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DUALITY THEOREMS FOR K-CONVEX FUNCTIONS

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Abstract: In this paper we have considered K-convex functions which are generalized convex functions and established the weak duality theorem, the strong duality theorem and the converse duality theorem for a pair of symmetric dual nonlinear programming problems.

Key words: K-convex function, weak duality, strong duality, converse duality, convex cone

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1.Introduction

Convex functions and convex sets are important in the theory of optimization. Doeringer [6],Jensen[9] and Nikodem[16] have discussed the properties of K-convex functions. These generalized functions are useful in economics and inventory models as shown by Gallego et.al. [7], Cass [3] and Hartl et.al. [8].Their properties in $\Re^n$ is presented in Gallego et.al.[7]. The concept of symmetric duality is vividly studied by Rockafellar[17], Mangasarian[10], Mishra et. al.[11], [12], Nayak[15] and Chandra et.al.[4].Bazaara and Goode [1],[2] have proved the duality theorems for usual convex and concave functions. Here we have proved some duality theorems in non linear programming using the K-convex functions. Mishra et. al. [12] with additional feasibility assumption have proved the same result by considering pseudo-convex functions.

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Symmetric duality theorems in nonlinear programming are proved by Dantziget. al. [5]. Our result is motivated by Mond[13], [14] and Wolfe [20].

We use the following notations and terminologies in this paper. Let $\psi(x, y)$ be a real valued, twice differentiable function defined on an open set in R^{n+m} containing $C_1 \times C_2$ where C_1 and C_2 are closed convex cones with nonempty interiors in R^n and R^m respectively.

Let $C_1^* = \{z \mid x^t z \leq 0$ for each $x \in C_1\}$ be the polar of C_1 and

$x^t =$ transpose of x.

C_2^* is defined similarly.

$\nabla_x \psi(x_0, y_0) =$ the gradient vector of ψ with respect to x at (x_0, y_0).

$\nabla_y \psi(x_0, y_0)$ is defined similarly.

$\nabla_{xx} \psi(x_0, y_0)$ denotes the Hessian matrix of second partial derivatives with respect to x at (x_0, y_0).

$\nabla_{xy} \psi(x_0, y_0)$, $\nabla_{yx} \psi(x_0, y_0)$ and $\nabla_{yy} \psi(x_0, y_0)$ are defined similarly.

2. Preliminaries

Definition 2.1

A function f on an interval I of the real line is K-convex, where K is a non-negative real number if and only if for $x, y \in I$, $x < y$ and $0 \leq \lambda \leq 1$, $f[\lambda x + (1-\lambda) y] \leq \lambda f(x) + (1-\lambda)[f(y) + K]$.

If $K = 0$, this becomes the usual definition of convexity. Similarly a function f is K-concave if $-f$ is K-convex.

We say that ψ is K-convex/K-concave on $C_1 \times C_2$ iff $\psi(.,y)$ is K-convex on C_1 for each given $y \in C_2$ i.e. for $x_1, x_2 \in C_1$, $x_1 \geq x_2 \Rightarrow \psi(x_1, y) \geq \psi(x_2, y)$ and $\psi(x)$ is K-concave on C_2 for each given $x \in C_1$, if for $y_1, y_2 \in C_2$, $y_1 \geq y_2 \Rightarrow \psi(x, y_1) \leq \psi(x, y_2)$.

Let us consider a pair of nonlinear programs as follows:

(P_0): Minimize

$$f(x,y) = \psi(x,y) - y^t \nabla_y \psi(x,y)$$

subject to

$$(x,y) \in C_1 \times C_2$$

$$\nabla_y \psi(x,y) \in C_2^*$$

(D_0): Maximize

$$g(u,v) = \psi(u,v) - u^t \nabla_u \psi(u,v)$$

subject to

$$(u,v) \in C_1 \times C_2$$

$$-\nabla_u \psi(u,v) \in C_1^*$$

Let P and D be the feasible solutions of P_0 and D_0 respectively.

So $$P = \{(x,y) \in C_1 \times C_2 : \nabla_y \psi(x,y) \in C_2^*\}$$

and $$D = \{(u,v) \in C_1 \times C_2 : -\nabla_u \psi(u,v) \in C_1^*\}.$$

3. Main results

Theorem.3.1 (Weak Duality)

Let ψ *be K-convex/K-concave on* $C_1 \times C_2$. *Then for any* $(x,y) \in P$ *and* $(u,v) \in D$ *with* $x-u \in C_1$ *and* $v-y \in C_2$, *then* $f(x,y) \geq g(u,v)$.

Proof: Let $x-u \in C_1$, then

$$-(x-u)^t \nabla_u \psi(u,v) \leq 0$$

$$\Rightarrow (x-u)^t \nabla_u \psi(u,v) \geq 0.$$

Since $\psi(.,y)$ is K-convex on C_1 for each given $y \in C_2$ and $(x-u)^t \nabla_u \psi(u,v) \geq 0$, so we have

$$\psi(x,v) \geq \psi(u,v) \quad (1.1)$$

Since $v-y \in C_2$ and $(x,y) \in P$

therefore, $(v-y)^t \nabla_y \psi(x,y) \leq 0$.

As $\psi(x,.)$ is K-concave C_2 for each given $x \in C_1$ and

$$(v-y)^t \nabla_y \psi(x,y) \leq 0$$

We get

$$\psi(x,v) \leq \psi(x,y) \quad (1.2)$$

From (1.1) and (1.2) we have

$$\psi(x,y) \geq \psi(x,v) \geq \psi(u,v)$$

i.e. $$\psi(x,y) \geq \psi(u,v) \quad (1.3)$$

Since $y \in C_2$, $\nabla_y \psi(x,y) \in C_2^*$

$$\Rightarrow y^t \nabla_y \psi(x,y) \le 0$$

$$\Rightarrow -y^t \nabla_y \psi(x,y) \ge 0 . \qquad (1.4)$$

Similarly $u \in C_1$

$$\nabla_u \psi(u,v) \in C_2^*$$

$$\Rightarrow -u^t \nabla_u \psi(u,v) \le 0 . \qquad (1.5)$$

From (1.4)

$$\psi(x,y) - y^t \nabla_y \psi(x,y) \ge \psi(u,v) - u^t \nabla_u \psi(u,v)$$

$$\Rightarrow f(x,y) \ge g(u.v)$$

This completes the proof.

Theorem.3.2 (Strong Duality)

If $(\bar{x},\bar{y})$ *solves* P_1 *and* $\nabla_{yy}\theta(\bar{x},\bar{y})$ *is negative definite then the following statements are true:*

(i) $f(\bar{x},\bar{y}) = g(\bar{x},\bar{y})$

(ii) $(\bar{y})^t \nabla_y \theta(\bar{x},\bar{y}) = (\bar{x})^t \nabla_x \theta(\bar{x},\bar{y}) = 0$

(iii) $(\bar{x},\bar{y})$ *solves* D_1

where θ *is a twice differentiable continuous real valued functionsatisfying K-convexity.*

Proof: The proof of (i) and (ii) require the arguments similar to Bazaraa and Goode [2]. We are presenting it here only for the sake of completeness.

Let $z=(x,y)$, $X=C_1\times C_2$, $C=C_2^*$ and $f(z)=\theta(x,y)-(y)^t\nabla_y\theta(x,y)$ and $g(z)=\nabla_y\theta(x,y)$.

Hence if z_0 solves the problem there exists a non zero (q_0,q)

such that

$$[q_0\nabla_x^t\theta(\bar{x},\bar{y})-q_0\bar{y}^t\nabla_{yx}\theta(\bar{x},\bar{y})+q^t\nabla_{yx}\theta(\bar{x},\bar{y})](x-\bar{x})$$

$$(-q_0\bar{y}^t+q^t)\nabla_{yy}\theta(\bar{x},\bar{y})(y-\bar{y})\geq 0,\ \text{for each}\ (x,y)\in C_1\times C_2 \tag{1.6}$$

and $q_0\geq 0$, $q\in\left(C_2^*\right)^*$

$=C_2$ (since C_2 is a closed convex cone)

$$\text{and}\ q^t\nabla_y\theta(\bar{x},\bar{y})=0 \tag{1.7}$$

We claim that $q_0\geq 0$. To show this let $x=\bar{x}$ in C_1, then we get

$$(-q_0(\bar{y})^t+q^t)\nabla_{yy}\theta(\bar{x},\bar{y})(y-\bar{y})\geq 0\ \text{for each}\ y\in C_2 \tag{1.8}$$

If $q_0=0$ and $y=\bar{y}+q$, we have from (1.8)

$q^t\nabla_{yy}\theta(\bar{x},\bar{y})q\geq 0$, which by negative definiteness of $\nabla_{yy}\theta(\bar{x},\bar{y})$ implies that $q=0$. But this impossible since $(q_0,q)\neq 0$ and therefore $q_0\geq 0$. Further let $q=q_0\bar{y}$, then (1.8) is valid.

If $q\neq q_0\bar{y}$, then (1.8) is not valid for $y=\dfrac{q}{q_0}\in C_2$. The relation (1.8) is

$$(-q_0\bar{y}^t+q^t)\nabla_{yy}\theta(\bar{x},\bar{y})(y-\bar{y})\geq 0$$

i.e. $$(-q_0\bar{y}^t + q^t)\nabla_{yy}\theta(\bar{x},\bar{y})\left(\frac{q}{q_0} - \bar{y}\right) \geq 0$$

$$\because y = \frac{q}{q_0}$$

i.e. $$(-q_0\bar{y}^t + q^t)\nabla_{yy}\theta(\bar{x},\bar{y})\left(\frac{q - q_0\bar{y}}{q_0}\right) \geq 0$$

which is not true as $\nabla_{yy}\theta(\bar{x},\bar{y})$ is negative definite.

Making use of this information and letting $y = \bar{y}$ in (1.6)we get

$$\nabla_x^t\theta(\bar{x},\bar{y})(x - \bar{x}) \geq 0 \text{ for each } x \in C_1$$

Let $x \in C_1$, then $\bar{x} + x \in C_1$, so that the last inequality implies that

$$x^t\nabla_x\theta(\bar{x},\bar{y}) \geq 0$$

i.e. $$-\nabla_x\theta(\bar{x},\bar{y}) \in C_1^*$$

By letting $X = 0$ and $X = \bar{x}$ in the last two inequalities, we obtain,

$$\bar{x}^t\nabla_x\theta(\bar{x},\bar{y}) = 0 \tag{1.9}$$

Since $q_0 \geq 0$, $q = q_0\bar{y}$, and $q^t\nabla_y\theta(\bar{x},\bar{y}) = 0$, then

$$\bar{y}^t\nabla_y\theta(\bar{x},\bar{y}) = 0 \tag{1.10}$$

This show that $f(\bar{x},\bar{y}) = g(\bar{x},\bar{y})$.

It remains to be shown that $(\bar{x},\bar{y})$ is indeed optimal of D_1. Since θ is K-convex /K-concave, by applying theorem 3.1, we observe that $(\bar{x},\bar{y})$ is indeed optimal solution of D_1 and the rest of the results follows from (1.9) and (1.10).

Theorem.3.3 (Converse Duality)

If $(\bar{x},\bar{y})$ *solves* D_1 *and* $\nabla_{xx}\theta(\bar{x},\bar{y})$ *is positive definite, then the following statements are true:*

(i) $f(\bar{x},\bar{y}) = g(\bar{x},\bar{y})$

(ii) $\bar{y}^t\nabla_y\theta(\bar{x},\bar{y}) = \bar{x}^t\nabla_x\theta(\bar{x},\bar{y}) = 0$

(iii) $(\bar{x},\bar{y})$ *solves* P_1

Proof: Here $z=(x,y)$, $X = C_1 \times C_2$, $C = C_1^*$ and $f(z) = -\theta(x,y) - x^t\nabla_y\theta(x,y)$ and $g(z) = -\nabla_x\theta(x,y)$. Hence if z_0 solves the problem there exists nonzero (q_0, q) such that

$$(q_0\bar{x}^t - q^t)\nabla_{xx}\theta(\bar{x},\bar{y})(x-\bar{x}) + [-q_0\nabla_y^t\theta(\bar{x},\bar{y}) + (q_0\bar{x}^t - q^t)\nabla_{xy}\theta(\bar{x},\bar{y})] \qquad (1.11)$$

$(y-\bar{y}) \geq 0$ for each $(x,y) \in C_1 \times C_2$

and $q_0 \geq 0$, $q \in \left(C_1^*\right)^* = C_1$ (Since C_1 is a closed convex cone) and

$$q^t\nabla_x\theta(\bar{x},\bar{y}) = 0. \qquad (1.12)$$

We claim that $q_0 \geq 0$. To show this let $y=\bar{y}$ in (1.11) then we get

$$(q_0\bar{x}^t - q^t)\nabla_{xx}\theta(\bar{x},\bar{y})(x-\bar{x}) \geq 0 \qquad (1.13)$$

for each given $x \in C_1$.

If $q_0 = 0$ and $x = \bar{x} + q$, we have from(1.13)

$$-q^t\nabla_{xx}\theta(\bar{x},\bar{y})q \geq 0$$

i.e. $q^t\nabla_{xx}\theta(\bar{x},\bar{y})q \leq 0$; which by positive definiteness of $\nabla_{xx}\theta(\bar{x},\bar{y})$ implies that $q=0$. But this is not possible since $(q,q_0) \neq 0$ and therefore $q_0 > 0$. Further let $q = q_0\bar{x}$, then (1.13) is valid.

If $q \neq q_0\bar{x}$ then the relation (1.13)is not valid for $x = \dfrac{q}{q_0} \in C_1$. The relation (1.13) is

$$(q_0\bar{x}^t - q^t)\nabla_{xx}\theta(\bar{x},\bar{y})(x-\bar{x}) \geq 0$$

i.e. $$(q_0\bar{x}^t - q^t)\nabla_{xx}\theta(\bar{x},\bar{y})\left(\frac{q}{q_0} - \bar{x}\right) \geq 0$$

$$\because x = \frac{q}{q_0}$$

i.e. $$(q_0\bar{x}^t - q^t)\nabla_{xx}\theta(\bar{x},\bar{y})\left(\frac{q - q_0\bar{x}}{q_0}\right) \geq 0$$

i.e. $$(q_0\bar{x}^t - q^t)\nabla_{xx}\theta(\bar{x},\bar{y})(q - q_0\bar{x}) \geq 0 \quad \text{as} \quad q_0 > 0$$

i.e. $$-(q_0\bar{x}^t - q^t)\nabla_{xx}\theta(\bar{x},\bar{y})(q - q_0\bar{x}) \geq 0,$$ which is not true since

$\nabla_{xx}\theta(\bar{x},\bar{y})$ is positive definite.

Using the fact and putting $x = \bar{x}$ in (1.11) we get

$$-q_0\nabla_y^t\theta(\bar{x},\bar{y})(y-\bar{y}) \geq 0 \text{ for each } \quad y \in C_2$$

Let $y \in C_2$, then $\bar{y} + y \in C_2$, so that the last inequality implies

$$-q_0 y^t\nabla_y\theta(\bar{x},\bar{y}) \geq 0$$

or $$y^t\nabla_y\theta(\bar{x},\bar{y}) \leq 0 \quad \text{as} \quad q_0 > 0$$

i.e. $$\nabla_y\theta(\bar{x},\bar{y}) \in C_2^*$$

Setting $y = 0$ and $y = \bar{y}$ in the last two inequalities, we obtain

$$\bar{y}^t \nabla_y \theta(\bar{x}, \bar{y}) = 0 \tag{1.14}$$

Since $q_0 > 0$, $q = q_0 \bar{x}$ and $q^t \nabla_x \theta(\bar{x}, \bar{y}) = 0$ then

$$\bar{x}^t \nabla_x \theta(\bar{x}, \bar{y}) = 0 \tag{1.15}$$

which implies that $f(\bar{x}, \bar{y}) = g(\bar{x}, \bar{y})$.

It remains to be shown that $(\bar{x}, \bar{y})$ is indeed optimal of P_1. Since θ is K-convex/K-concave on $C_1 \times C_2$ by theorem 3.1, we get that $(\bar{x}, \bar{y})$ is optimal of P_1 and the rest of the results follows from (1.14) and (1.15).

4.Conclusions

In this paper we have presented weak, strong and converse duality results for K-convex/K-concave functions in nonlinear programming with an additional feasibility condition.

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