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THE PROPERTIES OF GENERALIZED TRIANGLE ALGEBRAS

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Abstract. In this paper, we introduce the notions of generalized triangle algebras in a generalized residuated lattices. Moreover, we investigate their properties.

Keywords: generalized residuated lattices; generalized triangle algebras

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1. Introduction

Hájek [8] introduced a complete residuated lattice which is an algebraic structure for many valued logic. When we consider the conjunction to be non-commutative, generalized residuated lattice was introduced by Georgescu and Popescue [6,7]. Non-commutativity induces two implications. By using these concepts, information systems and decision rules are investigated [2,9,12]. Deschrijver, et.al. [3-5,10,11] introduced triangle algebras and interval-valued residuated lattices.

In this paper, we introduce the notions of generalized triangle algebras in a generalized residuated lattices. Moreover, we investigate their properties.

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Definition 1.1.[6,7] A structure $(L, \vee, \wedge, \odot, \rightarrow, \Rightarrow, \bot, \top)$ is called a *generalized residuated lattice* if it satisfies the following conditions:

(GR1) $(L, \vee, \wedge, \top, \bot)$ is a bounded where $\top$ is the universal upper bound and $\bot$ denotes the universal lower bound;

(GR2) $(L, \odot, \top)$ is a monoid;

(GR3) it satisfies a residuation , i.e.

$$a \odot b \leq c \text{ iff } a \leq b \rightarrow c \text{ iff } b \leq a \Rightarrow c.$$

We call that a generalized residuated lattice has the law of double negation if $a = (a^*)^0 = (a^0)^*$ where $a^0 = a \rightarrow \bot$ and $a^* = a \Rightarrow \bot$.

Remark 1.2.[6,7,12] (1) A generalized residuated lattice is a residuated lattice ($\rightarrow = \Rightarrow$) iff $\odot$ is commutative.

(2) A left-continuous t-norm $([0,1], \leq, \odot)$ defined by $a \rightarrow b = \bigvee\{c \mid a \odot c \leq b\}$ is a residuated lattice

(3) A pseudo MV-algebra is a generalized residuated lattice with the law of double negation.

Lemma 1.3.[4,5] Let $(L, \wedge, \vee, \odot, \rightarrow, \Rightarrow, \bot, \top)$ be a generalized residuated lattice with the law of double negation.

For each $x, y, z, x_i, y_i \in L$, we have the following properties.

(1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$ for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(2) $x \odot y \leq x \wedge y$.

(3) $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$ for $\rightarrow \in \{\rightarrow, \Rightarrow\}$.

(4) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$ and $(x \odot y) \Rightarrow z = y \Rightarrow (x \Rightarrow z)$.

(5) $(x \odot y)^0 = x \rightarrow y^0$ and $(x \odot y)^* = y \Rightarrow x^*$.

(6) $x \rightarrow (y \Rightarrow z) = y \Rightarrow (x \rightarrow z)$ and $x \Rightarrow (y \rightarrow z) = y \rightarrow (x \Rightarrow z)$.

(7) $x \odot (x \rightarrow y) \leq y$ and $(x \Rightarrow y) \odot x \leq y$.

(8) $(x \Rightarrow y) \odot (y \Rightarrow z) \leq x \Rightarrow z$ and $(y \rightarrow z) \odot (x \rightarrow y) \leq x \rightarrow z$.

(9) $x \rightarrow y = \top$ iff $x \leq y$ iff $x \Rightarrow y = \top$.

(10) $x \rightarrow y = y^0 \Rightarrow x^0$ and $x \Rightarrow y = y^* \rightarrow x^*$.

(11) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(12) $\bigwedge_{i\in\Gamma} x_i^0 = (\bigvee_{i\in\Gamma} x_i)^0$ and $\bigvee_{i\in\Gamma} x_i^0 = (\bigwedge_{i\in\Gamma} x_i)^0$.

Definition 1.4. A structure $(A, \wedge, \vee, \odot, \Rightarrow, \rightarrow, \nu_i, \mu_i, \bot, e_i, \top)$ for $i \in \{1,2\}$ is called a *generalized triangle algebra* if it satisfies the following conditions:

(R) $(A, \wedge, \vee, \odot, \Rightarrow, \rightarrow, \bot, \top)$ is a generalized residuated lattice.

(T1) $\nu_i(x) \leq x$ and $\nu_i(x) \leq \nu_i(\nu_i(x))$.

(T2) $\nu_i(x \wedge y) = \nu_i(x) \wedge \nu_i(y)$ and $\nu_i(x \vee y) = \nu_i(x) \vee \nu_i(y)$.

(T3) $\nu_i(e_i) = \bot$.

(T4) $\nu_i \circ \mu_i = \mu_i$.

(S1) $x \leq \mu_i(x)$ and $\mu_i(x) \geq \mu_i(\mu_i(x))$.

(S2) $\mu_i(x \wedge y) = \mu_i(x) \wedge \mu_i(y)$ and $\mu_i(x \vee y) = \mu_i(x) \vee \mu_i(y)$.

(S3) $\mu_i(e_i) = \top$.

(S4) $\mu_i \circ \nu_i = \nu_i$.

(T5) $\nu_1(x \Rightarrow y) \leq \nu_1(x) \Rightarrow \nu_1(x)$ and $\nu_2(x \rightarrow y) \leq \nu_2(x) \rightarrow \nu_1(x)$.

(T6) $(\nu_1(x) \Leftrightarrow \nu_1(y)) \odot (\mu_1(x) \Leftrightarrow \mu_1(y)) \leq (x \Leftrightarrow y)$ and $(\mu_2(x) \leftrightarrow \mu_2(y)) \odot (\nu_2(x) \leftrightarrow \nu_2(y)) \leq (x \leftrightarrow y)$.

(T7) $\nu_1(x) \Rightarrow \nu_1(y) \leq \nu_1(\nu_1(x) \Rightarrow \nu_1(y))$ and $\nu_2(x) \rightarrow \nu_2(y) \leq \nu_2(\nu_2(x) \rightarrow \nu_2(y))$.

Remark 1.5. (1) If $\odot$ is commutative (or $\Rightarrow=\rightarrow$), $\nu_1 = \nu_2$, $\mu_1 = \mu_2$ and $e_1 = e_2$ in Definition 1.4, then $(A, \wedge, \vee, \odot, \Rightarrow, \nu_1, \mu_1, \bot, e_1, \top)$ is a triangle algebra in [10].

(2) In Definition 1.4, $\nu_i(\top) = \top$ and $\mu_i(\bot) = \bot$ because $\nu_i(\top) = \nu_i(\mu_i(e_i)) = \mu_i(e_i) = \top$ and $\mu_i(\bot) = \mu_i(\nu_i(e_i)) = \nu_i(e_i) = \bot$.

2. The properties of generalized triangle algebras

Theorem 2.1. Let $(A, \wedge, \vee, \odot, \Rightarrow, \rightarrow, \bot, \top)$ be a generalized residuated lattice and ν_i be unary operator on A satisfying $\nu_i(\top) = \top$ for $i \in \{1,2\}$. Then ν_i satisfies (T5) iff $\nu_i(x \wedge y) \leq \nu_i(x)$ and $\nu_i(x) \odot \nu_i(y) \leq \nu_i(x \odot y)$ for $i \in \{1,2\}$ and $x, y \in A$.

Proof. ($\Rightarrow$) Since $\top = x \wedge y \to x = x \wedge y \Rightarrow x$, then

$$\top = \nu_1(1) = \nu_1((x \wedge y) \Rightarrow x) \leq \nu_1(x \wedge y) \Rightarrow \nu_1(x) \text{ iff } \nu_1(x \wedge y) \leq \nu_1(x)$$

$$\top = \nu_2(1) = \nu_2((x \wedge y) \to x) \leq \nu_2(x \wedge y) \to \nu_2(x) \text{ iff } \nu_2(x \wedge y) \leq \nu_2(x).$$

Moreover, $y \leq x$, then $\nu_i(y) \leq \nu_i(x)$ for $i \in \{1, 2\}$. Since $y \leq x \Rightarrow x \odot y$ and $x \leq y \to x \odot y$, then

$$\nu_1(y) \leq \nu_1(x \Rightarrow x \odot y) \leq \nu_1(x) \Rightarrow \nu_1(x \odot y) \text{ iff } \nu_1(x) \odot \nu_1(y) \leq \nu_1(x \odot y)$$

$$\nu_2(x) \leq \nu_2(y \to x \odot y) \leq \nu_2(x) \to \nu_2(x \odot y) \text{ iff } \nu_2(x) \odot \nu_2(y) \leq \nu_2(x \odot y).$$

($\Leftarrow$) Since $\nu_i(x \wedge y) \leq \nu_i(x)$ for each $i \in \{1, 2\}$, then ν_i is an increasing function. Since $x \odot (x \Rightarrow y) \leq y$ and $(x \to y) \odot x \leq y$, then

$$\begin{aligned} &\nu_1(x) \odot \nu_1(x \Rightarrow y) \leq \nu_1(x \odot (x \Rightarrow y)) \leq \nu_1(y) \\ &\text{ iff } \nu_1(x \Rightarrow y) \leq \nu_1(x) \Rightarrow \nu_1(y), \\ &\nu_2(x \to y) \odot \nu_2(x) \leq \nu_2((x \to y) \odot x) \leq \nu_2(y) \\ &\text{ iff } \nu_2(x \to y) \leq \nu_2(x) \to \nu_2(y). \end{aligned}$$

Remark 2.2. Let $(A, \wedge, \vee, \odot, \Rightarrow, \to, \nu_i, \mu_i, \bot, e_i, \top)$ for $i \in \{1, 2\}$ be a generalized residuated lattice. Since ν_i satisfies (T3) and (T4), by Remark 1.5 (2), ν_i satisfies (T5) iff $\nu_i(x) \odot \nu_i(y) \leq \nu_i(x \odot y)$ for $i \in \{1, 2\}$ and $x, y \in A$.

Theorem 2.3. Let $(A, \wedge, \vee, \odot, \Rightarrow, \to, \nu_i, \mu_i, \bot, e, \top)$ for $i \in \{1, 2\}$ be a generalized residuated lattice with $e \in A$ such that $e = e^0$. For each $i \in \{1, 2\}$, Define

$$\begin{aligned} &E_i(A) = \{x \in A \mid \nu_i(x) = x\}, \ \ \nu_i(A) = \{\nu_i(x) \mid x \in A\}, \\ &\nu_i(E_i(A)) = \{\nu_i(x) \mid \nu_i(x) = \nu_i(\nu_i(x))\}, \ F_i(A) = \{x \in A \mid \mu_i(x) = x\}, \\ &\mu_i(A) = \{\mu_i(x) \mid x \in A\}, \ \ \mu_i(F_i(A)) = \{\mu_i(x) \mid \mu_i(x) = \mu_i(\mu_i(x))\}. \end{aligned}$$

Then

(1) $E_i(A) = \nu_i(A) = \nu_i(E_i(A)) = F_i(A) = \mu_i(A) = \mu_i(F_i(A))$.

(2) $E_1(A)$ is closed under $\wedge, \vee, \odot$ and $\Rightarrow$.

(3) $E_2(A)$ is closed under $\wedge, \vee, \odot$ and $\to$.

(4) $E_i(A)$ is a complete bounded lattice with top elements $\top$ and bottom element $\bot$.

(5) $\nu_i(x) = \bigvee\{y \in E_i(A) \mid y \leq x\}$ and $\mu_i(x) = \bigwedge\{y \in E_i(A) \mid x \leq y\}$, for each $i \in \{1,2\}$.

(6) $\nu_i(x) \odot \nu_i(y) \leq \nu_i(x \odot y)$ and $\mu_i(x) \odot \mu_i(y) \geq \mu_i(x \odot y)$ for all $x, y \in A$.

(7) $x \leq y$ iff $\nu_1(x) \leq \nu_1(y)$ and $\mu_1(x) \leq \mu_1(y)$ iff $\nu_2(x) \leq \nu_2(y)$ and $\mu_2(x) \leq \mu_2(y)$.

Proof. (1) Let $x \in E_i(A)$. Then $\nu_i(x) = x \in \nu_i(A)$. So, $E_i(A) \subset \nu_i(A)$. Let $\nu_i(x) \in \nu_i(A)$. Since $\nu_i(x) = \nu_i(\nu_i(x))$, $\nu_i(x) \in \nu_i(E_i(A))$. Thus, $\nu_i(A) \subset \nu_i(E_i(A))$. Trivially, $\nu_i(E_i(A)) \subset E_i(A)$.

Let $x \in E_i(A)$. Since $\nu_i(x) = x$, by (T4),

$$\mu_i(x) = \mu_i(\nu_i(x)) = \nu_i(x) = x \in E_i(A).$$

Let $y \in F_i(A)$. Since $\mu_i(y) = y$, by (S4), $\nu_i(y) = \nu_i(\mu_i(y)) = \mu_i(y) = y$. $y \in E_i(A)$. Hence $E_i(A) = F_i(A)$. Similarly, $F_i(A) = \mu_i(A) = \mu_i(F_i(A))$.

(2) and (3). For $x, y \in E_i(A)$, $\nu_i(x \wedge y) = \nu_i(x) \wedge \nu_i(y) = x \wedge y$ and $\nu_i(x \vee y) = \nu_i(x) \vee \nu_i(y) = x \vee y$. Then $x \wedge y, x \vee y \in E_i(A)$. Since $x \Rightarrow y = \nu_1(x) \Rightarrow \nu_1(y) \leq \nu_1(\nu_1(x) \Rightarrow \nu_1(y)) \leq \nu_1(x) \Rightarrow \nu_1(y) = x \Rightarrow y$, then $x \Rightarrow y \in E_1(A)$. Similarly, $x \rightarrow y \in E_2(A)$. Since $x \odot y = \nu_i(x) \odot \nu_i(y) \leq \nu_i(x \odot y) \leq x \odot y$ from Theorem 2.1, then $x \odot y \in E_i(A)$.

(4) For $x_j \in E_i(A)$, since $\bigvee_{j \in \Gamma} \nu_i(x_j) \leq \nu_i(\bigvee_{j \in \Gamma} x_j) \leq \bigvee_{j \in \Gamma} x_j = \bigvee_{j \in \Gamma} \nu_i(x_j)$, then $\nu_i(\bigvee_{j \in \Gamma} x_j) = \bigvee_{j \in \Gamma} x_j \in E_i(A)$. Since $\bigwedge_{j \in \Gamma} \nu_i \leq \mu_i(\bigwedge_{j \in \Gamma} x_j) \leq \bigwedge_{j \in \Gamma} \mu_i(x_j) = \bigwedge_{j \in \Gamma} x_j$, then $\mu_i(\bigwedge_{j \in \Gamma} x_j) = \bigwedge_{j \in \Gamma} x_j \in E_i(A)$.

(5) Since $\nu_i(x) \leq x \leq \mu_i(x)$ and $\nu_i(x), \mu_i(x) \in E_i(A)$, then $\nu_i(x) \leq \bigvee\{y \in E_i(A) \mid y \leq x\}$ and $\mu_i(x) \geq \bigwedge\{y \in E_i(A) \mid x \leq y\}$. For $y \in E_i(A)$ and $y \leq x$, we have $y = \nu_i(y) \leq \nu_i(x) \leq x$. Hence $\nu_i(x) \geq \bigvee\{y \in E_i(A) \mid y \leq x\}$. For $y \in E_i(A)$ and $x \leq y$, we have $y = \mu_i(y) \geq \mu_i(x) \geq x$. Hence $\mu_i(x) \leq \bigwedge\{y \in E_i(A) \mid x \leq y\}$. So, $\nu_i(x) = \bigvee\{y \in E_i(A) \mid y \leq x\}$ and $\mu_i(x) = \bigwedge\{y \in E_i(A) \mid x \leq y\}$, for each $i \in \{1,2\}$.

(6) Since $\nu_i(x) \odot \nu_i(y) \leq x \odot y$ and $\nu_i(x) \odot \nu_i(y) \in E_i(A)$ from (4), $\nu_i(x) \odot \nu_i(y) \leq \nu_i(x \odot y)$. Since $\mu_i(x) \odot \mu_i(y) \geq x \odot y$ and $\mu_i(x) \odot \mu_i(y) \in E_i(A)$ from (4), $\mu_i(x) \odot \mu_i(y) \geq \mu_i(x \odot y)$ for all $x, y \in A$.

(7) If $x \leq y$, then $\nu_i(x) \leq \nu_i(y)$ and $\mu_i(x) \leq \mu_i(y)$ for $i \in \{1,2\}$.

Let $\nu_i(x) \leq \nu_i(y)$ and $\mu_i(x) \leq \mu_i(y)$ for $i \in \{1,2\}$. Then $\nu_i(x \wedge y) = \nu_i(x)$ and $\mu_i(x \wedge y) = \mu_i(x)$ By(T6), $\top \odot \top = (\nu_1(x \wedge y) \Leftrightarrow \nu_1(x)) \odot (\mu_1(x \wedge y) \Leftrightarrow \mu_1(x)) \leq (x \wedge y \Leftrightarrow x)$ implies $x \wedge y = x$. Thus $x \leq y$ Moreover, $\top \odot \top = (\mu_2(x \wedge y) \leftrightarrow \mu_2(x)) \odot (\nu_2(x \wedge y) \leftrightarrow \nu_2(x)) \leq (x \wedge y \leftrightarrow x)$ implies $x \wedge y = x$. Thus $x \leq y$.

Theorem 2.4. Let $(A, \wedge, \vee, \odot, \Rightarrow, \rightarrow, \nu_i, \mu_i, \bot, e_i, \top)$ be a generalized triangle algebra for $i \in \{1,2\}$. Then

(1) $\nu_i(x) \vee e_i = x \vee e$ and $\mu_i(x) \wedge e_i = x \wedge e_i$.

(2) $x = (\nu_i(x) \vee e_i) \wedge \mu_i(x) = (\mu_i(x) \wedge e_i) \vee \nu_i(x)$.

(3) $\nu_i(x \odot e_i) = \nu_i(e_i \odot x) = \bot$.

(4) $\nu_i(x \odot y) = \nu_i((x \vee e_i) \odot (y \vee e_i))$.

(5) $\nu_i(x \odot y) = \nu_i(x) \odot \nu_i(y)$.

Proof. (1) Since $\nu_i(\nu_i(x) \vee e_i) = \nu_i(\nu_i(x)) \vee \nu_i(e_i) = \nu_i(x) \vee \nu_i(e_i) = \nu_i(x \vee e_i)$ and $\mu_i(\nu_i(x) \vee e_i) = \mu_i(\nu_i(x)) \vee \mu_i(e_i) = \top = \mu_i(x) \vee \mu_i(e_i) = \mu_i(x \vee e_i)$, we have

$$\begin{aligned}
\top &= \top \odot \top = (\nu_1(\nu_1(x) \vee e_1) \leftrightarrow \nu_1(x \vee e_1)) \odot (\mu_1(\nu_1(x) \vee e_1) \leftrightarrow \mu_1(x \vee e_1)) \\
&\leq (\nu_1(x) \vee e_1 \leftrightarrow (x \vee e_1)) \\
\top &= \top \odot \top = (\nu_2(\nu_2(x) \vee e_2) \Leftrightarrow \nu_2(x \vee e_2)) \odot (\mu_2(\nu_2(x) \vee e_2) \Leftrightarrow \mu_2(x \vee e_2)) \\
&\leq (\nu_2(x) \vee e_2 \Leftrightarrow (x \vee e_2))
\end{aligned}$$

Hence $\nu_i(x) \vee e_i = x \vee e_i$.

Since $\nu_i(\mu_i(x) \wedge e_i) = \nu_i(\mu_i(x)) \wedge \nu_i(e_i) = \bot = \nu_i(x) \wedge \nu_i(e_i) = \nu_i(x \wedge e_i)$ and $\mu_i(\mu_i(x) \wedge e_i) = \mu_i(\mu_i(x)) \wedge \mu_i(e_i) = \mu_i(x) \wedge \mu_i(e_i) = \mu_i(x \wedge e_i)$, by a similar way in the above proof, $\mu_i(x) \wedge e_i = x \wedge e_i$.

(2) $\nu_i((\nu_i(x) \vee e_i) \wedge \mu_i(x)) = \nu_i((x \vee e_i) \wedge \mu_i(x)) = (\nu_i(x) \vee \nu_i(e_i)) \wedge \nu_i(\mu_i(x)) = \nu_i(x) \wedge \mu_i(x) = \nu_i(x)$

$\mu_i((\nu_i(x) \vee e_i) \wedge \mu_i(x)) = \mu_i((x \vee e_i) \wedge \mu_i(x)) = (\mu_i(x) \vee \mu_i(e_i)) \wedge \mu_i(\mu_i(x)) = \top \wedge \mu_i(x) = \mu_i(x)$

(3) Since $x \odot e_i \leq e_i$, then $\nu_i(x \odot e_i) \leq \nu_i(e_i) = \bot$ and $\nu_i(e_i \odot x) \leq \nu_i(e_i) = \bot$. Thus, $\nu_i(x \odot e_i) = \nu_i(e_i \odot x) = \bot$.

(4) Since $\nu_i(x \odot e_i) = \nu_i(e_i \odot x) = \bot$, we have

$$\begin{aligned} \nu_i(x \odot y) &= \nu_i(x \odot y) \vee \nu_i(x \odot e_i) = \nu_i((x \odot y) \vee (x \odot e_i)) \\ &= \nu_i(x \odot (y \vee e_i)), \\ \nu_i(x \odot (y \vee e_i)) &= \nu_i(x \odot (y \vee e_i)) \vee \nu_i(e_i \odot (y \vee e_i)) \\ &= \nu_i((x \odot (y \vee e_i)) \vee (e_i \odot (y \vee e))) = \nu_i((x \vee e_i) \odot (y \vee e_i)). \end{aligned}$$

(5) Since $\nu_i(x) \vee e_i = x \vee e_i$ and $\nu_i(y) \vee e_i = y \vee e_i$,

$$\begin{aligned} \nu_i(x \odot y) &= \nu_i((x \vee e_i) \odot (y \vee e_i)) = \nu_i((\nu_i(x) \vee e_i) \odot (\nu_i(x) \vee e_i)) \\ &= \nu_i(\nu_i(x) \odot \nu_i(y)) \leq \nu_i(x) \odot \nu_i(y). \end{aligned}$$

By Remark 2.2, since $\nu_i(x) \odot \nu_i(y) \leq \nu_i(x \odot y)$, we have $\nu_i(x) \odot \nu_i(y) = \nu_i(x \odot y)$.

Theorem 2.5. Let $(A, \wedge, \vee, \odot, \Rightarrow, \rightarrow, \nu_i, \mu_i, \bot, e, \top)$ be a generalized triangle algebra for $i \in \{1, 2\}$. Then the following statements are equivalent:

(1) $\mu_i(x \odot y) = \mu_i(x) \odot \mu_i(y)$ for all $x, y \in A$.

(2) $\mu_i(x \odot y) = \mu_i(x \odot z)$ if $\mu_i(y) = \mu_i(z)$.

Proof. (1)$\Rightarrow$(2). If $\mu_i(y) = \mu_i(z)$, then

$$\mu_i(x \odot y) = \mu_i(x) \odot \mu_i(y) = \mu_i(x) \odot \mu_i(z) = \mu_i(x \odot z).$$

(2)$\Rightarrow$(1). Since $\mu_i(\mu_i(y)) = \mu_i(y)$, we have $\mu_i(x \odot y) = \mu_i(x \odot \mu_i(y))$. Since $\mu_i(\mu_i(x)) = \mu_i(x)$, $\mu_i(x \odot \mu_i(y)) = \mu_i(\mu_i(x) \odot \mu_i(y))$. Thus

$$\mu_i(x) \odot \mu_i(y) \leq \mu_i(\mu_i(x) \odot \mu_i(y)) = \mu_i(x \odot y).$$

Since $\mu_i(x \odot y) \leq \mu_i(x) \odot \mu_i(y)$, then $\mu_i(x \odot y) = \mu_i(x) \odot \mu_i(y)$.

Theorem 2.6. Let $(A, \wedge, \vee, \odot, \Rightarrow, \rightarrow, \nu_i, \mu_i, \bot, e_i, \top)$ be a generalized triangle algebra for $i \in \{1, 2\}$. Then

(1) $\mu_i(x) \rightarrow \nu_i(y) \leq x \rightarrow y \leq \nu_i(x) \rightarrow \mu_i(y)$ for $\rightarrow \in \{\Rightarrow, \rightarrow\}$ and $i \in \{1, 2\}$.

(2) $\mu_1(x) \Rightarrow \nu_1(y) \leq \mu_1(x \Rightarrow y) \leq \nu_1(x) \Rightarrow \mu_1(y)$ and $\mu_2(x) \rightarrow \nu_2(y) \leq \mu_2(x \rightarrow y) \leq \nu_2(x) \rightarrow \mu_2(y)$.

(3) $\mu_1(\nu_1(x) \odot y) = \nu_1(x) \odot \mu_1(y)$ and $\mu_2(x \odot \nu_2(y)) = \mu_2(x) \odot \nu_2(y)$.

(4) For each $z \in E_1(A)$, $z \odot e_1 \leq y$ iff $z \leq \mu_1(y)$.

(5) For each $z \in E_2(A)$, $e_2 \odot z \leq y$ iff $z \leq \mu_2(y)$.

(6) $\nu_1(\nu_1(x) \Rightarrow y) = \nu_1(x) \Rightarrow \nu_1(y)$ and $\nu_2(\nu_2(x) \to y) = \nu_2(x) \to \nu_2(y)$.

(7) $\mu_1(\nu_1(x) \Rightarrow y) = \nu_1(x) \Rightarrow \mu_1(y)$ and $\mu_2(\nu_2(x) \to y) = \nu_2(x) \to \nu_2(y)$.

(8) $(x \wedge e_1) \Rightarrow (y \wedge e_1) = (x \wedge e_1) \Rightarrow y = (x \wedge e_1) \Rightarrow \mu_1(y)$.

(9) $(x \wedge e_2) \to (y \wedge e_2) = (x \wedge e_2) \to y = (x \wedge e_2) \to \mu_2(y)$.

(10) $\nu_1((x \wedge e_1) \Rightarrow y) = \mu_1(x) \Rightarrow \mu_1(y)$ and $\nu_2((x \wedge e_2) \to y) = \mu_2(x) \to \mu_2(y)$.

(11) $\mu_1((x \wedge e_1) \Rightarrow y) = \mu_1(x) \Rightarrow \mu_1(e_1 \to y)$ and $\mu_2((x \wedge e_2) \to y) = \mu_2(x) \to \mu_2(e_2 \Rightarrow y)$.

(12) $\mu_1(e_1 \Rightarrow y) = \mu_1(e_1 \odot e_1) \Rightarrow \mu_1(y)$ and $\mu_2(e_2 \to y) = \mu_2(e_2 \odot e_2) \to \mu_2(y)$

(13) $\nu_1(x \Rightarrow y) = (\nu_1(x) \Rightarrow \nu_1(y)) \wedge (\mu_1(x) \Rightarrow \mu_1(y))$ and $\nu_2(x \to y) = (\nu_2(x) \to \nu_2(y)) \wedge (\mu_2(x) \to \mu_2(y))$.

(14) $\mu_1(x \Rightarrow y) = (\mu_1(x) \Rightarrow (\mu_1(e_1) \Rightarrow \mu_1(y))) \wedge (\nu_1(x) \Rightarrow \mu_1(y))$ and $\mu_2(x \to y) = (\mu_2(x) \to (\mu_2(e_2) \to \mu_2(y))) \wedge (\nu_2(x) \to \mu_2(y))$.

(15) $\mu_1(x) \Rightarrow \nu_1(y) \leq \nu_1(x \Rightarrow y)$ and $\mu_2(x) \to \nu_2(y) \leq \nu_2(x \to y)$.

(16) $(\nu_1(x) \Rightarrow \nu_1(y)) \odot (\mu_1(x) \Rightarrow \mu_1(y)) \leq x \Rightarrow y$ and $(\nu_2(x) \to \nu_2(y)) \odot (\mu_2(x) \to \mu_2(y)) \leq x \to y$.

Proof. (1) Since $\nu_i(x) \leq x$ and $y \leq \mu_i(x)$, we have $\mu_i(x) \to \nu_i(y) \leq x \to y \leq \nu_i(x) \to \nu_i(y)$ for $\to \in \{\Rightarrow, \to\}$.

(2) Since $\nu_i(x), \mu_i(x), \nu_i(y), \mu_i(y) \in E_i(A)$, we have $\nu_1(x) \Rightarrow \mu_1(y), \mu_1(x) \Rightarrow \nu_1(y) \in E_1$ and $\nu_2(x) \to \mu_2(y), \mu_2(x) \to \nu_2(y) \in E_2$. Thus $\mu_1(x) \Rightarrow \nu_1(y) = \mu_1(\mu_1(x) \Rightarrow \nu_1(y)) \leq \mu_1(x \Rightarrow y) \leq \mu_1(\nu_1(x) \Rightarrow \mu_1(y)) = \nu_1(x) \Rightarrow \mu_1(y)$. Moreover, $\mu_2(x) \to \nu_2(y) = \mu_2(\mu_2(x) \to \nu_2(y)) \leq \mu_2(x \to y) \leq \mu_2(\nu_2(x) \to \mu_2(y)) = \nu_2(x) \to \mu_2(y)$.

(3) By Theorem 2.3 (6), $\mu_1(\nu_1(x) \odot y) \leq \mu_1(\nu_1(x)) \odot \mu_1(y) = \nu_1(x) \odot \mu_1(y)$.

Since $y \leq \nu_1(x) \Rightarrow \nu_1(x) \odot y$, by (2), we have

$$\mu_1(y) \leq \nu_1(\nu_1(x)) \Rightarrow \nu_1(x) \odot y) \leq \nu_1(x) \Rightarrow \mu_1(\nu_1(x) \odot y).$$

Thus, $\nu_1(x) \odot \mu_1(y) \leq \mu_1(\nu_1(x) \odot y)$. Hence $\nu_1(x) \odot \mu_1(y) = \mu_1(\nu_1(x) \odot y)$.

By Theorem 2.3 (6), $\mu_2(x \odot \nu_2(y)) \leq \mu_2(x) \odot \mu_2(\nu_2(y)) = \mu_2(x) \odot \nu_2(y)$.

Since $x \leq \nu_2(y) \to x \odot \nu_2(y)$, by (2), we have $\mu_2(x) \leq \mu_2(\nu_2(x) \to x \odot \nu_2(y)) \leq \nu_2(\nu_2(x)) \to \mu_2(x \odot \nu_2(y))$. Thus $\mu_2(x) \odot \nu_2(y) \leq \mu_2(x \odot \nu_2(y))$.

(4) For each $z \in E_1(A)$, if $z \odot e_1 \leq y$, by (3), then $\mu_1(z \odot e_1) = \mu_1(z) \odot \mu_1(e_1) = z \leq \mu_1(y)$. If $z \leq \mu_1(y)$, then $\mu_1(z \odot e_1) = z \leq \mu_1(y)$ and $\nu_1(z \odot e_1) = \perp \leq \nu_1(y)$. By Theorem 2.3(7), $z \odot e_1 \leq y$.

(5) It is similarly proved as (4).

(6) Since $\nu_1(x) \Rightarrow \nu_1(y) \leq \nu_1(x) \Rightarrow y$, then

$$\nu_1(x) \Rightarrow \nu_1(y) = \nu_1(\nu_1(x) \Rightarrow \nu_1(y)) \leq \nu_1(\nu_1(x) \Rightarrow y).$$

On the other hand, since $\nu_1(\nu_1(x) \Rightarrow y) \leq \nu_1(x) \Rightarrow y$, then $\nu_1(x) \odot \nu_1(\nu_1(x) \Rightarrow y) \leq y$. By Theorem 2.4 (5),

$$\nu_1(x) \odot \nu_1(\nu_1(x) \Rightarrow y) = \nu_1(\nu_1(x) \odot (\nu_1(x) \Rightarrow y)) \leq \nu_1(y).$$

Thus, $\nu_1(\nu_1(x) \Rightarrow y) \leq \nu_1(x) \Rightarrow \nu_1(y)$. Hence $\nu_1(\nu_1(x) \Rightarrow y) = \nu_1(x) \Rightarrow \nu_1(y)$.

Since $\nu_2(x) \to \nu_2(y) \leq \nu_2(x) \to y$, then

$$\nu_2(x) \to \nu_2(y) = \nu_2(\nu_2(x) \to \nu_2(y)) \leq \nu_2(\nu_2(x) \to y).$$

On the other hand, since $\nu_2(\nu_2(x) \to y) \leq \nu_2(x) \to y$, then $\nu_2(\nu_2(x) \to y) \odot \nu_2(x) \leq y$ implies

$$\nu_2(\nu_2(x) \to y) \odot \nu_2(x) = \nu_2(\nu_2(\nu_2(x) \to y) \odot \nu_2(x)) \leq \nu_2(y).$$

Thus, $\nu_2(\nu_2(x) \to y) \leq \nu_2(x) \to \nu_2(y)$. Hence $\nu_2(\nu_2(x) \to y) = \nu_2(x) \to \nu_2(y)$.

(7) We only show that $z \leq \mu_1(\nu_1(x) \Rightarrow y)$ iff $z \leq \nu_1(x) \Rightarrow \mu_1(y)$ for $z \in E_1(A)$. By (4), $z \leq \nu_1(x) \Rightarrow \mu_1(y)$ for $z \in E_1(A)$ iff $\nu_1(x) \odot z \leq \mu_1(y)$ iff $\nu_1(x) \odot z \odot e_1 \leq y$ iff $z \odot e_1 \leq \nu_1(x) \Rightarrow y$ iff $z \leq \mu_1(\nu_1(x) \Rightarrow y)$.

We only show that $z \leq \mu_2(\nu_2(x) \to y)$ iff $z \leq \nu_2(x) \to \mu_2(y)$ for $z \in E_2(A)$. By (5), $z \leq \nu_2(x) \to \mu_2(y)$ for $z \in E_2(A)$ iff $z \odot \nu_2(x) \leq \mu_2(y)$ iff $e_2 \odot z \odot \nu_2(x) \leq y$ iff $e_2 \odot z \leq \nu_2(x) \to y$ iff $z \leq \mu_2(\nu_2(x) \to y)$.

(8) It is similarly proved as the following (9).

(9) We have $(x \wedge e_2) \to y = (x \wedge e_2) \to \mu_2(y)$. Since

$$\nu_2((x \wedge e_2) \to \mu_2(y)) \odot (x \wedge e_2)) \leq \nu_2(x \wedge e_2) \leq \nu_2(e_2) = \perp \leq \nu_2(y)$$

$$\mu_2((x \wedge e_2) \to \mu_2(y)) \odot (x \wedge e_2)) \leq \mu_2(\mu_2(y)) = \mu_2(y),$$

by Theorem 2.3 (7), $((x \wedge e_2) \to \mu_2(y)) \odot (x \wedge e_2) \leq y$. Thus, $(x \wedge e_2) \to \mu_2(y) \leq (x \wedge e_2) \to y$. Hence $(x \wedge e_2) \to \mu_2(y) = (x \wedge e_2) \to y$. Moreover, put $y \wedge e_2$ instead of y, $(x\wedge e_2) \to \mu_2(y\wedge e_2) = (x\wedge e_2) \to \mu_2(y)\wedge\mu_2(e_2) = (x\wedge e_2) \to \mu_2(y) = (x\wedge e_2) \to (y\wedge e_2)$.

(10) Since $\nu_1((x \wedge e_1) \Rightarrow y) = \nu_1((x \wedge e_1) \Rightarrow \mu_1(y))$, we only show that $\nu_1((x \wedge e_1) \Rightarrow \mu_1(y)) = \mu_1(x) \Rightarrow \mu_1(y)$. Since $x \wedge e_1 \leq x \leq \mu_1(x)$, we have

$$\mu_1(x) \Rightarrow \mu_1(y) = \nu_1(\mu_1(x) \Rightarrow \mu_1(y)) \leq \nu_1((x \wedge e_1) \Rightarrow \mu_1(y)).$$

Since $(x \wedge e_1) \odot \nu_1((x \wedge e_1) \Rightarrow \mu_1(y)) \leq (x \wedge e_1) \odot ((x \wedge e_1) \Rightarrow \mu_1(y)) \leq \mu_1(y)$, then $\mu_1(x)\odot\nu_1((x\wedge e_1) \Rightarrow \mu_1(y)) \leq \mu_1(x\wedge e_1)\odot\nu_1((x\wedge e_1) \Rightarrow \mu_1(y)) = \mu_1((x\wedge e_1)\odot\nu_1((x\wedge e_1) \Rightarrow \mu_1(y)) \leq \mu_1(\mu_1(y)) = \mu_1(y)$. Thus, $\nu_1((x \wedge e_1) \Rightarrow \mu_1(y)) \leq \mu_1(x) \Rightarrow \mu_1(y)$. Other case is similarly proved.

(11) We will show that $z \leq \mu_1((x \wedge e_1) \Rightarrow y)$ iff $z \leq \mu_1(x) \Rightarrow \mu_1(e_1 \to y)$ for $z \in E_1(A)$.

$$\begin{aligned}
&z \leq \mu_1((x \wedge e_1) \Rightarrow y) \quad \text{iff} \;\; z \odot e_1 \leq (x \wedge e_1) \Rightarrow y \\
&\text{iff} \;\; (x \wedge e_1) \odot z \odot e_1 \leq y \quad \text{iff} \;\; (x \wedge e_1) \odot z \leq e_1 \to y \\
&\text{iff} \;\; \mu_1((x \wedge e_1) \odot z) \leq \mu_1(e_1 \to y), \;\; \bot = \nu_1((x \wedge e_1) \odot z) \leq \nu_1(e_1 \to y) \\
&\text{iff} \;\; \mu_1(x) \odot z \leq \mu_1(e_1 \to y) \quad \text{iff} \;\; z \leq \mu_1(x) \Rightarrow \mu_1(e_1 \to y).
\end{aligned}$$

We will show that $z \leq \mu_2((x \wedge e_2) \to y)$ iff $z \leq \mu_2(x) \to \mu_2(e_2 \Rightarrow y)$ for $z \in E_2(A)$.

$$\begin{aligned}
&z \leq \mu_2((x \wedge e_2) \to y) \quad \text{iff} \;\; e_2 \odot z \leq (x \wedge e_2) \to y \\
&\text{iff} \;\; e_2 \odot z \odot (x \wedge e_2) \leq y \quad \text{iff} \;\; z \odot (x \wedge e_2) \leq e_2 \Rightarrow y \\
&\text{iff} \;\; \mu_2(z \odot (x \wedge e_2)) \leq \mu_2(e_2 \Rightarrow y), \;\; \bot = \nu_2(z \odot (x \wedge e_2)) \leq \nu_2(e_2 \Rightarrow y) \\
&\text{iff} \;\; z \odot \mu_2(x) \leq \mu_2(e_2 \Rightarrow y) \quad \text{iff} \;\; z \leq \mu_2(x) \to \mu_2(e_2 \Rightarrow y).
\end{aligned}$$

(12) We will show that $z \leq \mu_1(e_1 \Rightarrow y)$ iff $z \leq \mu_1(e_1 \odot e_1) \Rightarrow \mu_1(y)$ for $z \in E_1(A)$.

$$\begin{aligned}
&z \leq \mu_1(e_1 \odot e_1) \Rightarrow \mu_1(y) \quad \text{iff} \;\; \mu_1(e_1 \odot e_1) \odot z \leq \mu_1(y) \\
&\text{iff} \;\; \mu_1(e_1 \odot e_1 \odot z) \leq \mu_1(y), \; \bot = \nu_1(e_1 \odot e_1 \odot z) \leq \nu_1(y) \\
&\text{iff} \;\; e_1 \odot e_1 \odot z \leq y \quad \text{iff} \;\; e_1 \odot z \leq e_1 \Rightarrow y \\
&\text{iff} \;\; z \leq \mu_1(e_1 \Rightarrow y)
\end{aligned}$$

Other case is similarly proved.

(13) Since $x = (\mu_1(x) \wedge e_1) \vee \nu_1(x) = (x \wedge e_1) \vee \nu_1(x)$ from Theorem 2.4 (1,2), we have

$$x \Rightarrow y = (x \wedge e_1) \vee \nu_1(x) \Rightarrow y = ((x \wedge e_1) \Rightarrow y) \wedge (\nu_1(x) \Rightarrow y).$$

By (6) and (10), we have

$$\begin{aligned} \nu_1(x \Rightarrow y) &= \nu_1(((x \wedge e_1) \Rightarrow y) \wedge (\nu_1(x) \Rightarrow y)) \\ &= \nu_1(((x \wedge e_1) \Rightarrow y)) \wedge \nu_1(\nu_1(x) \Rightarrow y) \\ &= (\mu_1(x) \Rightarrow \mu_1(y)) \wedge (\nu_1(x) \Rightarrow \nu_1(y)) \end{aligned}$$

Other case is similarly proved.

(14) Since $x \Rightarrow y = ((x \wedge e_1) \Rightarrow y) \wedge (\nu_1(x) \Rightarrow y)$, by (7) and (11),

$$\begin{aligned} \mu_1(x \Rightarrow y) &= \mu_1(((x \wedge e_1) \Rightarrow y) \wedge (\nu_1(x) \Rightarrow y)) \\ &= \mu_1(((x \wedge e_1) \Rightarrow y)) \wedge \mu_1(\nu_1(x) \Rightarrow y) \\ &= (\mu_1(x) \Rightarrow \mu_1(e_1 \rightarrow y)) \wedge (\nu_1(x) \Rightarrow \nu_1(y)) \end{aligned}$$

Other case is similarly proved.

(15) Since $\nu_1(x) \leq \mu_1(x)$, we have

$$\mu_1(x) \Rightarrow \nu_1(y) \leq \nu_1(x) \Rightarrow \nu_1(y), \;\; \mu_1(x) \Rightarrow \nu_1(y) \leq \mu_1(x) \Rightarrow \mu_1(y).$$

$$\begin{aligned} \mu_1(x) \Rightarrow \nu_1(y) &\leq (\nu_1(x) \Rightarrow \nu_1(y)) \wedge (\mu_1(x) \Rightarrow \mu_1(y)) \\ &= \nu_1(x \Rightarrow y). \end{aligned}$$

(16) By Lemma 1.3(2), we have

$$\begin{aligned} &(\nu_1(x) \Rightarrow \nu_1(y)) \odot (\mu_1(x) \Rightarrow \mu_1(y)) \\ &\leq (\nu_1(x) \Rightarrow \nu_1(y)) \wedge (\mu_1(x) \Rightarrow \mu_1(y)) = \nu_1(x \Rightarrow y) \leq x \Rightarrow y. \end{aligned}$$

Other case is similarly proved.

Theorem 2.7. Let $(A, \wedge, \vee, \odot, \Rightarrow, \rightarrow, \nu_i, \mu_i, \perp, e_i, \top)$ be a generalized triangle algebra for $i \in \{1, 2\}$. Then $e_1^0 \leq e_1$ and $e_2^* \leq e_2$. If $x^{*0} = x^{0*} = x$ for all $x \in A$, then $e_1^0 = e_1, e_2^* = e_2$ and $e_i \odot e_i = \perp$ for $i \in \{1, 2\}$.

Proof. Since $\nu_1(e_1^0) \leq e_1^0$ and $\nu_2(e_2^*) \leq e_2^*$, then

$$e_1 \leq e_1^{0*} \leq (\nu_1(e_1^0))^*, \;\; e_2 \leq e_2^{*0} \leq (\nu_2(e_2^*))^0.$$

Since $\nu_1(e_1^0) \in E_1(A)$ and $\nu_2(e_2^*) \in E_2(A)$, by Theorem 2.3 (2,3),

$$(\nu_1(e_1^0))^* \in E_1(A),\ (\nu_2(e_2^*))^0 \in E_2(A).$$

So,

$$\top = \mu_1(e_1) \leq \mu_1((\nu_1(e_1^0))^*) = (\nu_1(e_1^0))^*.$$

$$\top = \mu_2(e_2) \leq \mu_2((\nu_2(e_2^*))^0) = (\nu_2(e_2^*))^0.$$

Then $\nu_1(e_1^0) = \bot = \nu_2(e_2^*)$. Since $e_1 \vee e_1^0 = e_1 \vee \nu_1(e_1^0) = e_1$ and $e_2 \vee e_2^* = e_2 \vee \nu_2(e_2^*) = e_2$ from Theorem 2.4(1), $e_1^0 \leq e_1$ and $e_2^* \leq e_2$.

Let $x^{*0} = x^{0*} = x$ for all $x \in A$ be given. Since $\mu_1(e_1^0) \geq e_1^0$ and $\mu_2(e_2^*) \geq e_2^*$, then

$$e_1 = e_1^{0*} \geq (\mu_1(e_1^0))^*,\ \ e_2 = e_2^{*0} \geq (\mu_2(e_2^*))^0.$$

Since $\mu_1(e_1^0) \in E_1(A)$ and $\mu_2(e_2^*) \in E_2(A)$, by Theorem 2.3 (2,3),

$$(\mu_1(e_1^0))^* \in E_1(A),\ (\mu_2(e_2^*))^0 \in E_2(A).$$

So,

$$\bot = \nu_1(e_1) \geq \nu_1((\mu_1(e_1^0))^*) = (\mu_1(e_1^0))^*.$$

$$\bot = \nu_2(e_2) \leq \nu_2((\mu_2(e_2^*))^0) = (\mu_2(e_2^*))^0.$$

Then $\mu_1(e_1^0) = \top = \mu_2(e_2^*)$. Since $e_1 \wedge e_1^0 = e_1 \wedge \mu_1(e_1^0) = e_1$ and $e_2 \wedge e_2^* = e_2 \wedge \mu_2(e_2^*) = e_2$ from Theorem 2.4(1), $e_1^0 \geq e_1$ and $e_2^* \geq e_2$. Thus $e_1^0 = e_1$ and $e_2^* = e_2$. Moreover, since $e_1 \leq e_1 \to \bot$ and $e_2 \leq e_2 \Rightarrow \bot$, $e_i \odot e_i = \bot$ for $i \in \{1, 2\}$.

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