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A HIGHER ACCURACY EXPONENTIAL FINITE DIFFERENCE METHOD FOR THE NUMERICAL SOLUTION OF SECOND ORDER ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract: In this article, we have developed a new exponential finite difference method of order four for the numerical solution of second order elliptic partial differential equations. We have presented the derivation and development of the method as well as have estimated the local truncation error in the method. Numerical examples are given to illustrate the performance of the method and its accuracy. The proposed method compares well with compact nine points fourth order method, it can be observed in numerical section of the article.

Keywords: Exponential method, Elliptic equations, Finite difference method, Fourth order method, Mac Lauren series expansion.

2000 AMS Subject Classification: 65N06

1. Introduction:

Many problems in engineering and physics can be modeled in the form of partial differential equations. Most of the partial differential equations that arise in mathematical models of physical phenomena are difficult to solve analytically. So we have to use numerical methods to approximate the solution of such problems. There are numerous ways by which an approximate solution to these particular differential equations can be constructed. One of the methods is finite difference, the oldest and popular techniques for numerical solution of differential equations.

Consider the second order elliptic partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \tag{1.1}$$

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in square region $\Omega = \{(x,y): a \leq x \leq b\ , a \leq y \leq b\}$ with the boundary conditions

$$u(x,y) = g(x,y) \qquad on\ \partial\Omega \qquad (1.2)$$

where $\partial\Omega$ is boundary of Ω .

In past several years various authors such as Lynch etc. [1], Biosvert [2], Wilkinson [3] and many more have been used finite difference method for the numerical solution of problems (1.1) and have reported accurate and high order discretization methods. So much research has been reported on numerical solution of elliptic partial differential equations, many of them are excellent work. But a concept to develop a new approach to solve (1.1) cannot be overemphasized.

In this article, we develop a new algorithm capable of solving equation of form (1.1). To best of our knowledge, no similar method for the solution of (1.1) has been discussed in literature so for. In this paper using nine point compact cell, we discuss exponential finite difference, a new method of order four based on local assumption. Its development and analysis is based on Taylor series, Mac Lauren series and exponential expansion.

In the next section we discuss the derivation of the exponential finite difference method. A local truncation error estimated in Section 3 and finally the application of the developed method to the problem (1.1) has been presented and illustrative numerical results have been produced to show the efficiency of the new method in Section 4. We compare the computed results with the results obtained by the nine point fourth order method for problems. A discussion and conclusion on the performance of the method are presented in section 5.

2. **Derivation of the method:**

For the solution of problem (1.1), we consider square domain $\Omega = [a,b] \times [a,b]$ .First we generate a mesh $(x_i\ , y_j)$, $x_i = a \pm ih\ , i = 0,1,2\ \dots \dots .\ .\ N\ and\ \ y_j = a \pm jh$, $j = 0,1,2\ \dots \dots .\ .\ N$, where $h = \frac{(b-a)}{N}$ is the mesh size in the x and y directions of Cartesian coordinate system parallel to coordinate axes. Let denote the central mesh point $(x_i\ , y_j)$ by (i,j) and numerical solution of the problem (1.1) at this mesh point by u_{ij} . Similarly we can define other notations to in this article. Consider other eight mesh points $(i \pm 1, j), (i, j \pm 1) and\ (i \pm 1, j \pm 1)$ neighboring to central mesh point (i,j) . Assuming the local assumption that no previous truncation errors have been made in computation of solution at mesh point (i,j) i.e. $u_{i\pm1\,,j} = u\big(x_i \pm h\,, y_j\big)$,

$u_{i\,,j\pm1} = u\big(x_i\,, y_j \pm h\big)\ and\ u_{i\pm1\,,j\pm1} = u\big(x_i \pm h\,, y_j \pm h\big)$. Following the ideas in [4,5] , we propose an approximation to the theoretical solution $u\big(x_i\,, y_j\big)$ of the problem (1.1) nine point exponential finite difference scheme as

$$a_0\,(u_{i+1\,,j} + u_{i-1\,,j} + u_{i\,,j+1} + u_{i\,,j-1}) + a_1\,(u_{i+1\,,j+1} + u_{i-1\,,j+1} + u_{i+1\,,j-1} + u_{i-1\,,j-1}) + a_2 u_{i\,j} = b_0\, h^2 f_{ij}\, e^{\phi(h)} \tag{2.3}$$

where $a_0\,, a_1\,, a_2\ and\ b_0$ are unknown constants and $\phi(h)$,an unknown sufficiently differentiable function of mesh size h.

Let su define a function $F_{ij}(h,u)$ and associate it with (2.3) as,

$$F_{ij}(h,u) \equiv a_0\,\Big(u\big(x_i + h\,, y_j\big) + u\big(x_i\,, y_j + h\big) + u\big(x_i - h\,, y_j\big) + u\big(x_i\,, y_j - h\big)\Big)$$
$$+a_1\,\Big(u\big(x_i + h\,, y_j + h\big) + u\big(x_i - h\,, y_j + h\big) + u\big(x_i + h\,, y_j - h\big) + u\big(x_i - h\,, y_j - h\big)\Big)$$
$$+a_2\; u\big(x_i\,, y_j\big) - \; b_0\, h^2\; f\big(x_i\,, y_j\big)\, e^{\phi(h)} = 0 \tag{2.4}$$

We assume that the $\phi(h)$ can be expand in Mac Lauren's series. So we write Mac Lauren series expansion for the function $\phi(h)$.

$$\phi(h) = \phi(0) + h\phi'(0) + \frac{h^2}{2}\phi''(0) + O(h^3) \tag{2.5}$$

Thus the application of (2.5) in expansion of $e^{\phi(h)}$ will provide an $O(h^3)$ approximation of the form of

$$e^{\phi(h)} = 1 + \phi(0) + \frac{1}{2}\phi^2(0) + h\big(1 + \phi(0)\big)\phi'(0)$$
$$+\frac{h^2}{2}\Big\{\big(\phi'(0)\big)^2 + \big(1 + \phi(0)\big)\phi''(0)\Big\} + O(h^3) \tag{2.6}$$

Expand $F_{ij}(h,u)$ in Taylor series about mesh point $\big(x_i\,, y_j\big)$ and using (2.6) in it, we have

$$F_{ij}(h,u) \equiv \{(4a_0\ + 4a_1\ + a_2\,)u_{ij} + (a_0\ + 2a_1\,)h^2 f_{ij} +$$
$$\frac{h^4}{12}(a_0\ + 2a_1\,)\nabla^2 f_{ij} - \frac{h^4}{6}(a_0\ - 4a_1\,)(\frac{\partial^4 u}{\partial x^2 \partial y^2}\,)_{(i,j)}\}$$
$$-\, b_0\, h^2 f_{ij}\{\,1 + \phi(0) + \frac{1}{2}\phi^2(0) + h(\,1 + \phi(0)\,)\phi'(0)$$
$$+\frac{h^2}{2}(\,(\phi'(0))^2 + (\,1 + \phi(0)\,)\,\phi''(0)\,)\,\}\ = 0 \tag{2.6}$$

where $\nabla^2 = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ and we have used following relations as defined in [6,7], to simplify expressions (2.6).

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)_{(i,j)} = f_{ij} \quad and \quad \left\{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right\}_{(i,j)} = \nabla^2 f_{ij} \tag{2.7}$$

Comparing the coefficients of $h^p, p = 0, 2, 3, 4$ in (2.6) ,we will get following system of nonlinear equations

$$4a_0 + 4a_1 + a_2 = 0$$

$$(a_0 + 2a_1) - b_0\left(1 + \phi(0) + \frac{1}{2}\phi^2(0)\right) = 0$$

$$b_0 f_{ij}\big(1 + \phi(0)\big)\phi'(0) = 0$$

$$\frac{1}{12}(a_0 + 2a_1)\nabla^2 f_{ij} - \frac{1}{6}(a_0 - 4a_1)\left(\frac{\partial^4 u}{\partial x^2 \partial y^2}\right)_{(i,j)}$$

$$- b_0 f_{ij}\left\{\big(\phi'(0)\big)^2 + \big(1 + \phi(0)\big)\phi''(0)\right\} = 0 \tag{2.8}$$

To determine unknown constants in (2.8), we have to assign arbitrary values to some constants . So to simplify above system of equations, we have considered following

$$\phi(0) = 0$$

$$\phi'(0) = 0$$

$$a_0 - 4a_1 = 0 \tag{2.9}$$

So using (2.9) in (2.8) and solving the reduced system of equations, we obtained

$$a_0 = 4a_1$$

$$b_0 = 6a_1$$

$$a_2 = -20a_1$$

$$\phi''(0) = \frac{\nabla^2 f_{ij}}{6f_{ij}} \tag{2.10}$$

Substituting the values of $\phi(0), \phi'(0),\ and\ \phi''(0)$ etc. from (2.9) and (2.10) in (2.5), we have

$$\phi(h) = \frac{h^2\,\nabla^2 f_{ij}}{12\, f_{ij}} \tag{2.11}$$

Finally substitute the values of a_0 , b_0 , a_2 and $\phi(h)$ from (2.10) and (2.11) in (2.3),we will obtain our proposed exponential finite difference method as

$$4(u_{i+1\,,j} + u_{i-1\,,j} + u_{i\,,j+1} + u_{i\,,j-1}) + (u_{i+1\,,j+1} + u_{i-1\,,j+1} + u_{i+1\,,j-1} + u_{i-1\,,j-1})$$

$$-20\, \mathrm{u}_{ij} = 6\, h^2 f_{ij}\, e^{\left(\frac{h^2\,\nabla^2 f_{ij}}{12\, f_{ij}}\right)} \tag{2.12}$$

If we write system equations given by (2.12) for each mesh point, we can obtain the final linear system of equations and nonlinear system of equations in the case where $f(x,y,u)$.

For computations reported in Section 4, we have used second order finite difference approximation in place of $\nabla^2 f_{ij}$. We have made following substitution

$$h^2\nabla^2 f_{ij} = f_{i+1\,,j} + f_{i-1\,,j} - 4f_{ij} + f_{i\,,j+1} + f_{i\,,j-1}$$

3. Local Truncation Error :

The truncation error T_{ij} at the mesh point(i,j) may be written as

$$\mathrm{T}_{ij} = 4(u_{i+1\,,j} + u_{i-1\,,j} + u_{i\,,j+1} + u_{i\,,j-1}) - 20u_{ij} +$$

$$(u_{i+1\,,j+1} + u_{i-1\,,j+1} + u_{i+1\,,j-1} + u_{i-1\,,j-1}) - 6\, h^2 f_{ij}\, e^{\left(\frac{h^2\,\nabla^2 f_{ij}}{12\, f_{ij}}\right)} \tag{3.13}$$

Substituting Taylor series expansion of each term about mesh point (i,j) on the right side of (3.13) and simplify, we have

$$T_{ij} = [6h^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + \frac{h^4}{2}(\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4})$$

$$+\frac{h^6}{60}\,(\frac{\partial^6 u}{\partial x^6} + 5\frac{\partial^6 u}{\partial x^4 \partial y^2} + 5\frac{\partial^6 u}{\partial x^2 \partial y^6} + \frac{\partial^6 u}{\partial y^6})\,]_{(i\,,j)} - 6\, h^2 f_{ij}\, e^{\left(\frac{h^2\,\nabla^2 f_{ij}}{12\, f_{ij}}\right)} \tag{3.14}$$

Substituting second order approximation for the term $e^{\left(\frac{h^2\,\nabla^2 f_{ij}}{12\, f_{ij}}\right)}$ on the right side of (3.14) and using (2.8), we have

$$T_{ij} = \frac{h^6}{240}\{4(\frac{\partial^6 u}{\partial x^6} + 5\frac{\partial^6 u}{\partial x^4 \partial y^2} + 5\frac{\partial^6 u}{\partial x^2 \partial y^6} + \frac{\partial^6 u}{\partial y^6}) -$$

$$5(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})^{-1}(\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4})^2\}_{(i,j)} \quad (3.15)$$

Thus from (3.15) ,we conclude that the method (3.12) is of $O(h^4)$.

4. **Numerical Results:**

To illustrate our method and demonstrate computationally its efficiency, we consider linear and nonlinear examples, in which one is highly nonlinear elliptic problem. In each case we took the square as region of integration and covered it with a uniform mesh of size width h. In tables, we have shown maximum absolute error MAU, computed for different values of N, using the following formula

$$MAU = \max_{1\le i,j \le N-1} \left|u(x_i, y_j) - u_{ij}\right|$$

For shake of comparison we also solved examples by nine point finite difference method and computed results presented in tables. We have used iterative method Gauss-Seidel, Newton-Raphson to solve the system of linear and nonlinear equations respectively. The results obtained by our method compare favourably with those obtained by nine points method. The computations were performed on a MS Window 2007 professional operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Duo Core 2.20 Ghz PC . The stopping condition for iteration was either error of order 10^{-18} or number of iteration 10^5 . Thus we have also shown elapsed time in second and no. of iterations performed to achieve desired accuracy.

Example1. Consider the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{2}(e^u - 1) + f(x,y) \ , \qquad -1 \le x, y \le 1$$

with the boundary condition $u(x,y)$ on all sides of a square. The exact solution of the problem is $u(x,y) = \sin(x^2 + y^2)$. In table 1 ,we present the maximum absolute error MAU ,elapsed time in seconds and no. of iteration performed during the integration for various choices of mesh size h.

Example2. Consider the problem as in [8],

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-f(x,y,u)\ ,\qquad 0\le x,y\le 1$$

with the boundary condition $u(x,y)$ on all sides of a unit square. The exact solution of the problem is $u(x,y)=(1-x^2)\sin\left(\frac{\pi}{2}y\right)$. In table 2 ,we present the maximum absolute error MAU and no. of iteration performed during the integration for various choices of mesh size h.

Example 3. Consider the problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=x^2+y^2\ ,\qquad 0\le x,y\le 1$$

with the boundary condition $u(x,y)$ on all sides of a unit square. The exact solution of the problem is $u(x,y)=\frac{1}{2}x^2y^2$. In table 3 ,we present the maximum absolute error MAU and no. of iteration performed during the integration for various choices of mesh size h.

Example 4. Consider the problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=-\beta u+f(x,y)\ ,\qquad 0\le x,y\le 1$$

with the boundary condition $u(x,y)$ on all sides of a unit square. The exact solution of the problem is $u(x,y)=Sin(\lambda\pi x)\,Cos(\lambda\pi y)$. In table 4 ,we present the maximum absolute error MAU and no. of iteration performed during the integration for various choices of mesh size h , β and λ.

Table1.

N	Method (2.12)			Nine points Method		
	MAU	Etime*	Iterations	MAU	Etime*	Iterations
4	.12771636(-1)	0	18	.57487725(-2)	0	17
8	.25004745(-2)	0	55	.26421397(-	0	41

				3)		
16	.93191862(-4)	.1	146	.14724583(-4)	0	70
32	.24914742(-4)	.8	212	.86985528(-6)	.3	90
64	.11920929(-6)	.6	31	.59604645(-7)	.1	8
128	.59604647(-7)	4.1	54	.11920929(-6)	.7	12

***** Seconds

Table2.

N	Method (2.12)		Nine points Method	
	MAU	Iterations	MAU	Iterations
4	.28192997(-4)	11	.76889992(-5)	11
8	.17285347(-5)	22	.47683716(-6)	16
16	.59604645(-7)	2	.59604645(-7)	9

Table3.

N	Method (2.12)		Nine points Method	
	MAU	Iterations	MAU	Iterations
4	.51761977(-4)	15	**Exact	
8	.35460107(-5)	40	*	
16	.23737084(-6)	73	*	
32	.14849320(-7)	84	*	
64	.92813934(-9)	89	*	

Table4.

$\lambda = \frac{1}{5}\,, \beta = \frac{3}{25}\pi^2$				
N	Method (2.12)		Nine points Method	
	MAU	Iterations	MAU	Iterations
4	.44703484(-7)	6	.14901161(-7)	6
8	.29802322(-7)	5	.29802322(-7)	5
16	No change		No change	
$\lambda = \frac{1}{3}\,, \beta = -\frac{1}{3}\pi^2$				
4	.17583370(-5)	8	.46193600(-6)	8
8	.59604645(-7)	10	.59604645(-7)	4
16	No change		No change	

5. **Conclusion:**

In this article we have outlined a procedure for obtaining high order exponential difference method for the elliptic equations. The drawback to this method is that to obtain solutions to the resulting nonlinear system of equations. Numerical solution for model problems has been presented in this article. For shake of comparison, we have done with nine point method and find that present method compare favourably. Numerical results show that our method has the expected accuracy and is many times faster than the nine point fourth order method for nonlinear problems. But in case of linear problems there is some drawback. Method may be more competitive, if suitable solution techniques used to solve resulting system of nonlinear equations. The present technique may be open up new avenue of research in development of discretization method for other problems.

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