{m_s}} \end{aligned}$$

for all $0 < \rho_s < r_s; s \in I$, where

$$\sigma_{\mathbf{m}} = \inf_{|t|=1} \frac{1}{t^{\mathbf{m}}} = \frac{\{\langle \mathbf{m} \rangle\}^{\frac{\langle \mathbf{m} \rangle}{2}}}{\prod_{s=1}^n m_s^{\frac{m_s}{2}}} \quad (\text{see [12] and [15]}),$$

and $1 \leq \sigma_{\mathbf{m}} \leq (\sqrt{n})^{\langle \mathbf{m} \rangle}$ on the assumption that $m_s^{\frac{m_s}{2}} = 1$, whenever $m_s = 0; s \in I$. Thus, it follows that

$$\lim_{\langle \mathbf{m} \rangle \rightarrow \infty} \sup \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}} \prod_{s=1}^n (r_s)^{\langle \mathbf{m} \rangle - m_s}} \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} \leq \frac{1}{\prod_{s=1}^n \rho_s} \quad ; \quad \rho_s < r_s; s \in I$$

and since ρ_s can be chosen arbitrary near to r_s ; $s \in I$, we conclude that

$$\lim_{\langle \mathbf{m} \rangle \rightarrow \infty} \sup \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}} \prod_{s=1}^n (r_s)^{\langle \mathbf{m} \rangle - m_s}} \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} \leq \frac{1}{\prod_{s=1}^n r_s}.$$

For the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and its inverse $\{\bar{P}_{\mathbf{m}}[\mathbf{z}]\}$, we have

$$P_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}},$$

$$\bar{P}_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} \bar{P}_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}},$$

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \bar{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \bar{P}_{\mathbf{h}}[\mathbf{z}].$$

Let $N_{\mathbf{m}} = N_{m_1, m_2, \dots, m_n}$ be the number of non-zero coefficients $\bar{P}_{\mathbf{m},\mathbf{h}}$ in the last equality.

A basic set satisfying the condition

$$\lim_{\langle \mathbf{m} \rangle \rightarrow \infty} \{N_{\mathbf{m}}\}^{\frac{1}{\langle \mathbf{m} \rangle}} = a, \quad a > 1$$

is called a general basic set and if $a = 1$, then the basic set is called a Cannon set (see [14] and [15]). When this associated series converges uniformly to $f(\mathbf{z})$ in some domain it is said to represent $f(\mathbf{z})$ in that domain; in other words, as in the classical terminology of Whittaker for a single complex variable (see [21]), the basic set $P_{\mathbf{m}}[\mathbf{z}]$ will be effective in that domain. For more information about basic sets of polynomials we refer to ([1]-[21]). The convergence properties of basic sets of polynomials are classified according to the classes of functions represented by their associated basic series and also to the domain in which are represented.

Let

$$G(P_{\mathbf{m}}; r_s) = \prod_{s=1}^n r_s^{-m_s} \sum_{\mathbf{h}} |\bar{P}_{\mathbf{m},\mathbf{h}}| M[P_{\mathbf{h}}, r_s].$$

Then, the Cannon sum of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ for $\bar{\mathbf{E}}_{r_s}$ will be

$$\Omega(P_{\mathbf{m}}; r_s) = \sigma_{\mathbf{m}} G(P_{\mathbf{m}}; r_s).$$

Also, the Cannon function for the same set is

$$\Omega(P; r_s) = \lim_{\langle \mathbf{m} \rangle \rightarrow \infty} \{\Omega(P_{\mathbf{m}}; r_s)\}^{\frac{1}{\langle \mathbf{m} \rangle}}.$$

Concerning the effectiveness of the basic set of polynomials of several complex variables in hyperellipse, we have the following results from [7].

Theorem 1.1 [7] *The necessary and sufficient condition for the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of several complex variables to be effective in the closed hyperellipse $\bar{\mathbf{E}}_{r_s}$ is that*

$$(1) \quad \Omega(P; r_s) = r_s.$$

If $r_s \rightarrow 0^+$, then we obtain the effectiveness at the origin as in the following corollary:

Corollary 1.1 [7] *The necessary and sufficient condition for the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of several complex variables to be effective at the origin is that*

$$\Omega(P; 0^+) = \lim_{r_s \rightarrow 0^+} \Omega(P; r_s) = 0.$$

Convergence properties (effectiveness) for Hadamard product set simple monic sets of polynomials of a single complex variable is introduced by Melek and El-Said in [13]. In [18] Nassif and Rizk introduced an extension of this product in the case of two complex variables using spherical regions. In [6], the same author has studied this problem in \mathbb{C}^n using hepespherical regions. It should be mentioned here the study of this problem in Clifford analysis (see [1]). For more details on basic sets of polynomials in Clifford setting, we refer to [2],[3],[8],[19] and others. In the present paper, we aim to investigate the extent of a generalization of this Hadamard product set in \mathbb{C}^n using hyperspherical regions.

In [18], Nassif and Rizk introduced the following definition.

Definition 1.3 Let $\{P_{m,n}(z, w)\}$ and $\{q_{m,n}(z, w)\}$ be two simple monic sets of polynomials, where

$$P_{m,n}(z, w) = \sum_{(i,j)=0}^{(m,n)} P_{i,j}^{m,n} z^i w^j,$$

$$q_{m,n}(z, w) = \sum_{(i,j)=0}^{(m,n)} q_{i,j}^{m,n} z^i w^j.$$

Then the Hadamard product of the sets $\{P_{m,n}(z, w)\}$ and $\{q_{m,n}(z, w)\}$ is the simple monic set $\{U_{m,n}(z, w)\}$ given by

$$U_{m,n}(z, w) = \sum_{(i,j)=0}^{(m,n)} U_{i,j}^{m,n} z^i w^j,$$

where

$$U_{i,j}^{m,n} = \frac{\sigma_{m,n}}{\sigma_{i,j}} P_{i,j}^{m,n} q_{i,j}^{m,n}; \quad ((i, j) \leq (m, n)).$$

In this paper, we give an inevitable modification in the definition of Hadamard product of basic sets of polynomials of two complex variables as to yield favorable results in the case of several complex variables in hyperelliptical regions in \mathbb{C}^n , by using k basic sets of polynomials instead of two sets.

Now, we are in a position to extend the above product by using k basic sets of polynomials of several complex variables, so we will denote these polynomials by $\{P_{1,\mathbf{m}}[\mathbf{z}]\}$, $\{P_{2,\mathbf{m}}[\mathbf{z}]\}$, ..., $\{P_{k,\mathbf{m}}[\mathbf{z}]\}$ and in general write $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, \dots, k$.

Definition 1.4 Let $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$; $s_2 = 1, 2, 3, \dots, k$ be simple monic sets of polynomials of several complex variables, where

$$(2) \quad P_{s_2,\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=0}^{(\mathbf{m})} P_{s_2,\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}.$$

Then the Hadamard product of the sets $\{P_{s_2,\mathbf{m}}[\mathbf{z}]\}$ is the simple monic set $\{H_{\mathbf{m}}[\mathbf{z}]\}$ given by

$$(3) \quad H_{\mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=0}^{(\mathbf{m})} H_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}},$$

where

$$(4) \quad H_{\mathbf{m},\mathbf{h}} = \left(\frac{\sigma_{\mathbf{m}}}{\sigma_{\mathbf{h}}} \right)^{k-1} \left(\prod_{s_2=1}^k P_{s_2,\mathbf{m},\mathbf{h}} \right).$$

If we substitute by $k = 2$ and consider polynomials of two complex variables instead of several complex variables, then we will obtain Definition 1.3. It should be remarked here that Definition 1.4 is different from that used in [14].

2. Main results

In this section, we will study the effectiveness of the extended Hadamard product of simple monic sets of polynomials of several complex variables defined by (3) and (4) in closed hyperelliptical regions and at the origin.

Let $\{P_{s_2, \mathbf{m}}[\mathbf{z}]\}$ be simple monic sets of polynomials of several complex variables z_s ; $s \in I$, so that we can write

$$(5) \quad P_{s_2, \mathbf{m}}[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} P_{s_2, \mathbf{m}, \mathbf{h}} \mathbf{z}^{\mathbf{h}},$$

where

$$P_{s_2, m_1, m_2, \dots, m_n}^{m_1, m_2, \dots, m_n} = 1; \quad s_2 = 1, 2, \dots, k.$$

The normalizing functions of the sets $\{P_{s_2, \mathbf{m}}[\mathbf{z}]\}$ are defined by (see [18])

$$(6) \quad \mu(P_{s_2}; r_s) = \lim_{\langle \mathbf{m} \rangle \rightarrow \infty} \sup \left\{ \sigma_{\mathbf{m}} M[P_{s_2, \mathbf{m}}; r_s] \right\}^{\frac{1}{\langle \mathbf{m} \rangle}},$$

where $M[P_{s_2, \mathbf{m}}; r_s]$ are defined as follows:

$$M[P_{s_2, \mathbf{m}}; r_s] = \sup_{\mathbf{E}_{r_s}} |P_{s_2, \mathbf{m}}[\mathbf{z}]|.$$

Notice that the sets $\{P_{s_2, \mathbf{m}}[\mathbf{z}]\}$ are monic. By applying Cauchy's inequality in (2), we have

$$|P_{s_2, \mathbf{m}, \mathbf{h}}| \leq \frac{\sigma_{\mathbf{m}}}{r_s^{\langle \mathbf{m} \rangle}} \sup_{\mathbf{E}_{r_s}} |P_{s_2, \mathbf{m}}[\mathbf{z}]|,$$

which implies that

$$M[P_{s_2, \mathbf{m}}; r_s] \geq \frac{r_s^{\langle \mathbf{m} \rangle}}{\sigma_{\mathbf{m}}}.$$

It follows from (6) that

$$(7) \quad \mu(P_{s_2}; r_s) \geq r_s.$$

Next, we show if ρ are positive numbers greater than r_s , then

$$(8) \quad \mu(P_{s_2}; \rho_s) \leq \frac{\rho_s}{r_s} \mu(P_{s_2}; r_s), \quad \rho_s > r_s.$$

In fact, this relation follows by applying (6) to the inequality

$$M[P_{s_2, \mathbf{m}}; \rho_s] \leq K \left(\frac{\rho_s}{r_s} \right)^{\langle \mathbf{m} \rangle} M[P_{s_2, \mathbf{m}}; r_s],$$

which in its turn, is derivable from (6), Cauchy's inequality and the supremum of $\mathbf{z}^{\mathbf{m}}$, where $K = \mathcal{O}(\langle \mathbf{m} \rangle + 1)$.

Now, let $\{P_{s_2, \mathbf{m}}[\mathbf{z}]\}; s_2 = 1, 2, 3, \dots, k$ be simple monic sets of polynomials of several complex variables, and that $\{H_{\mathbf{m}}^*[\mathbf{z}]\}$ is the set defined as follows

$$(9) \quad H_{\mathbf{m}}^*[\mathbf{z}] = \prod_{s_2=1}^k P_{s_2, \mathbf{m}}[\mathbf{z}].$$

The following fundamental result is proved.

Theorem 2.1 *If, for any $r_s > 0$*

$$(10) \quad \mu(P_{s_2}; r_s) = r_s,$$

then

$$(11) \quad \mu(H^*; r_s) = r_s.$$

Proof. We first observe that, if ρ be any finite number greater than r , then by (6), (7) and (9), we obtain that

$$(12) \quad \mu(P_{s_2}; \rho_s) = \rho_s.$$

Now, given $r_s^* > r_s$, we choose finite number r'_s such that

$$(13) \quad r_s < r'_s < r_s^*.$$

Then by (6) and (10), we obtain that

$$(14) \quad M(P_{s_2, \mathbf{h}}; r_s) < \frac{\eta}{\sigma_{\mathbf{h}}} (r'_s)^{\langle \mathbf{h} \rangle} \quad \text{where } \eta > 1,$$

where $\langle \mathbf{h} \rangle = h_1 + h_2 + h_3 + \dots + h_n$. Also from (9), we can write

$$H_{\mathbf{m}}^*[\mathbf{z}] = \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} \prod_{s_2=1}^k P_{s_2, \mathbf{m}, \mathbf{h}} P_{s_2, \mathbf{h}}[\mathbf{z}].$$

Hence (13) and (14) lead to

$$M[H_{\mathbf{m}}^*; r_s] \leq \eta K \left(1 - \left(\frac{r'_s}{r_s^*} \right)^n \right)^{-n} M[P_{s_2, \mathbf{m}}; r_s^*],$$

Making $\langle \mathbf{m} \rangle \rightarrow \infty$ and applying (11), we get

$$\mu(H^*; r_s) = \lim_{\langle \mathbf{m} \rangle \rightarrow \infty} \sup \left\{ \sigma_{\mathbf{m}} M[H_{\mathbf{m}}^*; r_s] \right\}^{\frac{1}{\langle \mathbf{m} \rangle}} \leq \mu(P_{s_2}; r_s^*) = r_s^*,$$

which leads to the equality (10), by the choice of r_s^* near to r_s , and our theorem is therefore proved.

From Theorem 2.1 if we consider the simple monic sets $\{P_{s_2, \mathbf{m}}[\mathbf{z}]\}$ accord to condition (10), then it is not hard to prove by induction for the j -power sets $\{P_{s_2, \mathbf{m}}^{(j)}[\mathbf{z}]\}$ that

$$(15) \quad \mu(P_{s_2}^{(j)}; r_s) = r_s.$$

This completes the proof.

Now, we give the following result.

Theorem 2.2. *Let $\{P_{s_2, \mathbf{m}}[\mathbf{z}]\}; s_2 = 1, 2, 3, \dots, k$ be simple monic algebraic sets of polynomials of several complex variables, which accord to condition (10). Then the set will be effective in the closed hyperellipse $\bar{\mathbf{E}}_{r_s}$.*

Proof. Suppose that the monomial $\mathbf{z}^{\mathbf{m}}$ admit the representation

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \bar{P}_{\mathbf{m}, \mathbf{h}} P_{\mathbf{h}}[\mathbf{z}].$$

Since the set $\{P_{1, \mathbf{m}}[\mathbf{z}]\}$ is algebraic, we find there exists a relation of the form

$$(16) \quad \bar{P}_{1, \mathbf{m}, \mathbf{h}} = \sum_{j=1}^k \alpha_j P_{1, \mathbf{m}, \mathbf{h}}^{(j)}; \quad ((\mathbf{h}) \leq (\mathbf{m})),$$

where k is a finite positive integer which together with the coefficients $(\alpha_j)_{j=1}^k$, is independent of the indices $(\mathbf{m}), (\mathbf{h})$. The coefficients $P_{1, \mathbf{m}, \mathbf{h}}^{(j)}$ are defined by

$$P_{1, \mathbf{m}}^{(j)}[\mathbf{z}] = \sum_{(\mathbf{h})=1}^{(\mathbf{m})} P_{1, \mathbf{m}, \mathbf{h}}^{(j)} \mathbf{z}^{\mathbf{h}}; \quad 1 \leq j \leq k.$$

It follows that

$$(17) \quad \left| P_{1, \mathbf{m}, \mathbf{h}}^{(j)} \right| r_s^{\langle \mathbf{m} \rangle} \leq \sigma_{\mathbf{h}} M[P_{1, \mathbf{m}}^{(j)}; r_s].$$

According to (15) for given $r_s^* > r_s$ and from the definition corresponding to $\mu(P_1^{(j)}; r_s)$, we deduce that

$$(18) \quad M[P_{1,\mathbf{h}}^{(j)}; r_s] < \frac{K}{\sigma_{\mathbf{h}}}(r_s^*)^{(\mathbf{h})}.$$

Applying (17) and (18) in (16), we obtain that

$$(19) \quad |\overline{P}_{1,\mathbf{m},\mathbf{h}}^{(j)}| < \zeta K \beta \frac{\sigma_{\mathbf{h}}(r_s^*)^{(\mathbf{m})}}{\sigma_{\mathbf{m}}(r_s)^{(\mathbf{m})}},$$

where

$$(20) \quad \beta = \max\{|\alpha_j|; 0 \leq j \leq k\} \quad \text{and} \quad \zeta \quad \text{is a constant.}$$

In view of the representation

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}],$$

the Cannon sum of the set $\{P_{1,\mathbf{m}}^{(j)}[\mathbf{z}]\}$ will be

$$(21) \quad \Omega(P_{1,\mathbf{m}}^{(j)}; r_s) = \sigma_{\mathbf{m}} \sum_{(\mathbf{h})=\mathbf{0}}^{(\mathbf{m})} |\overline{P}_{\mathbf{m},\mathbf{h}}^{(j)}| M[P_{1,\mathbf{h}}^{(j)}; r_s],$$

where,

$$(22) \quad M[P_{1,\mathbf{h}}^{(j)}; r_s] = \sup_{\overline{\mathbf{E}}_{r_s}} |P_{1,\mathbf{m}}^{(j)}[\mathbf{z}]|.$$

Therefore (18), (19) and (21) (for $r_s^* > r_s$) give

$$(23) \quad \Omega(P_{1,\mathbf{m}}^{(j)}; r_s) < \zeta K \beta (r_s^*)^{(\mathbf{m})}.$$

Hence the Cannon function of the set $\{P_{1,\mathbf{m}}^{(j)}[\mathbf{z}]\}$ turns out to be

$$\Omega(P_1^{(j)}; r_s) = \lim_{(\mathbf{m}) \rightarrow \infty} \left\{ \Omega(P_{1,\mathbf{m}}^{(j)}; r_s) \right\}^{\frac{1}{(\mathbf{m})}} = r_s^*,$$

which, by the choice of r_s^* , implies that

$$\Omega(P_1^{(j)}; r_s) = r_s.$$

As very similar, we can obtain that the sets $\{P_{\nu,\mathbf{m}}^{(j)}[\mathbf{z}]\}; \nu = 2, 3, 4, \dots, k$ will be effective in the hyperellipse $\overline{\mathbf{E}}_{r_s}$. Our theorem is therefore proved.

As a consequence of Theorem 2.1, we give the following results.

Corollary 2.1 *Let $\{P_{s_2, \mathbf{m}}[\mathbf{z}]\}; s_2 = 1, 2, 3, \dots, k$ be simple monic sets of polynomials of several complex variables, which accord to the following condition:*

$$(24) \quad \mu(P_{s_2}; 0^+) = 0 \quad , \quad s_2 = 1, 2, 3, \dots, k.$$

Then, for the set $\{H_{\mathbf{m}}^[\mathbf{z}]\}$ of (9), we have*

$$(25) \quad \mu(H^*; 0^+) = 0,$$

and

$$(26) \quad \mu(P_{s_2}; 0^+) = \lim_{r \rightarrow 0^+} \mu(P_{s_2}; r_s).$$

Proof. The proof of this result is much akin to that of Lemma 3 in [18]. We can again, apply Corollary 2.1, to prove by induction that, if the simple monic sets $\{P_{s_2, \mathbf{m}}[\mathbf{z}]\}; s_2 = 1, 2, 3, \dots, k$ accord to condition (24), then for the j -power set $\{P_{s_2, \mathbf{m}}^{(j)}[\mathbf{z}]\}$, we can obtain that

$$(27) \quad \mu(P_{s_2}^{(j)}; 0^+) = 0.$$

Corollary 2.2 *Let $\{P_{s_2, \mathbf{m}}[\mathbf{z}]\}; s_2 = 1, 2, 3, \dots, k$ are algebraic sets of polynomials of several complex variables and accord to the following condition*

$$(28) \quad \mu(P_{s_2}; 0^+) = 0.$$

Then the sets will be effective at the origin.

Proof. The proof of Corollary 2.2 is similar to that of Lemma 4 in [18], so it is omitted.

REFERENCES

- [1] M.A. Abul-Ez, Hadamard product of bases of polynomials in Clifford analysis, Complex variables theory Appl. 43 (2000) 109-128.
- [2] L. Aloui, M.A. Abul-Ez, G.F. Hassan, On the order of the difference and sum bases of polynomials in Clifford setting, Complex Var. Elliptic Equ, 55 (2010) 1117-1130.

- [3] L. Aloui, G.F. Hassan, Hypercomplex derivative bases of polynomials in Clifford analysis, *Math. Methods Appl. Sci.* 33 (2010) 350-357.
- [4] A. El-Sayed Ahmed, On some classes and spaces of holomorphic and hyperholomorphic functions, *Dissertationes, Bauhaus University at Weimar-Germany* (2003).
- [5] A.El-Sayed Ahmed, On derived and integrated sets of basic sets of polynomials of several complex variables, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* 19 (2003) 195-204.
- [6] A.El-Sayed Ahmed, Extended results of the Hadamard product of simple sets of polynomials in hypersphere, *Annales Societatis Mathematicae Polonae Seria I, Commentationes Mathematicae, XLVI* (2006) 201-214.
- [7] A.El-Sayed Ahmed, Z.M.G. Kishka, On the effectiveness of basic sets of polynomials of several complex variables in elliptical regions, In: Begehr et al. (Eds) *Progress in Analysis, proceedings of the 3rd International ISAAC Congress, Freie Universitaet Berlin-Germany, August, 20-25, 2001.* Kluwer. Acad. Publ. Vol I (2003), 265-278.
- [8] G.F. Hassan, A note on the growth order of the inverse and product bases of special monogenic polynomials, *Math. Methods Appl. Sci.* 35 (2012) 286-292.
- [9] Z.M.G. Kishka, Power set of simple set of polynomials of two complex variables, *Bull. Soc. R. Sci. Liege.* 62 (1993) 361-372.
- [10] Z.M.G. Kishka, A.El-Sayed Ahmed, On the order and type of basic and composite sets of polynomials in complete Reinhardt domains, *Period. Math. Hung.* 46 (2003) 67-79.
- [11] W.F. Kumuyi, M. Nassif, Derived and integrated sets of simple sets of polynomials in two complex variables, *J. Approx. Theory* 47 (1986) 270-283.
- [12] R.H. Makar, B.H. Makar, On algebraic simple monic sets of polynomials, *Proc. Amer. Math. Soc.* 2 (1951) 526-537.
- [13] S.Z. Melek, A.E. El-Said, On Hadamard product of basic sets of polynomials, *Bull. Fac. of Engineering, Ain Shams Univ.* 16 (1985) 1-14.
- [14] M. S. Metwally, On the generalized Hadamard product functions of two complex matrices, *Int. Math. J.* 1 (2002) 171-183.
- [15] M. Mursi, B. H. Makar, Basic sets of polynomials of several complex variables I, *Proceedings of the Second Arab Science Congress, Cairo* (1955), 51-60.
- [16] M. Mursi, B. H. Makar, Basic sets of polynomials of several complex variables II, *Proceedings of the Second Arab Science Congress, Cairo* (1955) 61-68.
- [17] M. Nassif, Composite sets of polynomials of several complex variables, *Publ. Math. Debrecen*, 18 (1971) 43-53.

- [18] M. Nassif, S. W. Rizk, Hadamard product of simple sets of polynomials of two complex variables, Bull. Fac. of Engineering, Ain Shams Univ. 18 (1988) 97-116.
- [19] M.A. Saleem, M.A. Abul-Ez, M. Zayed, On polynomial series expansions of Cliffordian functions, Math. Methods Appl. Sci. 35 (2012) 134-143.
- [20] K.A.M. Sayyed, M.S. Metwally, Effectiveness of similar sets of polynomials of two complex variables in polycylinders and in Faber regions, Int. J. Math. Math. Sci. 21 (1998) 587-593.
- [21] J. M. Whittaker, Sure les series de Base de polynomes quelconques, Paris 1949.