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DOUBLE INTEGRAL CHARACTERIZATIONS BY BEREZIN TRANSFORM IN SOME MÖBIUS INVARIANT SPACES

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Abstract. In this paper, we give some characterizations of the analytic $Q_{K,\omega}$ space in terms of double integrals. The obtained results are proved using Berezin transform in the unit disk. Our results extend and generalize some results in [18, 39].

Keywords: Berezin transform, $Q_{K,\omega}$ space, Möbius invariant spaces.

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1. Introduction

Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc of the complex plane $\mathbb{C}$. The Green's function in the unit disk Δ with singularity at $a \in \Delta$ is given by $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. For $0 < r < 1$, let $\Delta(a,r) = \{z \in \Delta : |\varphi_a(z)| < r\}$ be the pseudo-hyperbolic disk with the center $a \in \Delta$ and radius r. For a given reasonable function $\omega : (0,1] \to (0,\infty)$ and for $0 < \alpha < \infty$. An analytic function f on $\mathbb{D}$ is said to belong to the $\alpha-$weighted Bloch space $\mathscr{B}^{\alpha}_{\omega}$ (see [35, 36]) if

$$\|f\|_{\mathscr{B}^{\alpha}_{\omega}} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)^{\alpha}}{\omega(1-|z|)} |f'(z)| < \infty.$$

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Also, for a given reasonable function $\omega : (0,1] \to (0,\infty)$ and for $0 < \alpha < \infty$. An analytic function f on $\mathbb{D}$ is said to belong to the little weighted Bloch space $\mathscr{B}^{\alpha}_{\omega,0}$ (see [35, 36]) if

$$\|f\|_{\mathscr{B}^{\alpha}_{\omega,0}} = \lim_{|z|\to 1^-} \frac{(1-|z|)^{\alpha}}{\omega(1-|z|)}|f'(z)| = 0.$$

Throughout this paper and for some techniques we consider the case of $\omega \not\equiv 0$.

Through this paper, we assume that $K : [0,\infty) \to [0,\infty)$ is a right continuous and nondecreasing function. For $0 < p < \infty$ and $-2 < q < \infty$, we say that a function f analytic in Δ belongs to the space $Q_{K,\omega}(p,q)$ (see [35, 36]) if

$$\|f\|^p_{K,\omega,p,q} \;=\; \sup_{a\in\Delta}\int_{\Delta}\left|f'(z)\right|^p\left(1-|z|^2\right)^q\frac{K(1-|\varphi_a(z)|^2)}{\omega(1-|z|)}dA(z) < \infty,$$

where $dA(z)$ is the Euclidean area element on Δ. It is clear that $Q_{K,\omega}(p,q)$ is a Banach space with the norm $\|f\| = |f(0)| + \|f\|_{K,\omega,p,q}$ where $p \geq 1$. If $q+2 = p$, $Q_{K,\omega}(p,q)$ is Möbius invariant, i.e., $\|f\circ\varphi_a\| = \|f\|_{K,\omega,p,q}$ for all $a \in \Delta$. Now we consider some special cases. If $p = 2$, and $q = 0$ and $\omega \equiv 1$, we obtain that $Q_K(p,q) = Q_K$ (cf. [26, 38]). If $K(t) = t^s$ and $\omega \equiv 1$, then $Q_{K,1}(p,q) = F(p,q,s)$ (cf. [40]) that $F(p,q,s)$ is contained in the weighted $\frac{q+2}{p}$-Bloch space. The space $Q_{K,\omega,0}(p,q)$ consists of analytic function f in Δ with the property that (see [35, 36])

$$\lim_{|a|\to 1^-}\int_{\Delta}|f'(z)|^p(1-|z|^2)^q\frac{K(1-|\varphi_a(z)|^2)}{\omega(1-|z|)}dA(z) = 0.$$

It can be checked that $Q_{K,\omega,0}(p,q)$ is a closed subspace in $Q_{K,\omega}(p,q)$.

In this paper, for simplicity, we consider the class $Q_{K,\omega}$, which is defined as follows:

$$\|f\|^2_{K,\omega} = \|f\|^2_{K,\omega,2,0} = \sup_{a\in\Delta}\int_{\Delta}\left|f'(z)\right|^2\frac{K(1-|\varphi_a(z)|^2)}{\omega(1-|z|)}dA(z) < \infty,$$

The following identity is easily verified:

$$1-|\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2} = (1-|z|^2)|\varphi_a'(z)|.$$

For $a \in \Delta$, the substitution $z = \varphi_a(w)$ results in the Jacobian change in measure given by $dA(w) = |\varphi_a(z)|^2 dA(z)$. For a Lebesgue integrable or a non-negative Lebesgue measurable function h on Δ we thus have the following change-of-variable formula:

$$\int_{\Delta(0,r)} h(\varphi_a(w))dA(w) = \int_{\Delta(a,r)} h(z)\left(\frac{1-|\varphi_a(z)|^2}{1-|z|^2}\right)^2 dA(z)\;.$$

Note that $\varphi_a(\varphi_a(z)) = z$ and thus $\varphi_a^{-1}(z) = \varphi_a(z)$. For $a, z \in \Delta$ and $0 < r < 1$, the pseudo-hyperbolic disk $\Delta(a,r)$ is defined by $\Delta(a,r) = \{z \in \Delta : |\varphi_a(z)| < r\}$. We will also need to use the so-called Berezin transform. More specifically, for any function $f \in L^1(\Delta, dA)$, we define a function Bf by

$$Bf(z) = \int_\Delta \frac{(1-|z|^2)^2}{|1-zw|^4} f(w)dA(w), \quad z \in \Delta.$$

We call Bf the Berezin transform of f. By a change of variables, we can also write

$$Bf(z) = \int_\Delta f \circ \varphi_z(w)dA(w), \quad z \in \Delta$$

see [22, 24, 33, 27] and [41] for basic properties of the Berezin transform.

The following estimate is the key to the main results of this paper.

Lemma 1.1. [41] *For any $R > 0$, there esists a positive constant C (depending on R) such that*

$$|f(z)|^2 \le \frac{C}{|\Delta(z,R)|} \int_{\Delta(z,R)}^{\infty} |f(w)|^2 dA(w), \tag{1}$$

for all $z \in \Delta$ and analytic function f in Δ.

If K is only defined on $(0,1]$, then we extend it to $(0,\infty)$ by setting $K(t) = K(1)$ for $t > 0$. We can then define an auxiliary function as

$$\varphi_{K,\omega}(s) = \sup_{0<t\le t} \frac{\omega(t)K(st)}{\omega(st)K(t)}, \quad 0 < s < \infty.$$

Now we prove the following result.

Lemma 1.2. *Let K be any nonnegative and Lebesgue measurable function on $(0,\infty)$ and $f(z) = \frac{K(1-|z|^2)}{\omega(1-|z|^2)}$ with $\omega(1-|z|^2) \sim \omega(1-|z|)$. If*

$$\int_0^\infty \frac{\varphi_{K,\omega}(x)}{(1+x)^3} dx < \infty, \tag{2}$$

then there exists a positive constant C such that $Bf(z) \le Cf(z)$ for all $z \in \Delta$.

Proof. From the definition of Berezin transform, we have

$$Bf(z) = \int_\Delta \frac{K(1-|\varphi_z(w)|^2)}{\omega(1-|\varphi_z(w)|^2)} dA(w).$$

Since,

$$1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{w}z|^2},$$

we find

$$\frac{K(1-|\varphi_z(w)|^2)}{\omega(1-|\varphi_z(w)|^2)} \le \frac{K(1-|z|^2)}{\omega(1-|z|^2)}\varphi_{K,\omega}\left(\frac{1-|w|^2}{|1-\bar{w}z|^2}\right).$$

It follows that

$$\begin{aligned} Bf(z) &\le f(z)\int_\Delta \varphi_{K,\omega}\left(\frac{1-|\bar{w}z|^2}{|1-\bar{w}z|^2}\right)dA(w) \\ &\le \frac{f(z)}{|z|^2}\int_{|w|<|z|}\varphi_{K,\omega}\left(\frac{1-|w|^2}{|1-\bar{w}z|^2}\right)dA(w) \\ &\le \frac{f(z)}{|z|^2}\int_{|w|<|z|}\varphi_{K,\omega}\left(\frac{1-|w|^2}{|1-w|^2}\right)dA(w) = \frac{2f(z)}{|z|^2}\int_0^\infty \varphi_{K,\omega}\left(\frac{r}{(1+r)^3}\right)dr. \end{aligned}$$

This completes the proof.

2. A double integral characterization in $Q_{K,\omega}$ space

In this section, we characterize the space $Q_{K,\omega}$ in terms of a double integral that does not involve the use of derivatives. We begin with the following estimate.

Theorem 2.1 *Let* $0 < p < \infty$. *For a given reasonable function* $\omega : (0,1] \to (0,\infty)$, *there exists a constant* $C > 0$ *(independent of* K *and* ω*) such that*

$$\int_\Delta \left|f'(z)\right|^2 \frac{K(1-|z|^2)}{\omega(1-|z|)}dA(z) \le CI(f)$$

for all analytic functions f *in* Δ, *where*

$$I(f) = \int_\Delta\int_\Delta \frac{\left|f(z)-f(w)\right|^2}{\left|1-z\bar{w}\right|^4}\frac{K(1-|z|^2)}{\omega(1-|z|)}dA(w).$$

Proof. We write the double integral $I(f)$ as an iterated integral

$$I(f) = \int_\Delta \frac{K(1-|z|^2)}{(1-|z|^2)^2\omega(1-|z|)}dA(z)\int_\Delta \frac{\left(1-|z|^2\right)^2}{\left|1-z\bar{w}\right|^4}\left|f(z)-f(w)\right|^2 dA(w).$$

Making a change of variables in the inner integral, we obtain

$$I(f) = \int_\Delta \frac{K(1-|z|^2)}{(1-|z|^2)^2\omega(1-|z|)}dA(z)\int_\Delta \left|f(\varphi_z(w))-f(z)\right|^2 dA(w). \tag{3}$$

It is well known that

$$\int_\Delta |g(w)-g(0)|^2 dA(w) \sim \int_\Delta |g'(w)|^2(1-|w|^2)^2 dA(w), \tag{4}$$

for analytic functions g in Δ. Applying (4) to the inner integral in (3) with the function $g(w) = f(\varphi_z(w))$, we deduce that

$$I(f) \sim \int_\Delta \frac{K(1-|z|^2)}{(1-|z|^2)^2\omega(1-|z|)} dA(z) \int_\Delta \left|(f\circ\varphi_z)'(w)\right|^2 \left(1-|w|^2\right)^2 dA(w).$$

Therefore, by the chain rule and a change of variables, we get

$$I(f) \quad \sim \int_\Delta \frac{K(1-|z|^2)}{\omega(1-|z|)} dA(z) \int_\Delta \left|f'(w)\right|^2 \frac{(1-|w|^2)^2}{|1-z\bar{w}|^4} dA(w). \tag{5}$$

Fix any positive radius R. Then there exists a constant $C > 0$ such that

$$I(f) \geq C \int_\Delta \frac{K(1-|z|^2)}{\omega(1-|z|)} dA(z) \int_{\Delta(z,R)} \left|f'(w)\right|^2 \frac{(1-|w|^2)^2}{|1-z\bar{w}|^4} dA(w).$$

It is well known that (see e.g [37, 41])

$$\frac{(1-|w|^2)}{|1-z\bar{w}|^2} \sim \frac{1}{(1-|z|^2)} \sim \frac{1}{\sqrt{|\Delta(z,R)|}}.$$

for $w \in \Delta(z,R)$. It is follows that there exists a positive constant C such that

$$I(f) \geq C \int_\Delta \frac{K(1-|z|^2)}{\omega(1-|z|)} dA(z) \frac{1}{|\Delta(z,R)|} \int_{\Delta(z,R)} \left|f'(w)\right|^2 dA(w).$$

Then using lemma 1.1, we obtain

$$I(f) \geq C \int_\Delta \left|f'(z)\right|^2 (1-|z|^2)^{p-2} \frac{K(1-|z|^2)}{\omega(1-|z|)} dA(z).$$

The proof of the theorem is therefore established.

Theorem 2.2 *Let $0 < p < \infty$. If the function K satisfies condition (1), for a given reasonable function $\omega : (0,1] \to (0,\infty)$, there exists a constant $C > 0$ such that*

$$\int_\Delta \left|f'(z)\right|^2 \frac{K(1-|z|^2)}{\omega(1-|z|)} dA(z) \geq CI(f)$$

for all analytic functions f in Δ, where $I(f)$ is as given in Theorem 2.1.

Proof. By Fubini's theorem, we can rewrite (5) as

$$I(f) \sim \int_\Delta \left|f'(w)\right|^2 dA(w) \int_\Delta \frac{\left(1-|w|^2\right)^2}{|1-z\bar{w}|^4} \frac{K(1-|z|^2)}{\omega(1-|z|)} dA(z).$$

$$\sim \int_\Delta \left|f'(w)\right|^p (1-|w|^2)^{p-2} dA(w) \int_\Delta \frac{\left(1-|w|^2\right)^2}{|1-z\bar{w}|^4} \frac{K(1-|z|^2)}{\omega(1-|z|)} dA(z). \tag{6}$$

The inner integral above is nothing but the Berezin transform of the function $\frac{K(1-|z|^2)}{\omega(1-|z|)}$ at the point w. The desired estimate now follows from Lemma 1.2

We can now prove the main result of this section

Theorem 2.3 *Suppose K satisfies condition (1) and satisfies all conditions of Theorems 2.1 and 2.2, then an analytic function f in Δ belongs to $Q_{K,\omega}$ if and only if*

$$\sup_{a\in\Delta}\int_\Delta\int_\Delta \frac{\left|f(z)-f(w)\right|^2}{|1-z\bar{w}|^4}\frac{K(1-|\varphi_a(z)|^2)}{\omega(1-|\varphi_a(z)|)}dA(z)dA(w)<\infty. \tag{7}$$

Proof. We know that $f\in Q_{K,\omega}$ if and only if

$$\sup_{a\in\Delta}\int_\Delta f'(z)\big|^2\frac{K(1-|\varphi_a(z)|^2)}{\omega(1-|\varphi_a(z)|)}dA(z)<\infty.$$

By a change of variables, we have $f\in Q_{K,\omega}$ if and only if

$$\sup_{a\in\Delta}\int_\Delta \left|(f\circ\varphi_a)'(z)\right|^2\frac{K(1-|z|^2)}{\omega(1-|z|)}dA(z).$$

Replacing f by $f\circ\varphi_a$ in Theorems 2.1 and 2.2, we conclude that $f\in Q_{K,\omega}$ if and only if

$$\sup_{a\in\Delta}\int_\Delta\int_\Delta \frac{\left|f\circ\varphi_a(z)-f\circ\varphi_a(w)\right|^2}{|1-z\bar{w}|^4}\frac{K(1-|z|^2)}{\omega(1-|z|)}dA(z)dA(w)<\infty.$$

Changing variables and simplifying the result, we find that the double integral above is the same as

$$\sup_{a\in\Delta}\int_\Delta\int_\Delta \frac{\left|f(z)-f(w)\right|^2}{|1-z\bar{w}|^4}\frac{K(1-|\varphi_a(z)|^2)}{\omega(1-|\varphi_a(z)|)}dA(z)dA(w).$$

Therefore, $f\in Q_{K,\omega}$ if and only if the condition (7) holds.

Remark 2.2. It is still an open problem to study the results of this paper in generalized Hardy spaces of analytic functions, for information on classes of generalized Hardy spaces we refer to [7, 34, 28].

Remark 2.1. It is still an open problem to extend the results of this paper to the classes $Q_K(p,q)$ and $Q_{K,\omega}(p,q)$ of hyperbolic functions. For recent studies on spaces of hyperbolic functions, we refer to [6, 11, 34] and others. For some studies on analytic or meromorphic $Q_{K,\omega}(p,q)$ and $Q_K(p,q)$ classes, we refer to [9, 10, 14, 15, 16, 18, 19, 20, 35, 36].

Remark 2.2. It is still an open problem to extend the results of this paper to Clifford analysis setting. For information on function spaces in Clifford analysis, we refer to [1, 2, 3, 4, 5, 8, 12, 21, 23, 25, 32, 29, 30, 31] and others.

Conflict of Interests

The authors declare that there is no conflict of interests.

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