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NORMAL JACOBI FIELD ON RIEMANNIAN MANIFOLD

NASSAR H. ABDEL-ALL[1], E. I. ABDEL-GALIL[2*]

[1]Department of Mathematics, Faculty of Science, Assiut University, Assiut 61516, Egypt

[2]Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt

Abstract: In this paper we presents a brief instructions to find the Normal Jacobi field on a two dimensional Riemannian manifold (surfaces) by exact solution for the Jacobi field ordinary differential equation. The resulting solution can be easy plotted on the geodesics for some surfaces by using the Mathematica program command.

Keywords: Differential Geometry, vector field, Jacobi vector field, Riemannian geometry

2000 AMS Subject Classification: 47H17; 47H05; 47H09

1. Introduction

In Riemannian geometry, a Jacobi field is a vector field along a geodesic γ on a Riemannian manifold describing the difference between the geodesic and an "infinitesimally close" geodesic. In other words, the Jacobi fields along a geodesic form the tangent space to the geodesic in the space of all geodesics [2].

Jacobi vector fields are important in the study of Riemannian geometry, in particular they are a very powerful tool for an elegent study of the extrinsic and intrinsic geometry of geodesic spheres and tubes about curves and submanifolds of a Riemannian manifold. Moreover, many properties of the geometry of a Riemannian manifold $(M; g)$ may be studied using Jacobi vector fields[13].

*Corresponding author
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Jacobi fields are vector fields defined along a geodesic in a Riemannian Manifold which satisfies a second order differential equation involiving curvature operator and are associated to variation of geodesics. The study of Jacobi vector fields leads to the study of the Jacobi differential equation which is very useful in the study of the geometry.

An important starting point for our work the covariant derivatives along the Curve γ and introduce to Jacobi vector field on the infinite dimensional Riemmanian manifold of diffeomorphisms. Several authors have studied the solution of the exponential map inverse problem and we will study to solve the (IVP) Jacobi equation in special cases for surface with constant Gaussian Curvature and some cases for revolution surface with non constant Gaussian Curvature

2. Normal Jacobi field

Definition 2.1:-

A curve $\gamma(\mathrm{t}) : [a,b] \to M$ on a Riemannian manifold $(M;\ g)$ with the parameterization of the surface $X(u,v)$ and Cherestofiel symbols Γ_{ij}^{λ} is called a geodesic if its tangent vectors are parallel along the curve ($\frac{D\gamma'(t)}{dt} = 0$). In coordinates, the equation of geodesics can be written as

$$u'' + \Gamma_{11}^{1}(u')^2 + 2\Gamma_{12}^{1}u'v' + \Gamma_{22}^{1}(v')^2 = 0 \qquad (2.1)$$
$$v'' + \Gamma_{11}^{2}(u')^2 + 2\Gamma_{12}^{2}u'v' + \Gamma_{22}^{2}(v')^2 = 0$$

In other words, $\gamma(\mathrm{t}): \mathrm{I} \longrightarrow \mathrm{M}$ is a geodesic if and only if the system of second order differential equations (2.1) is satisfied for every interval $\mathrm{J} \subset \mathrm{I}$ such that $\gamma(\mathrm{J})$ is contained in a coordinate neighbourhood [8]

Definition 2.2:-

Let $\gamma : [a,b] \to M$ be a geodesic on an m-dimensional Riemmanian manifold (M,g) and a variation of γ is a differential map $h(s,t)$: $(-\epsilon,\epsilon) \times I \to M$, Jacobi fields can be obtained in the following way: Take a smooth one parameter family of geodesics $\gamma_s(t)$ with $\gamma_0(t) = h(0,t) = \gamma(t)$, then

$$J(t) = \left.\frac{\partial h(s,t)}{\partial s}\right|_{s=0} = \frac{d\gamma(t)}{dt} \tag{2.2}$$

is a Jacobi field, which describes the behavior of the geodesics in an infinitesimal neighborhood of a given geodesic $\gamma(t)$ and Any solution to the Jacobi field comes from a variation of geodesics.

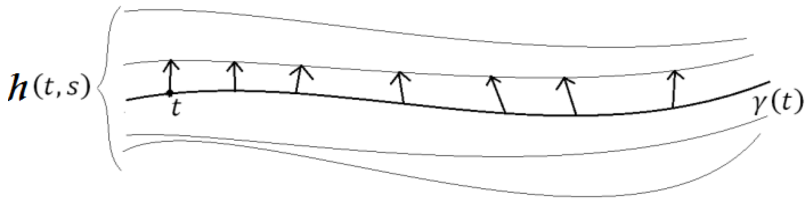


Fig 2.1

Definition 2.3:-

Let $p \in M$ and $v \in T_pM$ and $\gamma(t) = exp_p(tv)$ be a geodesic and $w \in T_pM$ Then the Jacobi field $J(t)$ along $\gamma(t)$ such that $J(0) = 0$ and $J'(0) = w$ is given by the formula [8]

$$J(t) = \left(exp_p\right)_{tv}(tw) \tag{2.3}$$

Definition 2.4:-

Let $\gamma : [a,b] \to M$ be a geodesic on an m-dimensional Riemannian manifold (M,g). A vector field $J = J(t)$ along γ is called Jacobi field if it satisfies the following Jacobi equation:

$$J''(t) + R\big(J(t),\gamma'(t)\big)\gamma'(t) = 0 \tag{2.4}$$

Where R is the Riemannian Curvature operator.

From this definition we conclude that

i) If $J(t)$ is a Jacobi field along γ then Normal component Jacobi field $J^{\perp} = J - \langle J, T \rangle T$ is also Jacobi field along γ where $= \gamma'$.

ii) One of the important result about Jacobi field is the following :

$$\frac{d^2}{dt^2}\langle J, \gamma' \rangle = \langle J'', \gamma' \rangle = -\langle R\big(J(t), \gamma'(t)\big)\gamma'(t), \gamma' \rangle = 0 \qquad (2.5)$$

So $\langle J, \gamma' \rangle$ is linear function of t $(i.e$ the tangential component Jacobi field are of the form

$$J^T = \langle J, \gamma' \rangle \gamma' = (\lambda t + \mu)\gamma' \qquad (2.6)$$

for some constants λ and μ)

Jacobi equation is a linear, second order ordinary differential equation; in particular, values of J and $\frac{DJ}{dt}$ at one point of γ uniquely determine the Jacobi field. Furthermore, the set of Jacobi fields along a given geodesic forms a real vector space of dimension twice the dimension of the manifold.

Theorem 2.1:-

Let $\gamma: I \to M$ be a geodesic on an m-dimensional Riemmanian manifold (M, g), and let $a \in I$

1) A Jacobi field along γ is normal if and only if

$$J(a) \perp \gamma'(a), \qquad J'^{(a)} \perp \gamma'(a) \qquad (2.7)$$

2) Any Jacobi field orthogonal to γ' at two points is normal

Definition 2.5:-

If J is orthogonal on γ everywhere , then J is called a normal Jacobi field.

3. Jacobi field equation for 2-dimensional Riemmanian manifold

Let $\gamma : [a,b] \to M$ be a geodesic on an 2-dimensional Riemmanian manifold (M,g) parametrized by the natural parameter and let $V(t) = \gamma'(t)$ be the field of unit tangent vectors. A vector field $J = J(t)$ along γ is called Jacobi field if it is the field of initial velocities of some family of geodesics $\gamma_s(t)$, (i.e , $J = \frac{\partial \gamma_s(t)}{\partial s}\Big|_{s=0}$)

Each Jacobi field J satisfies the equation (2.4) and if J is decomposed as the sum of the tangent and normal components. Then both is Jacobi field. The tangent component is responsible for the reparametrizations of the geodesic. Therefore, we may assume that it is equal to zero, that is

$$J(t) = y(t)N(t) \tag{3.1}$$

where $N(t)$ is the unit normal vector and $N(t)$ is covariantly constant along γ since $\gamma'(t)$ is covariantly constant. Besides , $R(N,\gamma')\gamma' = KN$ (where K is the Gaussian curvature) .

Hence

$$J''(t) + R\big(J(t),\gamma'(t)\big)\gamma'(t) = (y''(t) + Ky)N = 0 \tag{3.2}$$

the function $y(t)$ satisfies the equation

$$y''(t) + Ky = 0 \tag{3.3}$$

A unique solution of this equation demands the initial values $y(0)$ and $y'(0)$

From theorem 2.1 it is easy to see that the normal Jacobi field $J \perp \gamma'$ and this leads to $J = y(t)\gamma'^{\perp}$ where $\gamma'^{\perp}$ is tangent parallel vector field along geodesic γ and the function $y(t)$ satisfies Jacobi equation (3.3) on 2-dimensional Riemmanian manifold

4. Examples on Jacobi field

i) As trivial examples of Jacobi fields one can consider $\gamma'(t)$ and $t\gamma'(t)$

ii) Any Jacobi field J can be represented in a unique way as a sum $\mathrm{T} + \mathrm{I}$, where

$$\mathrm{T} = a\gamma'(t) + bt\gamma'(t) \qquad (4.1)$$

is a linear combination of trivial Jacobi fields and $I(t)$ is orthogonal to $\gamma'(t)$, for all t. Then field I(t) corresponds to the same variation of geodesics as J, only with changed parameterizations.

iii) To find normal Jacobi field on any surface with constant Gaussian curvature as we find in equation (3.3) for any geodesic γ with unit -speed vector where $J(t) = y(t)N(t)$ and by assuming that $J(0) = 0$, $J'(0) = 1$ as initial value and let $K = c\ (constant)$ then

$$y'' + cy = 0 \qquad (4.2)$$

A unique solution of this equation depends on the initial values $y(0)$ and $y'(0)$ and also have a different solution on every case ($c = 0$ or $c > 0$ or < 0) which is given by

$$y(t) = \begin{cases} At + B & for\ c = 0 \\ A\cos\sqrt{c}t + B\sin\sqrt{c}t & for\ c > 0 \\ A\cosh\sqrt{-c}t + B\sinh\sqrt{-c}t & for\ c < 0 \end{cases} \qquad (4.3)$$

where $A.B$ are constants can be determined by the initial conditions of ordinary differential equation.

Case .1 Jacobi field on a surface with zero Gaussian Curvature ($K = c = 0$)

This case can be found for any developable surface as (plane , cylinder ,cone, …. etc)

Example 4.1

in case of plane with parameterization

$$X(u, v) = (u, v, 0), \quad -\infty \leq u, v \leq \infty \qquad (4.4)$$

,

the geodesic are straight line since the Gaussian curvature is vanished and ordinary Jacobi differential equation is reduced to

$$y''(t) = 0 \tag{4.5}$$

and the solution is given by

$$y(t) = At + B \tag{4.6}$$

where A, B are two constant vector fields on the Plane can be determined by the initial condition of ordinary differential equation $y(0) = 0$ and $y'(0) = 1$ then the solution of equation (3.3) become

$$y(t) = t \tag{4.7}$$

and Jacobi field can be easy determine $J(t) = y(t)N(t) = tN(t)$ where $N(t) = (0,0,1)$ on the plane then Jacobi field is given by

$$J(t) = (0,0,t) \tag{4.8}$$

as shown in Fig . 4.1

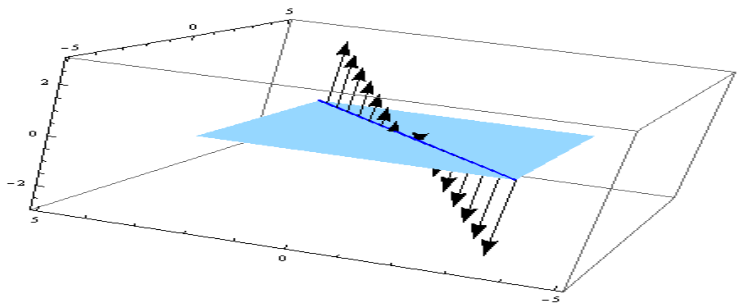


Fig 4.1:- Jacobi vector field on the plane ($-\infty < t < \infty$)

Example 4.2

In case of the cylinder with parameterization

$$X(u,v) = (\cos u, \sin u, v) \quad , -\infty \le v \le \infty \ , \quad -\pi \le u \le \pi \tag{4.9}$$

the geodesic may be straight line (a) or an parallel circles (b) or an helix (c) as a similar way in example (4.1) then Jacobi field can be given by

$$J(t) = tN(t) = t(\cos u(t), \sin u(t), 0)$$

Or equivalently for the different geodesics (lines , circles , helix) as shown in Fig (4.2)a, (4.2)b,(4.2)c

$J(t) = t(\cos h, \sin h, 0)$, h $(const)$ for straight line (a) , $J(t) = t(\cos t, \sin t, 0)$ for parallel circles (b) and helix (c)

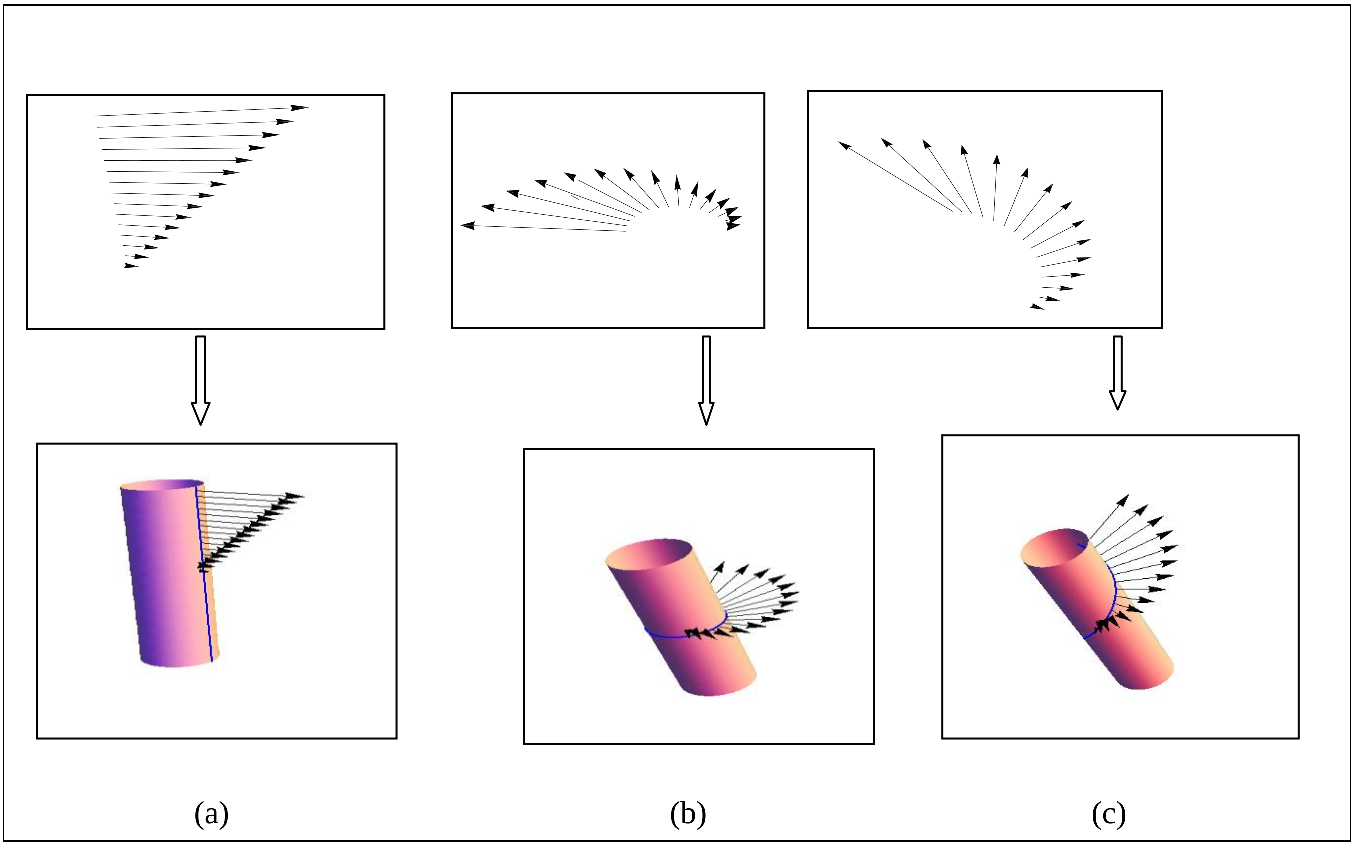


Fig 4.2:- Jacobi vector field on geodesic curves (a),(b),(c) on cylinder , $(0 \le t \le \pi \quad , \quad h = 0)$

Case .2 Jacobi field on a surface with positive constant Gaussian Curvature ($K = c > 0$)

Example 4.3

Consider the unit sphere S^2 with the graphic parameterization

$$X(u,v) = (cos\, v\, cos\, u\, , \cos v \sin u\, , \sin v)$$

$$for\ \ 0 \le v \le 2\pi \qquad , \qquad 0 \le u \le 2\pi \qquad (4.10)$$

Have positive Gaussian equal to 1 and geodesic curve on the unit sphere parameterized by

$$\gamma(t) \to X(u(t), v(t))$$

and $u(t), v(t)$ are satisfy the following geodesic differential equation on the unit sphere

$$\begin{aligned} u'' - 2\tan v\ u'v' &= 0 \\ v'' + \sin v \cos v\ u'^2 &= 0 \end{aligned} \qquad (4.11)$$

This is a nonlinear system of second order differential equations which can be solving analytically in the following cases

a) the first equation of the system is trivially satisfied when $u(t)$ is equal to zero $(u'(t) = 0\,,\ u''(t) = 0)$. then the second equation becomes $(v''(t) = 0\,)$ in which case a solution is $v(t) = t$ for $t \in [-\pi, \pi]$ and the equation $u(t) = 0$ define a plane through the origin of the space . this plane intersects the unit sphere in a great circle and the value $v(t) = t$ moves you around the great circle (longitude curve) with unit speed and Parameterized by

$$\gamma_{Sa}(t) = (\cos t\, , 0, \sin t)$$

b) the second equation of the system is trivially satisfied when $v(t) = \frac{\pi}{2}$ is constant $(v'(t) = 0\,,\ v''(t) = 0)$. then the first equation becomes $(u''(t) = 0\,)$ in which case a solution is $u(t) = t$ for $t \in [-\pi, \pi]$ and the equation $v(t) = \frac{\pi}{2}$ define a plane through the origin of the space . this plane intersects the unit sphere in a great circle and the value $u(t) = t$ moves you around the great circle (latitude curve) with unit speed and Parameterized by

$$\gamma_{Sb}(t) = (\cos t\, , \sin t\, , 0) \quad .$$

then the ordinary Jacobi differential equation (3.3) is reduced to

$$y''(t) + y(t) = 0 \qquad (4.12)$$

and the solution can be given as

$$y(t) = A\ \cos t + B\ \sin t \qquad (4.13)$$

where A, B are constants can be determined by the initial conditions $y(0) = 0$ and $y'(0) = 1$ then the solution of equation (3.3) become

$$y(t) = \sin t$$

This the unit normal vector on the unit Sphere S^2 is

$$N\big(u(t), v(t)\big) = \big(\cos v(t)\cos u(t), \cos v(t)\sin u(t), \sin v(t)\big) \qquad (4.14)$$

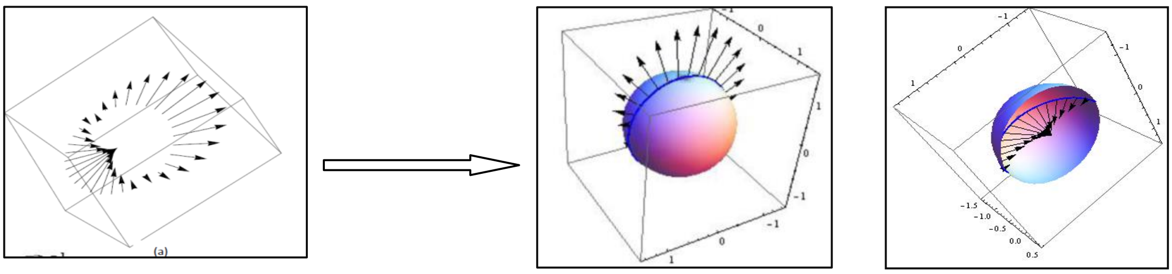
Consequently Jacobi field can be easy determined by

$$J(t) = y(t)N(t) = \sin t\, N(t) \qquad (4.15)$$

From (4.14) and (4.15) ,one can see that the Jacobi fields along the longitude $\gamma_{Sa}(t)$ and latitude $\gamma_{Sb}(t)$ are given respectively

$$J(t)|_{\gamma_{Sa}(t)} = \sin t(\cos t, 0, \sin t)\ , \quad J(t)|_{\gamma_{Sb}(t)} = (0,0,\sin t)$$

As shown in figures (4.3)(a) and (4.3)(b)



$J(t)|_{\gamma_{Sa}(t)}$ for $t \in [-\pi, \pi]$ for $t \in [0, \pi]$ for $t \in [-\pi, 0]$

Fig 4.3 (a):- Jacobi vector field on longitude geodesic curve $J(t)|_{\gamma_{Sa}(t)}$ on unit sphere S^2

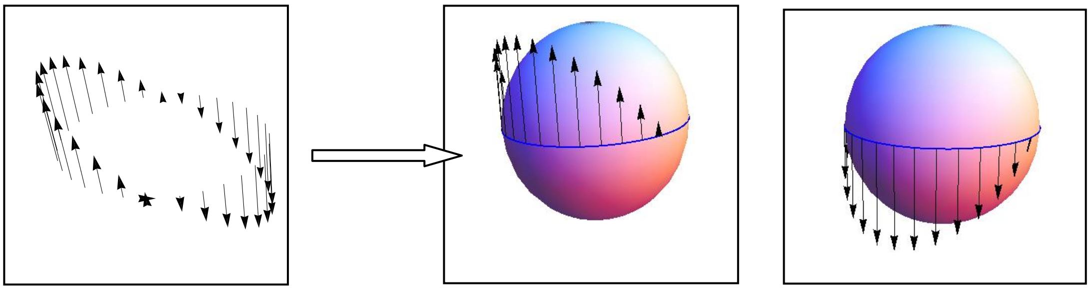


Fig 4.3 (b):- Jacobi vector field on latitude geodesic curve $J(t)|_{\gamma_{Sb}(t)}$ on unit sphere S^2

Case .3 Jacobi field on a surface with negative Gaussian Curvature ($K = c < 0$)

Example 4.4

Consider the Pseudosphere surface parameterized by

$$X(u, v) = (\operatorname{sech} v \cos u , \operatorname{sech} v \sin u , v - \tanh v) \quad \text{for} \quad 0 \le v \le 4 , \quad 0 \le u \le \pi \tag{4.16}$$

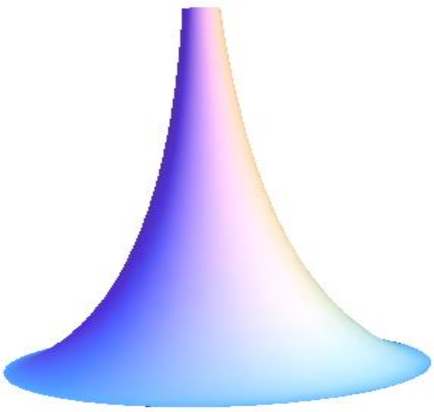


Fig 4.4:- The Pseudosphere Surface

The geodesic curve $\gamma(t) \to X(u(t), v(t))$ on Pseudosphere is given from the solution of geodesics differential equation

$$\begin{aligned} u''(t) - 2\tanh v(t)\; u'(t)v'(t) = 0 \\ v''(t) + \operatorname{csch} v(t) \operatorname{sech} v(t) \left(u'(t)\right)^2 + \operatorname{csch} v(t) \operatorname{sech} v(t) \left(v'(t)\right)^2 = 0 \end{aligned} \quad (4.17)$$

This is a nonlinear system of second order differential equations which can be solving analytically in the following cases

a) First, as expected, the meridians $u = const.$ and $v = v(s)$, parametrized by arc length s , are geodesics. Indeed, the first equation of (4.17) is trivially satisfied by $u = const$ and then $u' = 0$. The second equation becomes

$$v'' + \operatorname{csch} v(t) \operatorname{sech} v(t)\, (v')^2 = 0 \qquad (4.18)$$

Since the first fundamental form along the meridian $u = const. v = v(s)$ yields $\tanh^2 v(t)\, v'^2 = 1$ we conclude that

$$v'^2 = 1/\tanh^2 v(t)$$

Therefore, by derivation

$$2v'v'' = -2 \operatorname{csch} v(t) \operatorname{sech} v(t)\, v'$$

or, since for $v' \neq 0$

$$v'' = -\operatorname{csch} v(t) \operatorname{sech} v(t)\, (v')^2$$

that is, along the meridian $\tanh^2 v(t)\, v'^2 = 1$ the second equation of the equations (4.17) is also satisfied ,which shows that in fact the meridians are geodesics and parameterized for function

$$u(t) = h\ (const) \quad , \quad v(t) = \cosh^{-1} e^t$$

This function give the tractrix which parametrized by curve

$$\gamma_{Pa}(t) = (\operatorname{sech}[\cosh^{-1} e^t] \cos h, \operatorname{sech}[\cosh^{-1} e^t] \sin h, \cosh^{-1} e^t - \tanh[\cosh^{-1} e^t])$$

for $0 < t < \infty$ (4.19)

b) Second ,as also expected the parallels $v = const.\ u = u(t)$,parametrized by arc length, are geodesics. The first of the equations (4.17) gives $u' = const.$ and the second becomes

$$\operatorname{csch} v(t) \operatorname{sech} v(t) \, (u'(t))^2 = 0 \tag{4.20}$$

In order that the parallel $v = const., u = u(s)$ be a geodesic it is necessary that $u' \neq 0$. Since $\operatorname{sech} v(t) \neq 0$, we conclude from Equation (4.19) that

$$\operatorname{csch} v(t) = 0$$

In other words, a necessary condition for a parallel on a surface of Revolution to be a geodesic is that such a parallel be generated by the rotation of a point of the generating curve where the tangent is parallel to the axis of revolution. This condition is clearly sufficient, since it implies that the normal line of the parallel agrees with the normal line to the surface then the orthogonal latitudes are therefore a geodesically parallel parameterized by functions

$$u(t) = t \quad , \quad v(t) = 0$$

This function give a family of "horocycles" parametrized by curve

$$\gamma_{Pb}(t) = (\cos t, \sin t, 0) \tag{4.21}$$

then the ordinary Jacobi differential equation (3.3) is reduced to

$$y''(t) - y(t) = 0 \tag{4.22}$$

and the solution can be given by

$$y(t) = A \cosh t + B \sinh t \tag{4.23}$$

where A, B are constants can be determined by the initial conditions $y(0) = 0$ and $y'(0) = 1$ then the solution of equation (3.3) become

$$y(t) = \sinh t$$

This the unit normal vector on Pseudosphere Surface is

$$N(u(t), v(t)) = (cos\, u(t) \tanh v(t)\,, \sin u(t) \tanh v(t)\,, \operatorname{sech} v(t)) \qquad (4.24)$$

Then Jacobi field is given by

$$J(t) = y(t)N(t) = \sinh t\, N(t) \qquad (4.25)$$

From (4.24) and (4.25) ,one can see that the Jacobi fields along the geodesics curves on Pseudosphere $\gamma_{Pa}(t)$ and $\gamma_{Pb}(t)$ are given respectively by

$$J(t)|_{\gamma_{Pa}(t)} = \sinh t \left(cos\,h\ e^{-t} \sqrt{\frac{-1+e^t}{1+e^t}}(1+e^t)\ , \sin h\ e^{-t} \sqrt{\frac{-1+e^t}{1+e^t}}(1+e^t)\ , e^{-t} \right) ,$$

$$J(t)|_{\gamma_{Pb}(t)} = \sinh t\,(0,0,1) \qquad (4.26)$$

As shown in figures (4.5)(a) and (4.5)(b)

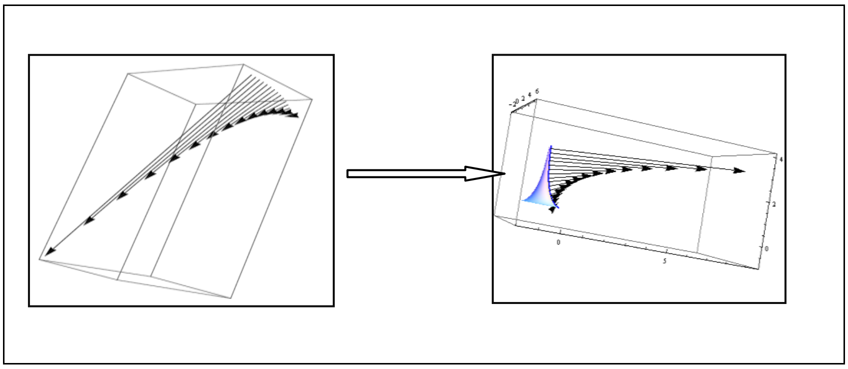


Fig 4.5 (a):- Jacobi vector field on geodesic curve $J(t)|_{\gamma_{Pa}(t)}$ on Pseudosphere, ($t \in [0, \pi]$)

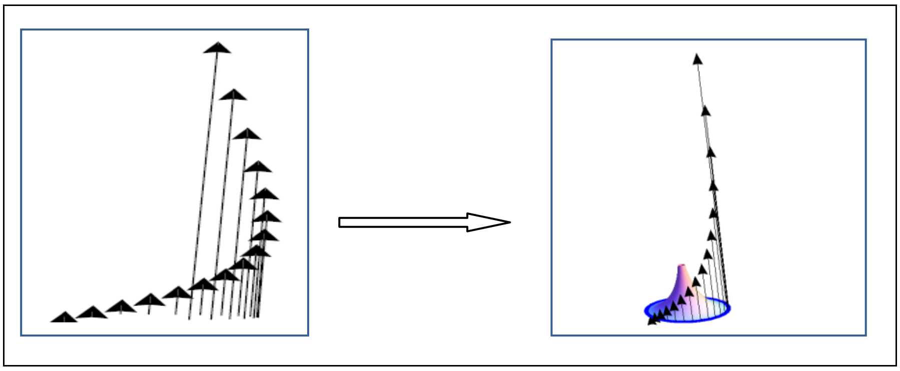


Fig 4.5 (b):- Jacobi vector field on geodesic curve $J(t)|_{\gamma_{Pb}(t)}$ on Pseudosphere, $(t \in [0,\pi])$

iv) To find normal Jacobi field on any revolution surface with nonconstant Gaussian curvature $K(u(t), v(t)) = \beta(v(t))$ *or* $\mu(u(t))$ as we find in equation (3.3) for any special cases to geodesic $\gamma(t)$ on revolution surface with unit speed vector where $J(t) = y(t)N(t)$ and by assuming that $J(0) = 0$,$J'(0) = 1$ we can solve the initial value problem equation (3.3) in the form

$$y'' + \beta(v(t))y = 0 \qquad (4.27)$$

or in the form

$$y'' + \mu(u(t))y = 0 \qquad (4.28)$$

with initial $y(0) = 0$, $y'(0) = 1$

Example 4.5

For example of revolution surface in this case the Torus surface with parameterization

$$X(u, v) = ((2 + \cos v) \cos u, (2 + \cos v) \sin u \ , \sin v) \quad \text{for } u, v \in [-\pi, \pi] \qquad (4.29)$$

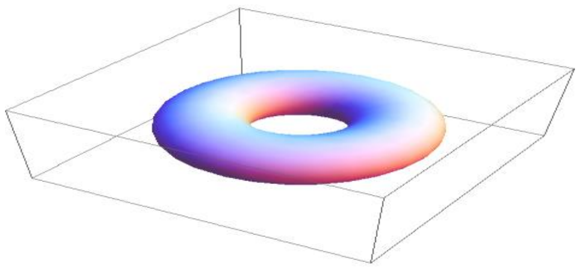


Fig 4.6 :- the tour surface for $u, v \in [-\pi, \pi]$

Which have a non constant Gaussian curvature along any curve $(u = u(t), v = v(t))$ on it is given by

$$K\big(u(t), v(t)\big) = \beta\big(v(t)\big) = \frac{\cos v(t)}{2 + \cos v(t)} \tag{4.30}$$

And unit normal vector field on surface along any curve $(u = u(t), v = v(t))$ on it is define by

$$N\big(u(t), v(t)\big) = \left(\frac{\cos u(t)\,[1 + 4\cos v(t) + \cos 2v(t)]}{2[2 + \cos v(t)]}, \frac{\sin u(t)\,[1 + 4\cos v(t) + \cos 2v(t)]}{2[2 + \cos v(t)]}, \sin \mathrm{v(t)}\right) \tag{4.31}$$

the geodesic curve on tour is parameterized with

$$\gamma(t) \to X(u(t), v(t))$$

And $u(t), v(t)$ are satisfied the geodesic differential equation (2.1) for Tours

$$\begin{aligned} u''(t) - \frac{2\sin v(t)}{2 + \cos v(t)} u'(t)v'(t) &= 0 \\ v''(t) + (2 + \cos v(t))\sin v(t)\ \big(u'(t)\big)^2 &= 0 \end{aligned} \tag{4.32}$$

This is a nonlinear system of second order differential equations which can be solving analytically in the following cases .

a) First for $v'(t) = 0$ we have from the first equation of (4.32) that $u'(t) = const \neq 0$, $u(t) = t$ for initial condition $u'(0) = 1, u(0) = 0$ and from second equation of (4.32) we have

$$(2 + \cos v(t))\sin v(t)\ \big(u'(t)\big)^2 = 0$$

then $\sin v(t) = 0$ then parallel are geodesic curve when $v(t) = 0\,,\ \pi$ (i.e a necessary condition for a parallel on a Tours to be a geodesic is that such a parallel be generated by the rotation of a point of the generating curve where the tangent is parallel to the axis of revolution and the normal line of the parallel agree with the normal line to the surface)

In case $v(t) = 0$, $u(t) = t$ the one geodesic curve is the outer equator for the Tours parameterized by curve

$$\gamma_{To}(t) = (3\,cos\,t\,, 3\,sin\,t\ ,0) \quad \text{for} \quad t \in [-\pi, \pi] \qquad (4.33)$$

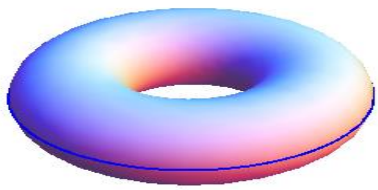


Fig 4.7 :- the outer equator on a tour surface

In case $v(t) = \pi$, $u(t) = t$ the one geodesic curve is the inner equator for the Tours parameterized by curve

$$\gamma_{Ti}(t) = (cos\,t\,, sin\,t\ ,0) \quad \text{for} \quad t \in [-\pi, \pi] \qquad (4.34)$$

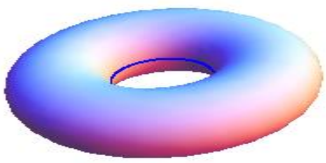


Fig 4.8 :- the inner equator on a Tours surface

b) Second for $u'(t) = 0$ then the first equation of (4.32) is trivially satisfied by $u(t) = h(\,const)$ and The second equation becomes $v''(t) = 0$ then which satisfy $v'(t) = 1$ and $v(t) = t$ for initial condition $v'(0) = 1, v(0) = 0$ then the meridian on a Tours are a geodesic curve which is parameterized by family of curve

$$\gamma_{Tm}(t) = \big((2 + cos\, t)\, cos\, h\,, (2 + cos\, t)\, sin\, h\ \ , sin\, t\big) \quad \text{for } \ t \in [-\pi, \pi]\,, h \text{ is constsnt} \tag{4.35}$$

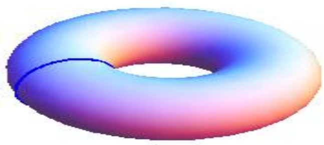


Fig 4.8 :- the meridian geodesic on a Tours surface

To find Jacobi vector field on three special cases of geodesic $\gamma_{To}(t)$, $\gamma_{Ti}(t)$ and $\gamma_{Tm}(t)$ for the Tours surface

For outer equator geodesic curve $\gamma_{To}(t)$

we should solve ordinary differential equation (3.3) and by obtained for $K\big(u(t), v(t)\big)$ and $N\big(u(t), v(t)\big)$ restricted on $\gamma_{To}(t)$ where

$$K\big(u(t), v(t)\big)\big|_{\gamma_{To}(t)} = \frac{1}{3} = c > 0 \ \ ,$$

$$N\big(u(t), v(t)\big)\big|_{\gamma_{To}(t)} = (cos\, t\,, sin\, t\ \ ,0) \tag{4.36}$$

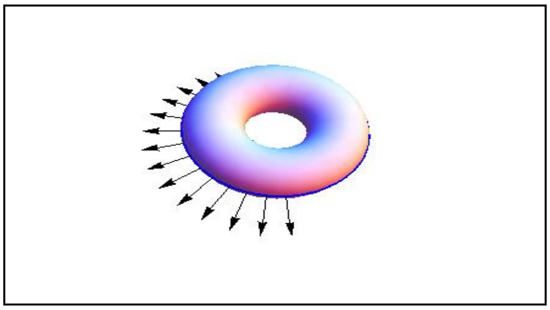
and from solution of equation (3.3) for constant positive Gaussian curvature equation (4.3) we have

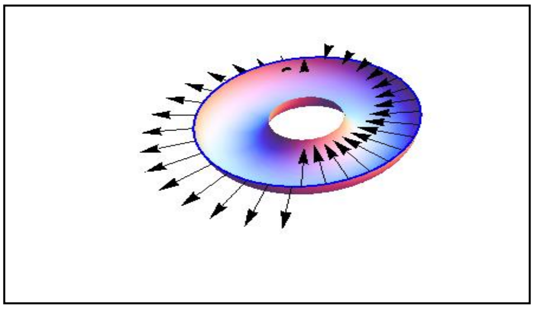
$$y(t) = \sqrt{3}\sin\frac{t}{\sqrt{3}} \tag{4.37}$$

with initial condition for equation (3.3) $y(0) = 0$, $y'^{(0)} = 1$. Then from (4.36), (4.37), one can see that the Jacobi vector field $J(t)$ along outer equator geodesic curve $\gamma_{To}(t)$ on a Tours surface as shown in figure (4.9) is given by

$$J(t)|_{\gamma_{To}(t)} = y(t)N(t)|_{\gamma_{To}(t)}$$

$$= \sqrt{3}\sin\frac{t}{\sqrt{3}}(cos\, t\,, sin\, t\ ,0) \qquad (4.38)$$





For $t \in [0,\pi]$ For $t \in [-\pi,\pi]$

Fig 4.9 :- Jacobi vector field on the outer equator curve $\gamma_{To}(t)$ on a Tours surface

For inner equator geodesic curve $\boldsymbol{\gamma_{Ti}(t)}$

To obtained for $K(u(t),v(t))$ and $N(u(t),v(t))$ restricted on $\gamma_{Ti}(t)$ where

$$K(u(t),v(t))\big|_{\gamma_{Ti}(t)} = -1 = c < 0 \ ,$$

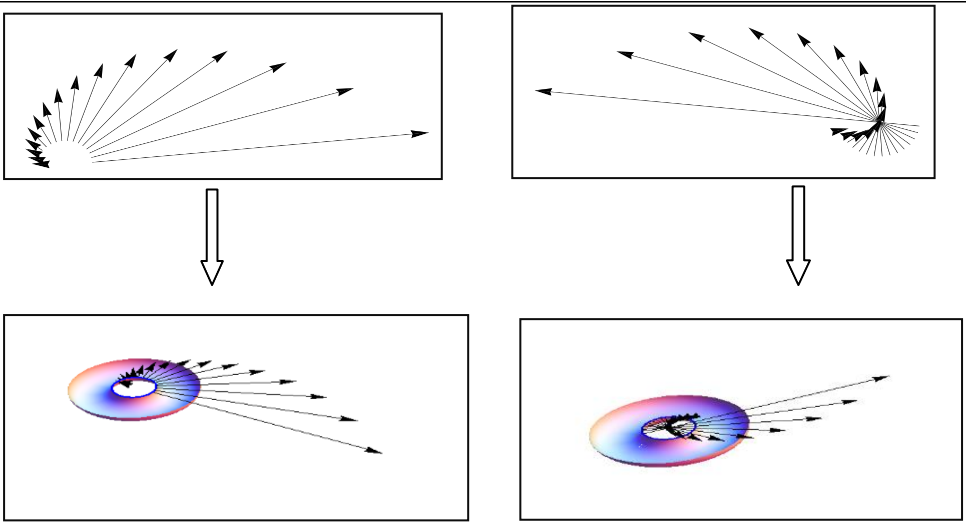
$$N(u(t),v(t))\big|_{\gamma_{Ti}(t)} = -(cos\, t\,, sin\, t\ ,0) \qquad (4.39)$$

from solution of equation (3.3) for constant negative Gaussian curvature (4.3) we have

$$y(t) = \sinh t \qquad (4.40)$$

with initial condition for equation (3.3) $y(0) = 0$, $y'(0) = 1$ .Then from (4.39) and (4.40), one can see that the Jacobi vector field $J(t)$ along inner geodesic curve $\gamma_{Ti}(t)$ on a Tours surface as shown in fig (4.10) is represented by

$$J(t)|_{\gamma_{Ti}(t)} = y(t)N(t)|_{\gamma_{Ti}(t)} = -\sinh t\,(cos\, t\,, sin\, t\ ,0) \qquad (4.41)$$



$J(t)|_{\gamma_{Ti}(t)}$ for $t \in [-\pi, 0]$ $J(t)|_{\gamma_{Ti}(t)}$ for $t \in [0, \pi]$

Fig 4.10 :- Jacobi vector field for the inner equator $J(t)|_{\gamma_{Ti}(t)}$ on a Tours surface

For meridian geodesic curve $\gamma_{Tm}(t)$

To obtained for $K(u(t), v(t))$ and $N(u(t), v(t))$ restricted on $\gamma_{Tm}(t)$ where

$$K(u(t), v(t))\big|_{\gamma_{Tm}(t)} = \frac{\cos t}{2 + \cos t} \quad ,$$

$$N(u(t), v(t))\big|_{\gamma_{Tm}(t)} = \left(\frac{\cos h\,[1 + 4\cos t + \cos 2t]}{2[2 + \cos t]}, \frac{\sin h\,[1 + 4\cos t + \cos 2t]}{2[2 + \cos t]}, \sin t \right) \quad (4.42)$$

The ordinary differential equation (3.3) become

$$y''(t) + \frac{\cos t}{2 + \cos t}\, y(t) = 0 \quad , y(0) = 0 \ , \quad y'(0) = 1 \quad (4.43)$$

It is easy solved initial value problem and the solution $y(t)$ is given by

$$y(t) = i\sqrt{-1 + \cos t^2} + \frac{2}{3}\sqrt{3}(\cos t + 2)(\psi(t) - \varphi(t) - \lambda + \pi) \quad (4.44)$$

Where

$$\psi(t) = i\log\left[3\left(3 - 2(2+cos\,t) + \sqrt{3}\sqrt{3 - 4(2+cos\,t) + (2+cos\,t)^2}\right)\right],$$
$$\varphi(t) = i\log\left[\sqrt{3}(2+cos\,t)\right] - 12i\sqrt{3}\,cos\,t\log\left[\sqrt{3}(2+cos\,t)\right], \qquad (4.45)$$
$$\lambda = \frac{i}{12}\log[729]$$

From (4.42) and (4.44) ,one can see that the Jacobi fields along meridian geodesic curve $\gamma_{Tm}(t)$ on a Tours surface as show in fig (4.11) is represented by

$$J(t)|_{\gamma_{Tm}(t)} = y(t)N(t)$$
$$= \left(i\sqrt{-1+cos\,t^2} + \frac{2}{3}\sqrt{3}(cos\,t+2)(\psi(t) - \varphi(t) - \lambda + \pi)\right)$$
$$* \left(\frac{\cos h\,[1+4\cos t+\cos 2t]}{2[2+\cos t]}, \frac{\sin h\,[1+4\cos t+\cos 2t]}{2[2+\cos t]}, \sin t\right) \qquad (4.46)$$

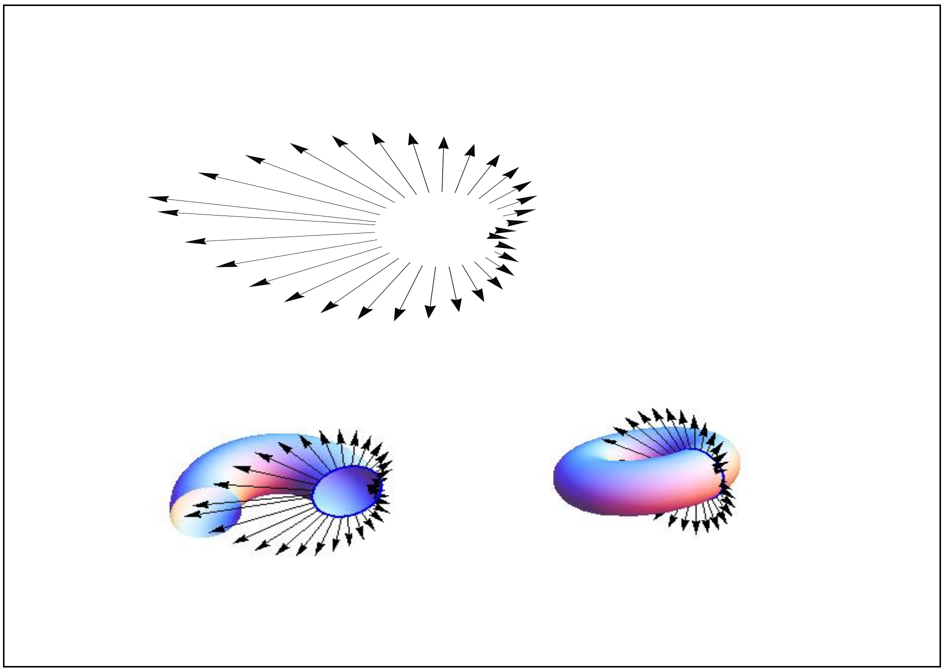


Fig 4.11 :- Jacobi vector field for the meridian $J(t)|_{\gamma_{Tm}(t)}$ on a Tours surface where $t \in [-\pi, \pi]$, $h = 0$

Conflict of Interests

The author declares that there is no conflict of interests.

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