Available online at http://scik.org
J. Math. Comput. Sci. 3 (2013), No. 6, 1389-1404
ISSN: 1927-5307

PROPERTIES OF INTERVAL IMPLICATIONS

YONG CHAN KIM

Department of Mathematics, Gangneung-Wonju National University,
Gangneung, Gangwondo 210-702, Korea

Abstract. In this paper, we construct pairs of interval negations and interval implications from pairs of negations and implications. Moreover, we investigate their properties and give examples.

Keywords: pairs of negations; pairs of implications; pairs of interval negations; pairs of interval implications

2000 AMS Subject Classification: 03E72; 03G10; 06A15; 06F07

1. Introduction

Bedregal and Takahashi [4] introduced interval fuzzy connectives as an extension for fuzzy connectives. This concept provides tools for approximate reasoning and decision making with a frame work to deal with uncertainty and incompleteness of information [1-3]. Georgescu and Popescue [5-7] introduced pseudo t-norms and generalized residuated lattices in a sense as non-commutative property. Kim [11] introduced pairs of (interval) negations and (interval) implications. which are induced by non-commutative property. Let $(L,\wedge,\vee,\odot,\rightarrow,\Rightarrow,\top,\bot)$ be a complete generalized residuated lattice with the law of double negation defined as $a = n_1(n_2(a)) = n_2(n_1(a))$ where $n_1(a) = a \Rightarrow \bot$ and $n_2(a) = a \rightarrow \bot$ (ref. [5-7,11]). We consider

Received August 5, 2013

a pair of two implications defined by $a \Rightarrow b = \bigvee\{c \mid a \odot c \leq b\}$ and $a \rightarrow b = \bigvee\{c \mid c \odot a \leq b\}$. Moreover, we consider a pair of two negations defined by $a \Rightarrow \perp$ and $a \rightarrow \perp$.

In this paper, we construct pairs of interval negations and interval implications from pairs of negations and implications. Moreover, we investigate their properties and give examples.

2. Preliminaries

In this paper, we assume that $(L, \vee, \wedge, \perp, \top)$ is a bounded lattice with a bottom element $\perp$ and a top element $\top$. Moreover, we define the following definitions in a sense as non-commutative [5-7] and interval property [1-4].

Definition 2.1.[11] A pair (n_1, n_2) with maps $n_i : L \to L$ is called a *pair of negations* if it satisfies the following conditions:

(N1) $n_i(\top) = \perp, n_i(\perp) = \top$ for all $i \in \{1,2\}$.

(N2) $n_i(x) \geq n_i(y)$ for $x \leq y$ and $i \in \{1,2\}$.

(N3) $n_1(n_2(x)) = n_2(n_1(x)) = x$ for all $x \in X$.

Definition 2.2.[11] A pair (I_1, I_2) with maps $I_1, I_2 : L \times L \to L$ is called a *pair of implications* if it satisfies the following conditions:

(I1) $I_i(\top,\top) = I_i(\perp,\top) = I_i(\perp,\perp) = \top, I_i(\top,\perp) = \perp$ for all $i \in \{1,2\}$.

(I2) If $x \leq y$, then $I_i(x,z) \geq I_i(y,z)$ for all $i \in \{1,2\}$.

(I3) $I_i(\top, x) = x$ for all $x \in L$ and $i \in \{1,2\}$.

(I4) $I_1(x, I_2(y,z)) = I_2(y, I_1(x,z))$ for all $x,y,z \in X$.

(I5) $I_1(I_2(x,\perp),\perp) = I_2(I_1(x,\perp),\perp) = x$.

Let $L^{[2]} = \{[x_1,x_2] \mid x_1 \leq x_2, x_1, x_2 \in L\}$ where $[x_1,x_2] = \{x \in L \mid x_1 \leq x \leq x_2\}$. We define

$$[x_1,x_2] \leq [y_1,y_2], \text{ iff } x_1 \leq y_1,\ x_2 \leq y_2$$

$$[x_1,x_2] \subset [y_1,y_2], \text{ iff } y_1 \leq x_1 \leq x_2 \leq y_2$$

$$l([x_1,x_2]) = x_1, \quad r([x_1,x_2]) = x_2.$$

Definition 2.3.[11] A pair $(\mathbf{N}_1, \mathbf{N}_2)$ with maps $\mathbf{N}_i : L^{[2]} \to L^{[2]}$ is called a *pair of interval negations* if it satisfies the following conditions:

(IN1) $\mathbf{N}_i([\top,\top]) = [\bot,\bot]$, $\mathbf{N}_i([\bot,\bot]) = [\top,\top]$ for all $i \in \{1,2\}$.

(IN2) If $[x_1,x_2] \leq [y_1,y_2]$, then $\mathbf{N}_i([y_1,y_2]) \leq \mathbf{N}_i([x_1,x_2])$ for all $i \in \{1,2\}$.

(IN3) If $[x_1,x_2] \subset [y_1,y_2]$, then $\mathbf{N}_i([x_1,x_2]) \subset \mathbf{N}_i([y_1,y_2])$ for all $i \in \{1,2\}$.

(IN4) $\mathbf{N}_1(\mathbf{N}_2([x_1,x_2])) = \mathbf{N}_2(\mathbf{N}_1([x_1,x_2])) = [x_1,x_2]$ for all $[x_1,x_2] \in L^{[2]}$.

Definition 2.4.[11] A pair $(\mathbf{I}_1,\mathbf{I}_2)$ with maps $\mathbf{I}_1,\mathbf{I}_2 : L^{[2]} \times L^{[2]} \to L^{[2]}$ is called a *pair of interval implications* if it satisfies the following conditions:

(II1) $\mathbf{I}_i([\top,\top],[\top,\top]) = \mathbf{I}_i([\bot,\bot],[\top,\top]) = \mathbf{I}_i([\bot,\bot],[\bot,\bot]) = [\top,\top]$, $\mathbf{I}_i([\top,\top],[\bot,\bot]) = [\bot,\bot]$ for all $i \in \{1,2\}$.

(II2) If $[x_1,x_2] \leq [y_1,y_2]$, then $\mathbf{I}_i([x_1,x_2],[z_1,z_2]) \geq \mathbf{I}_i([y_1,y_2],[z_1,z_2])$ for all $i \in \{1,2\}$.

(II3) If $[x_1,x_2] \subset [y_1,y_2]$, then $\mathbf{I}_i([x_1,x_2],[z_1,z_2]) \subset \mathbf{I}_i([y_1,y_2],[z_1,z_2])$ for all $i \in \{1,2\}$.

(II4) $\mathbf{I}_i([\top,\top],[x_1,x_2]) = [x_1,x_2]$ for all $i \in \{1,2\}$.

(II5) $\mathbf{I}_1([x_1,x_2],\mathbf{I}_2([y_1,y_2],[z_1,z_2])) = \mathbf{I}_2([y_1,y_2],\mathbf{I}_1([x_1,x_2],[z_1,z_2]))$ for all $[x_1,x_2],[y_1,y_2],[z_1,z_2] \in L^{[2]}$.

(II6) $\mathbf{I}_1(\mathbf{I}_2([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = \mathbf{I}_2(\mathbf{I}_1([x_1,x_2],[\bot,\bot]),[\bot,\bot]) = [x_1,x_2]$.

Theorem 2.5.[11] *Let $\mathbf{N}_i : L^{[2]} \to L^{[2]}$ be a pair of interval negations. Then we have the following properties.*

(1) *Define maps $\underline{\mathbf{N}_i}, \overline{\mathbf{N}_i} : L \to L$ as*

$$\underline{\mathbf{N}_i}(x) = l(\mathbf{N}_i([x,x])),\ \ \overline{\mathbf{N}_i}(x) = r(\mathbf{N}_i([x,x])).$$

Then $\mathbf{N}_i([x_1,x_2]) = [\underline{\mathbf{N}_i}(x_2),\overline{\mathbf{N}_i}(x_1)]$.

(2) $(\underline{\mathbf{N}_1},\underline{\mathbf{N}_2})$ *is a pair of negations such that*

$$\underline{\mathbf{N}_1} = \overline{\mathbf{N}_1},\ \ \underline{\mathbf{N}_2} = \overline{\mathbf{N}_2}.$$

(3) *We define maps $\mathbf{I}_i : L^{[2]} \times L^{[2]} \to L^{[2]}$ as*

$$\mathbf{I}_1([x_1,x_2],[y_1,y_2]) = \mathbf{N}_1([x_1,x_2]) \vee [y_1,y_2],$$

$$\mathbf{I}_2([x_1,x_2],[y_1,y_2]) = \mathbf{N}_2([x_1,x_2]) \vee [y_1,y_2].$$

Then $(\mathbf{I}_1,\mathbf{I}_2)$ is a pair of interval implications.

Theorem 2.6.[11] *Let $(\mathbf{I}_1,\mathbf{I}_2)$ be a pair of interval implications on $L^{[2]}$. We define*

$$\underline{\mathbf{I}_i}(x,y)=l(\mathbf{I}_i([x,x],[y,y])),\ \ \overline{\mathbf{I}_i}(x,y)=r(\mathbf{I}_i([x,x],[y,y])).$$

Then we have the following properties:

(1) *If $[y_1,y_2]\le[z_1,z_2]$, then*

$$\mathbf{I}_1([x_1,x_2],[y_1,y_2])\le\mathbf{I}_1([x_1,x_2],[z_1,z_2]).$$

(2) *If $[y_1,y_2]\subset[z_1,z_2]$, then*

$$\mathbf{I}_1([x_1,x_2],[y_1,y_2])\subset\mathbf{I}_1([x_1,x_2],[z_1,z_2]).$$

(3) $\mathbf{I}_i([x_1,x_2],[y_1,y_2])=[\underline{\mathbf{I}_i}(x_2,y_1),\overline{\mathbf{I}_i}(x_1,y_2)]$.

(4) *If , for each $x,y\in L$, there exists $z\in L$ such that $\mathbf{I}_i([x,x],[y,y])=[z,z]$, $i=1,2$, then $(\underline{\mathbf{I}_1},\underline{\mathbf{I}_2})$ is a pair of implications such that*

$$\underline{\mathbf{I}_1}=\overline{\mathbf{I}_1},\ \underline{\mathbf{I}_2}=\overline{\mathbf{I}_2}.$$

(5) *Define maps $\mathbf{N}_i:L^{[2]}\to L^{[2]}$ as*

$$\mathbf{N}_1([x_1,x_2])=\mathbf{I}_1([x_1,x_2],[\perp,\perp]),$$

$$\mathbf{N}_2([x_1,x_2])=\mathbf{I}_2([x_1,x_2],[\perp,\perp]).$$

Then $(\mathbf{N}_1,\mathbf{N}_2)$ is a pair of interval negations.

(6)

$$\mathbf{I}_1(\mathbf{N}_2([y_1,y_2]),\mathbf{N}_2([x_1,x_2]))=\mathbf{I}_2([x_1,x_2],[y_1,y_2]),$$

$$\mathbf{I}_2(\mathbf{N}_1([y_1,y_2]),\mathbf{N}_1([x_1,x_2]))=\mathbf{I}_1([x_1,x_2],[y_1,y_2]).$$

3. Properties of interval implications

Theorem 3.1. *Let (n_1,n_2) be a pair of negations on L. Then we have the following properties.*

(1) *Define maps $I_i:L\times L\to L$ as*

$$I_1(x,y)=n_1(x)\vee y,\ I_2(x,y)=n_2(x)\vee y.$$

Then (I_1, I_2) is a pair of implications.

(2) *Define maps $\mathbf{N}_i : L^{[2]} \to L^{[2]}$ as*

$$\mathbf{N}_1([x_1,x_2]) = [n_1(x_2), n_1(x_1)], \mathbf{N}_2([x_1,x_2]) = [n_2(x_2), n_2(x_1)]$$

Then $(\mathbf{N}_1, \mathbf{N}_2)$ is a pair of interval negations such that

$$\underline{\mathbf{N}_i}(x) = \overline{\mathbf{N}_i}(x) = n_i(x),$$

$$\mathbf{N}_i([x_1,x_2]) = [\underline{\mathbf{N}_i}(x_2), \overline{\mathbf{N}_i}(x_1)].$$

(3) *For maps I_i in (1), we define maps $\mathbf{I}_i : L^{[2]} \times L^{[2]} \to L^{[2]}$ as*

$$\mathbf{I}_1([x_1,x_2],[y_1,y_2]) = [n_1(x_2) \vee y_1, n_1(x_1) \vee y_2],$$

$$\mathbf{I}_2([x_1,x_2],[y_1,y_2]) = [n_2(x_2) \vee y_1, n_2(x_1) \vee y_2].$$

Then $(\mathbf{I}_1, \mathbf{I}_2)$ is a pair of interval implications such that $\underline{\mathbf{I}_i}(x,y) = n_i(x) \vee y = \overline{\mathbf{I}_i}(x,y)$ and

$$\mathbf{I}_i([x_1,x_2],[y_1,y_2]) = [\underline{\mathbf{I}_i}(x_2,y_1), \overline{\mathbf{I}_i}(x_1,y_2)].$$

Proof. (1) (I1) $I_i(\top,\bot) = n_i(\top) \vee \bot = \bot$, $I_i(\bot,\bot) = n_i(\bot) \vee \bot = \bot = I_i(\bot,\top) = I_i(\top,\top)$.

(I2) If $x \le y$, then $n_1(x) \ge n_i(y)$. Then $I_i(x,z) \ge I_i(y,z)$.

(I3) $I_i(\top,x) = n_i(\top) \vee x = x$.

(I4) $I_1(x, I_2(y,z)) = n_1(x) \vee n_2(y) \vee z = I_2(y, I_1(x,z))$.

(I5) $I_1(I_2(x,\bot),\bot) = n_1(n_2(x)) = x = n_2(n_1(x)) = I_2(I_1(x,\bot),\bot)$.

Hence (I_1, I_2) is a pair of implications.

(2) (IN1) $\mathbf{N}_i([\bot,\bot]) = [\top,\top]$ and $\mathbf{N}_i([\top,\top]) = [\bot,\bot]$.

(IN2) If $[x_1,x_2] \le [y_1,y_2]$, then $\mathbf{N}_i([y_1,y_2]) = [n_i(y_2), n_i(y_1)] \le [n_i(x_2), n_i(x_1)] = \mathbf{N}_i([x_1,x_2])$ for all $i \in \{1,2\}$.

(IN3) If $[x_1,x_2] \subset [y_1,y_2]$, then $y_1 \le x_1 \le x_2 \le y_2$. So, $n_i(y_1) \ge n_i(x_1) \ge n_i(x_2) \ge n_i(y_2)$. Thus, $\mathbf{N}_i([x_1,x_2]) \subset \mathbf{N}_i([y_1,y_2])$ for all $i \in \{1,2\}$,

(IN4)

$$\begin{aligned}\mathbf{N}_1(\mathbf{N}_2([x_1,x_2])) &= \mathbf{N}_1([(n_2(x_2), n_2(x_1)]) \\ &= [n_1(n_2(x_1)), n_1(n_2(x_2))] = [x_1,x_2]\end{aligned}$$

Similarly, $\mathbf{N}_2(\mathbf{N}_1([x_1,x_2])) = [x_1,x_2]$. for all $[x_1,x_2] \in L^{[2]}$.

$$\begin{aligned}\underline{\mathbf{N}_i}(x) &= l(\mathbf{N}_i([x,x]) = l([n_i(x),n_i(x)])\\ &= r([n_i(x),n_i(x)]) = \overline{\mathbf{N}_i}(x) = n_i(x).\end{aligned}$$

$$\mathbf{N}_1([x_1,x_2]) = [n_1(x_2),n_1(x_1)] = [\underline{\mathbf{N}_i}(x_2),\overline{\mathbf{N}_i}(x_1)].$$

(3) (II1)

$$\begin{aligned}&\mathbf{I}_i([\top,\top],[\bot,\bot]) = [n_i(\top)\vee\bot, n_i(\top)\vee\bot] = [\bot,\bot],\\ &\mathbf{I}_i([\bot,\bot],[\top,\top]) = [n_i(\bot)\vee\top, n_i(\bot)\vee\top] = [\top,\top],\\ &\mathbf{I}_i([\bot,\bot],[\bot,\bot]) = [\top,\top] = \mathbf{I}_i([\top,\top],[\top,\top]).\end{aligned}$$

(II2) If $[x_1,x_2]\le[y_1,y_2]$, then $n_i(y_1)\le n_i(x_1)$ and $n_i(y_2)\le n_i(x_2)$. Thus,

$$\begin{aligned}&\mathbf{I}_i([x_1,x_2],[z_1,z_2]) = [n_i(x_2)\vee z_1, n_i(x_1)\vee z_2]\\ &\ge [n_i(y_2)\vee z_1, n_i(y_1)\vee z_2] = \mathbf{I}_1([y_1,y_2],[z_1,z_2]).\end{aligned}$$

(II3) If $[x_1,x_2]\subset[y_1,y_2]$, then $y_1\le x_1\le x_2\le y_2$ and $n_i(y_1)\ge n_i(x_1)\ge n_i(x_2)\ge n_i(y_2)$. So,

$$\begin{aligned}&\mathbf{I}_1([x_1,x_2],[z_1,z_2]) = [n_i(x_2)\vee z_1, n_i(x_1)\vee z_2]\\ &\subset [n_i(y_2)\vee z_1, n_i(y_1)\vee z_2] = \mathbf{I}_1([y_1,y_2],[z_1,z_2]).\end{aligned}$$

(II4)

$$\mathbf{I}_i([\top,\top],[z_1,z_2]) = [n_i(\top)\vee z_1, n_i(\top)\vee z_2] = [z_1,z_2].$$

(II5)

$$\begin{aligned}&\mathbf{I}_1([x_1,x_2],\mathbf{I}_2([y_1,y_2],[z_1,z_2]))\\ &= \mathbf{I}_1([x_1,x_2],[n_2(y_2)\vee z_1, n_2(y_1)\vee z_2])\\ &= [n_1(x_2)\vee n_2(y_2)\vee z_1, n_1(x_1)\vee n_2(y_1)\vee z_2]\\ &= [n_2(y_2)\vee n_1(x_2)\vee z_1, n_2(y_1)\vee n_1(x_1)\vee z_2]\\ &= \mathbf{I}_2([y_1,y_2],\mathbf{I}_1([x_1,x_2],[z_1,z_2])).\end{aligned}$$

(II6)

$$\begin{aligned}&\mathbf{I}_1(\mathbf{I}_2([x_1,x_2],[\bot,\bot]),[\bot,\bot])\\ &= \mathbf{I}_1([n_2(x_2)\vee\bot, n_2(x_1)\vee\bot],[\bot,\bot])\\ &= [n_1(n_2(x_1))\vee\bot, n_1(n_2(x_2))\vee\bot]\\ &= [x_1,x_2].\end{aligned}$$

Thus $(\mathbf{I}_1,\mathbf{I}_2)$ is a pair of interval implications.

$$\begin{aligned}\mathbf{I}_i([x_1,x_2],[y_1,y_2]) &= [n_i(x_2)\vee y_1, n_i(x_1)\vee y_2]\\ &= [n_i(x_2),n_i(x_1)]\vee[y_1,y_2]\\ &= \mathbf{N}_i([x_1,x_2])\vee[y_1,y_2].\end{aligned}$$

Moreover, $\underline{\mathbf{I}_i}(x,y) = n_i(x)\vee y = \overline{\mathbf{I}_i}(x,y)$ from

$$\begin{aligned}\underline{\mathbf{I}_i}(x,y) &= l(\mathbf{I}_i([x,x],[y,y])) = l([n_i(x)\vee y, n_i(x)\vee y])\\ &= r(\mathbf{I}_i([x,x],[y,y])) = n_i(x)\vee y = \overline{\mathbf{I}_i}(x,y),\end{aligned}$$

$$\begin{aligned}\mathbf{I}_i([x_1,x_2],[y_1,y_2]) &= [n_i(x)\vee y, n_i(x)\vee y]\\ &= [\underline{\mathbf{I}_i}(x_2,y_1),\overline{\mathbf{I}_i}(x_1,y_2)].\end{aligned}$$

Example 3.2. Let $(L,\wedge,\vee,\odot,\to,\Rightarrow,\top,\bot)$ be a complete generalized residuated lattice with the law of double negation defined as $a = n_1(n_2(a)) = n_2(n_1(a))$ where $n_1(a) = a\Rightarrow\bot$ and $n_2(a) = a\to\bot$ (ref. [5,6]).

(1) A pair (n_1,n_2) is a pair of negations.

(2) By Theorem 3.1, (I_1,I_2) is a pair of implications such that

$$I_1(x,y) = n_1(x)\vee y = (x\Rightarrow\bot)\vee y,$$

$$I_2(x,y) = n_2(x)\vee y = (x\to\bot)\vee y.$$

(3) Define maps $\mathbf{N}_i : L^{[2]}\to L^{[2]}$ as

$$\mathbf{N}_1([x_1,x_2]) = [x_2\Rightarrow\bot, x_1\Rightarrow\bot], \mathbf{N}_2([x_1,x_2]) = [x_2\to\bot, x_1\to\bot].$$

By Theorem 3.1, $(\mathbf{N}_1,\mathbf{N}_2)$ is a pair of interval negations such that

$$\underline{\mathbf{N}_1}(x) = \overline{\mathbf{N}_1}(x) = n_1(x) = x\Rightarrow\bot,$$

$$\underline{\mathbf{N}_2}(x) = \overline{\mathbf{N}_2}(x) = n_2(x) = x\to\bot.$$

(4) For maps I_i in (2), we define maps $\mathbf{I}_i : L^{[2]}\times L^{[2]}\to L^{[2]}$ as

$$\mathbf{I}_1([x_1,x_2],[y_1,y_2]) = [(x_2\Rightarrow\bot)\vee y_1, (x_1\Rightarrow\bot)\vee y_2],$$

$$\mathbf{I}_2([x_1,x_2],[y_1,y_2]) = [(x_2\to\bot)\vee y_1, (x_1\to\bot)\vee y_2].$$

By Theorem 3.1, $(\mathbf{I}_1,\mathbf{I}_2)$ is a pair of interval implications such that

$$\underline{\mathbf{I}_1}(x,y)=(x\Rightarrow\bot)\vee y=\overline{\mathbf{I}_1}(x,y),$$

$$\underline{\mathbf{I}_2}(x,y)=(x\rightarrow\bot)\vee y=\overline{\mathbf{I}_2}(x,y).$$

Example 3.3. Put $L=\{(x,y)\in R^2\mid(\frac{1}{2},1)\leq(x,y)\leq(1,0)\}$ with a bottom element $(\frac{1}{2},1)$ and a top element $(1,0)$ where

$$(x_1,y_1)\leq(x_2,y_2)\Leftrightarrow x_1<x_2 \text{ or } x_1=x_2,y_1\leq y_2.$$

Put $n_1(x,y)=(\frac{1}{2x},\frac{1-y}{x})$, $n_2(x,y)=(\frac{1}{2x},1-\frac{y}{2x})$. Then (n_1,n_2) is a pair of negations from:

$$n_1(n_2(x,y))=(x,y),\ n_2(n_1(x,y))=(x,y).$$

From Theorem 3.1, we obtain a pair of implications (I_1,I_2) as follows:

$$\begin{aligned}I_1((x_1,y_1),(x_2,y_2)) &= n_1(x_1,y_1)\vee(x_2,y_2)\\ &=(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1})\vee(x_2,y_2)\\ I_2((x_1,y_1),(x_2,y_2)) &= n_2(x_1,y_1)\vee(x_2,y_2)\\ &=(\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1})\vee(x_2,y_2).\end{aligned}$$

From Theorem 3.1, a pair of interval negations $(\mathbf{N}_1,\mathbf{N}_2)$ is defined $\mathbf{N}_i:L^{[2]}\to L^{[2]}$ as

$$\begin{aligned}\mathbf{N}_1([(x_1,y_1),(x_2,y_2)]) &= [n_1(x_2,y_2),n_1(x_1,y_1)]\\ &=[(\tfrac{1}{2x_2},\tfrac{1-y_2}{x_2}),(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1})],\\ \mathbf{N}_2([(x_1,y_1),(x_2,y_2)]) &= [n_2(x_2,y_2),n_2(x_1,y_1)]\\ &=[(\tfrac{1}{2x_2},1-\tfrac{y_2}{2x_2}),(\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1})].\end{aligned}$$

From Theorem 3.1, a pair of interval implications $(\mathbf{I}_1,\mathbf{I}_2)$ is defined $\mathbf{I}_i:L^{[2]}\times L^{[2]}\to L^{[2]}$ as

$$\begin{aligned}&\mathbf{I}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\ &=[n_1(x_2,y_2)\vee(z_1,w_1),n_1(x_1,y_1)\vee(z_2,w_2)]\\ &=[(\tfrac{1}{2x_2},\tfrac{1-y_2}{x_2})\vee(z_1,w_1),(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1})\vee(z_2,w_2)]\\ &=\mathbf{N}_1([(x_1,y_1),(x_2,y_2)])\vee[(z_1,w_1),(z_2,w_2)].\end{aligned}$$

$$\begin{aligned}
&\mathbf{I}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)]) \\
&= [n_2(x_2,y_2)\vee(z_1,w_1), n_2(x_1,y_1)\vee(z_2,w_2)] \\
&= [(\tfrac{1}{2x_2}, 1-\tfrac{y_2}{2x_2})\vee(z_1,w_1), (\tfrac{1}{2x_1}, 1-\tfrac{y_1}{2x_1})\vee(z_2,w_2)] \\
&= \mathbf{N}_2([(x_1,y_1),(x_2,y_2)])\vee[(z_1,w_1),(z_2,w_2)].
\end{aligned}$$

Since $\mathbf{I}_1([(x,y),(x,y)],[(z,w),(z,w)]) = [(\frac{1}{2x},\frac{1-y}{x})\vee(z,w),(\frac{1}{2x},\frac{1-y}{x})\vee(z,w)]$, it satisfies the condition of Theorem 2.6(4). Thus $(\underline{\mathbf{I}_1},\underline{\mathbf{I}_2})$ is a pair of implications such that

$$\begin{aligned}
&\underline{\mathbf{I}_1}((x,y),(z,w)) = l(\mathbf{I}_1([(x,y),(x,y)],[(z,w),(z,w)]) \\
&= l([(\tfrac{1}{2x},\tfrac{1-y}{x})\vee(z,w),(\tfrac{1}{2x},\tfrac{1-y}{x})\vee(z,w)]) \\
&= r([(\tfrac{1}{2x},\tfrac{1-y}{x})\vee(z,w),(\tfrac{1}{2x},\tfrac{1-y}{x})\vee(z,w)]) \\
&= (\tfrac{1}{2x},\tfrac{1-y}{x})\vee(z,w) = \overline{\mathbf{I}_1}((x,y),(z,w)).
\end{aligned}$$

$$\begin{aligned}
&\underline{\mathbf{I}_2}((x,y),(z,w)) = l(\mathbf{I}_2([(x,y),(x,y)],[(z,w),(z,w)]) \\
&= l([(\tfrac{1}{2x},1-\tfrac{y}{2x})\vee(z,w),(\tfrac{1}{2x},1-\tfrac{y}{2x})\vee(z,w)]) \\
&= r([(\tfrac{1}{2x},1-\tfrac{y}{2x})\vee(z,w),(\tfrac{1}{2x},1-\tfrac{y}{2x})\vee(z,w)]) \\
&= (\tfrac{1}{2x},1-\tfrac{y}{2x})\vee(z,w) = \underline{\mathbf{I}_2}((x,y),(z,w)).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathbf{I}_i([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)]) \\
&= [\underline{\mathbf{I}_i}((x_2,y_2),(z_1,w_1)),\overline{\mathbf{I}_i}((x_1,y_1),(z_2,w_2))].
\end{aligned}$$

Theorem 3.4. *Let $(L,\vee,\wedge,\top,\bot)$ be a bounded lattice and (I_1,I_2) an pair of implications on L. We define*

$$n_1(x) = I_1(x,\bot),\ n_2(x) = I_2(x,\bot).$$

(1) *(n_1,n_2) is a pair of negations.*

(2) *$I_1(n_2(y),n_2(x)) = I_2(x,y)$ and $I_2(n_1(y),n_1(x)) = I_1(x,y)$.*

(3)*If $y \le z$, then $I_i(x,y) \le I_i(x,z)$.*

(4) *For maps I_i in (1), we define maps $\mathbf{I}_i : L^{[2]}\times L^{[2]} \to L^{[2]}$ as*

$$\mathbf{I}_1([x_1,x_2],[y_1,y_2]) = [I_1(x_2,y_1),I_1(x_1,y_2)],$$

$$\mathbf{I}_2([x_1,x_2],[y_1,y_2]) = [I_2(x_2,y_1),I_2(x_1,y_2)].$$

Then $(\mathbf{I}_1,\mathbf{I}_2)$ *is a pair of interval implications such that* $\underline{\mathbf{I}_i}(x,y)=I_i(x,y)=\overline{\mathbf{I}_i}(x,y)$ *and*

$$\mathbf{I}_i([x_1,x_2],[y_1,y_2])=[\underline{\mathbf{I}_i}(x_2,y_1),\overline{\mathbf{I}_i}(x_1,y_2)].$$

(5) *Define maps* $\mathbf{N}_i : L^{[2]}\to L^{[2]}$ *as*

$$\mathbf{N}_1([x_1,x_2])=[I_1(x_2,\perp),I_1(x_1,\perp)],$$

$$\mathbf{N}_2([x_1,x_2])=[I_2(x_2,\perp),I_2(x_1,\perp)].$$

Then $(\mathbf{N}_1,\mathbf{N}_2)$ *is a pair of interval negations such that*

$$\underline{\mathbf{N}_i}(x)=\overline{\mathbf{N}_i}(x)=I_i(x,\perp),$$

$$\mathbf{N}_i([x_1,x_2])=[\underline{\mathbf{N}_i}(x_2),\overline{\mathbf{N}_i}(x_1)].$$

Proof. (1) (N1) By (I1), $n_i(\perp)=I_1(\perp,\perp)=\top$ and $n_i(\top)=I_i(\top,\perp)=\perp$.

(N2) If $x\le y$, by (I2), $n_i(x)=I_i(x,\perp)\ge I_i(y,\perp)=n_i(y)$.

(N3) $n_1(n_2(x))=I_1(I_2(x,\perp),\perp)=x=I_2(I_1(x,\perp),\perp)=n_2(n_1(x))$.

(2)

$$\begin{aligned} I_1(n_2(y),n_2(x)) &= I_1(I_2(y,\perp),I_2(x,\perp)) \\ &= I_2(x,I_1(I_2(y,\perp),\perp)) \text{ (by (I3))} \\ &= I_2(x,y) \end{aligned}$$

Similarly, $I_2(n_1(y),n_1(x))=I_1(x,y)$.

(3) If $y\le z$, then $n_1(z)\le n_1(y)$ and $n_2(z)\le n_2(y)$.

$$\begin{aligned} I_1(x,y) &= I_2(n_1(y),n_1(x))\le I_2(n_1(z),n_1(x))=I_1(x,z). \\ I_2(x,y) &= I_1(n_2(y),n_2(x))\le I_1(n_2(z),n_2(x))=I_2(x,z). \end{aligned}$$

(4) (II1)

$$\begin{aligned} &\mathbf{I}_i([\top,\top],[\perp,\perp])=[I_i(\top,\perp),I_i(\top,\perp)]=[\perp,\perp], \\ &\mathbf{I}_i([\perp,\perp],[\top,\top])=[I_i(\perp,\top),I_i(\perp,\top)]=[\top,\top], \\ &\mathbf{I}_i([\perp,\perp],[\perp,\perp])=[\top,\top]=\mathbf{I}_i([\top,\top],[\top,\top]). \end{aligned}$$

(II2) If $[x_1,x_2]\le[y_1,y_2]$, then $x_1\le y_1$ and $x_2\le y_2$. For $i\in\{1,2\}$,

$$\begin{aligned} &\mathbf{I}_i([x_1,x_2],[z_1,z_2])=[I_i(x_2,z_1),I_i(x_1,z_2)] \\ &\ge[I_i(y_2,z_1),I_i(y_1,z_2)]=\mathbf{I}_i([y_1,y_2],[z_1,z_2]). \end{aligned}$$

(II3) If $[x_1,x_2] \subset [z_1,z_2]$, then $z_1 \le x_1 \le x_2 \le z_2$. So, $I_i(z_2,y_1) \le I_i(x_2,y_1)$ and $I_i(x_1,y_2) \le I_i(z_1,y_2)$ for $i \in \{1,2\}$. Hence

$$\begin{aligned}\mathbf{I}_i([x_1,x_2],[y_1,y_2]) &= [I_i(x_2,y_1),I_i(x_1,y_2)]\\ &\subset [I_i(z_2,y_1),I_i(z_1,y_2)] = \mathbf{I}_i([z_1,z_2],[y_1,y_2]).\end{aligned}$$

(II4)

$$\mathbf{I}_i([\top,\top],[z_1,z_2]) = [I_i(\top,z_1),I_i(\top,z_2)] = [z_1,z_2].$$

(II5)

$$\begin{aligned}&\mathbf{I}_1([x_1,x_2],\mathbf{I}_2([y_1,y_2],[z_1,z_2]))\\ &= \mathbf{I}_1([x_1,x_2],[I_2(y_2,z_1),I_2(y_1,z_2)])\\ &= [I_1(x_2,I_2(y_2,z_1)),I_1(x_1,I_2(y_1,z_2))]\\ &= [I_2(y_2,I_1(x_2,z_1)),I_2(y_1,I_1(x_1,z_2))]\\ &= \mathbf{I}_2([y_1,y_2],\mathbf{I}_1([x_1,x_2],[z_1,z_2])).\end{aligned}$$

(II6)

$$\begin{aligned}&\mathbf{I}_1(\mathbf{I}_2([x_1,x_2],[\bot,\bot]),[\bot,\bot])\\ &= \mathbf{I}_1([I_2(x_2,\bot),I_2(x_1,\bot)],[\bot,\bot])\\ &= [I_1(I_2(x_1,\bot),\bot),I_1(I_2(x_2,\bot),\bot)]\\ &= [x_1,x_2].\end{aligned}$$

Hence $(\mathbf{I}_1,\mathbf{I}_2)$ is a pair of interval implications. Moreover, $\underline{\mathbf{I}_i}(x,y) = I_i(x,y) = \overline{\mathbf{I}}_i(x,y)$ from

$$\begin{aligned}\underline{\mathbf{I}_i}(x,y) &= l(\mathbf{I}_i([x,x],[y,y])) = l([I_i(x,y),I_i(x,y)])\\ &= r(\mathbf{I}_i([x,x],[y,y])) = I_i(x,y) = \overline{\mathbf{I}}_i(x,y),\end{aligned}$$

$$\begin{aligned}\mathbf{I}_i([x_1,x_2],[y_1,y_2]) &= [I_i(x_2,y_1),I_i(x_1,y_2)]\\ &= [\underline{\mathbf{I}_i}(x_2,y_1),\overline{\mathbf{I}}_i(x_1,y_2)].\end{aligned}$$

(5) (IN1)

$$\mathbf{N}_i([\bot,\bot]) = [I_i(\bot,\bot),I_i(\bot,\bot)] = [\top,\top],$$

$$\mathbf{N}_i([\top,\top]) = [I_i(\top,\bot),I_i(\top,\bot)] = [\bot,\bot].$$

(IN2) If $[x_1,x_2] \le [y_1,y_2]$, then $x_1 \le y_1$ and $x_2 \le y_2$. So, $I_i(x_1,\bot) \ge I_i(y_1,\bot)$ and $I_i(x_2,\bot) \ge I_i(y_2,\bot)$. Thus, for all $i \in \{1,2\}$,

$$\mathbf{N}_i([x_1,x_2]) = [I_i(x_2,\bot),I_i(x_1,\bot)] \ge [I_i(y_2,\bot),I_i(y_1,\bot)] = \mathbf{N}_i([y_1,y_2]).$$

(IN3) If $[x_1,x_2] \subset [y_1,y_2]$, then $y_1 \le x_1 \le x_2 \le y_2$. Since $I_i(y_2,\perp) \le I_i(x_2,\perp) \le I_i(x_1,\perp) \le I_i(y_1,\perp)$ for all $i \in \{1,2\}$, then

$$\mathbf{N}_i([x_1,x_2]) = [I_i(x_2,\perp), I_i(x_1,\perp)] \subset [I_i(y_2,\perp), I_i(y_1,\perp)] = \mathbf{N}_i([y_1,y_2]).$$

(IN4) $\mathbf{N}_1(\mathbf{N}_2([x_1,x_2])) = \mathbf{N}_2(\mathbf{N}_1([x_1,x_2])) = [x_1,x_2]$ for all $[x_1,x_2] \in L^{[2]}$.

$$\begin{aligned} \mathbf{N}_1(\mathbf{N}_2([x_1,x_2])) &= \mathbf{N}_1([I_2(x_2,\perp), I_2(x_1,\perp)]) \\ &= [I_1(I_2(x_1,\perp),\perp), I_1(I_2(x_2,\perp),\perp)] \\ &= [x_1,x_2]. \end{aligned}$$

$$\begin{aligned} \underline{\mathbf{N}_i}(x) &= l(\mathbf{N}_i([x,x]) = l([I_i(x,\perp), I_i(x,\perp)]) \\ &= r([I_i(x,\perp), I_i(x,\perp)]) = \overline{\mathbf{N}_i}(x) = I_i(x,\perp). \end{aligned}$$

$$\mathbf{N}_i([x_1,x_2]) = [I_i(x_2,\perp), I_i(x_1,\perp)] = [\underline{\mathbf{N}_i}(x_2), \overline{\mathbf{N}_i}(x_1)].$$

Example 3.5. Let $(L,\wedge,\vee,\odot,\rightarrow,\Rightarrow,\top,\perp)$, n_1 and n_2 be given in Example 3.2. We define

$$I_1(a,b) = a \Rightarrow b, \ \ I_1(a,b) = a \rightarrow b.$$

(1) A pair (I_1,I_2) is a pair of implications because $a \rightarrow (b \Rightarrow c) = b \Rightarrow (a \rightarrow c)$.

(2) A pair (n_1,n_2) is a pair of negations.(ref. [5,6]).

(3) Define maps $\mathbf{I}_i : L^{[2]} \times L^{[2]} \rightarrow L^{[2]}$ as

$$\mathbf{I}_1([x_1,x_2],[y_1,y_2]) = [I_1(x_2,y_1), I_1(x_1,y_2)] = [x_2 \Rightarrow y_1, x_1 \Rightarrow y_2],$$

$$\mathbf{I}_2([x_1,x_2],[y_1,y_2]) = [x_2 \rightarrow y_1, x_1 \rightarrow y_2].$$

Then $(\mathbf{I}_1,\mathbf{I}_2)$ is a pair of interval implications such that $\underline{\mathbf{I}_i}(x,y) = I_i(x,y) = \overline{\mathbf{I}_i}(x,y)$ and

$$\mathbf{I}_i([x_1,x_2],[y_1,y_2]) = [\underline{\mathbf{I}_i}(x_2,y_1), \overline{\mathbf{I}_i}(x_1,y_2)].$$

(4) Define maps $\mathbf{N}_i : L^{[2]} \rightarrow L^{[2]}$ as

$$\mathbf{N}_1([x_1,x_2]) = [I_1(x_2,\perp), I_1(x_1,\perp)],$$

$$\mathbf{N}_2([x_1,x_2]) = [I_2(x_2,\perp), I_2(x_1,\perp)].$$

Then $(\mathbf{N}_1, \mathbf{N}_2)$ is a pair of interval negations such that

$$\underline{\mathbf{N}_1}(x) = \overline{\mathbf{N}_1}(x) = x \Rightarrow \bot,$$

$$\underline{\mathbf{N}_2}(x) = \overline{\mathbf{N}_2}(x) = x \to \bot.$$

Example 3.6. Put $L = \{(x,y) \in R^2 \mid (\frac{1}{2}, 1) \leq (x,y) \leq (1,0)\}$ with a bottom element $(\frac{1}{2}, 1)$ and a top element $(1,0)$ where

$$(x_1,y_1) \leq (x_2,y_2) \Leftrightarrow x_1 < x_2 \text{ or } x_1 = x_2, y_1 \leq y_2.$$

(1) Define $I_1, I_2 : L \times L \to L$ as follows:

$$\begin{aligned} I_1((x_1,y_1),(x_2,y_2)) &= (\tfrac{x_2}{x_1}, \tfrac{y_2-y_1}{x_1}) \wedge (1,0) \\ I_2((x_1,y_1),(x_2,y_2)) &= (\tfrac{x_2}{x_1}, y_2 - \tfrac{x_2 y_1}{x_1}) \wedge (1,0). \end{aligned}$$

Then it satisfies (I1)-(I3) and (I4) from:

$$\begin{aligned} I_1((x_1,y_1), I_2((x_2,y_2),(x_3,y_3))) &= I_1((x_1,y_1), (\tfrac{x_3}{x_2}, y_3 - \tfrac{x_3 y_2}{x_2}) \wedge (1,0)) \\ &= (\tfrac{x_3}{x_1 x_2}, \tfrac{x_2 y_3 - x_3 y_2 - x_2 y_1}{x_1 x_2}) \wedge (1,0) \\ I_2((x_2,y_2), I_1((x_1,y_1),(x_3,y_3))) &= I_2((x_2,y_2), (\tfrac{x_3}{x_1}, \tfrac{y_3-y_1}{x_1}) \wedge (1,0)) \\ &= (\tfrac{x_3}{x_1 x_2}, \tfrac{x_2 y_3 - x_3 y_2 - x_2 y_1}{x_1 x_2}) \wedge (1,0) \end{aligned}$$

(I5) $I_2(I_1((x_1,y_1),(\frac{1}{2},1)),(\frac{1}{2},1)) = (x_1,y_1) = I_1(I_2((x_1,y_1),(\frac{1}{2},1)),(\frac{1}{2},1))$ from

$$\begin{aligned} I_1((x_1,y_1),(\tfrac{1}{2},1)) &= (\tfrac{1}{2x_1}, \tfrac{1-y_1}{x_1}) = n_1(x_1,y_1) \\ I_2((x_1,y_1),(\tfrac{1}{2},1)) &= (\tfrac{1}{2x_1}, 1 - \tfrac{y_1}{2x_1}) = n_2(x_1,y_1) \end{aligned}$$

Hence (I_1, I_2) is a pair of implications. Moreover, (n_1, n_2) is a pair of implications. By Theorem 3.4 (4), we obtain a pair $(\mathbf{I}_1, \mathbf{I}_2)$ of interval implications defined as $\mathbf{I}_1, \mathbf{I}_2 : L^{[2]} \times L^{[2]} \to L^{[2]}$ as follows:

$$\begin{aligned} &\mathbf{I}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)]) \\ &= [I_1((x_2,y_2),(z_1,w_1), I_1((x_1,y_1),(z_2,w_2))] \\ &= [(\tfrac{z_1}{x_2}, \tfrac{w_1-y_2}{x_2}) \wedge (1,0), (\tfrac{z_2}{x_1}, \tfrac{w_2-y_1}{x_1}) \wedge (1,0)] \\ &\mathbf{I}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)]) \\ &= [I_2((x_2,y_2),(z_1,w_1), I_2((x_1,y_1),(z_2,w_2))] \\ &= [(\tfrac{z_1}{x_2}, w_1 - \tfrac{z_1 y_2}{x_2}) \wedge (1,0), (\tfrac{z_2}{x_1}, w_2 - \tfrac{z_2 y_1}{x_1}) \wedge (1,0)] \end{aligned}$$

Since $\mathbf{I}_1([(x,y),(x,y)],[(z,w),(z,w)]) = [(\frac{1}{2x},\frac{1-y}{x})\vee(z,w),(\frac{1}{2x},\frac{1-y}{x})\vee(z,w)]$, it satisfies the condition of Theorem 2.6(4). Thus $(\underline{\mathbf{I}_1},\underline{\mathbf{I}_2})$ is a pair of implications such that

$$\begin{aligned}
&\underline{\mathbf{I}_1}((x,y),(z,w)) = l(\mathbf{I}_1([(x,y),(x,y)],[(z,w),(z,w)])\\
&= l([(\tfrac{z}{x},\tfrac{w-y}{x})\wedge(1,0),(\tfrac{z}{x},\tfrac{w-y}{x})\wedge(1,0)])\\
&= r([(\tfrac{z}{x},\tfrac{w-y}{x})\wedge(1,0),(\tfrac{z}{x},\tfrac{w-y}{x})\wedge(1,0)])\\
&= (\tfrac{z}{x},\tfrac{w-y}{x})\wedge(1,0) = \overline{\mathbf{I}_1}((x,y),(z,w)).
\end{aligned}$$

$$\begin{aligned}
&\underline{\mathbf{I}_2}((x,y),(z,w)) = l(\mathbf{I}_2([(x,y),(x,y)],[(z,w),(z,w)])\\
&= l([(\tfrac{z}{x},w-\tfrac{zy}{x})\wedge(1,0),(\tfrac{z}{x},w-\tfrac{zy}{x})\wedge(1,0)])\\
&= r([(\tfrac{z}{x},w-\tfrac{zy}{x})\wedge(1,0),(\tfrac{z}{x},w-\tfrac{zy}{x})\wedge(1,0)])\\
&= (\tfrac{z}{x},w-\tfrac{zy}{x})\wedge(1,0) = \underline{\mathbf{I}_2}((x,y),(z,w)).
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\mathbf{I}_i([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)])\\
&= [\underline{\mathbf{I}_i}((x_2,y_2),(z_1,w_1)),\overline{\mathbf{I}_i}((x_1,y_1),(z_2,w_2))].
\end{aligned}$$

$\mathbf{N}_1,\mathbf{N}_2 : L^{[2]} \to L^{[2]}$ as follows:

$$\begin{aligned}
\mathbf{N}_1([(x_1,y_1),(x_2,y_2)]) &= [I_1((x_2,y_2),(\tfrac{1}{2},1)),I_1((x_1,y_1),(\tfrac{1}{2},1))]\\
&= [(\tfrac{1}{2x_2},\tfrac{1-y_2}{x_2}),(\tfrac{1}{2x_1},\tfrac{1-y_1}{x_1})]\\
\mathbf{N}_2([(x_1,y_1),(x_2,y_2)]) &= [I_2((x_2,y_2),(\tfrac{1}{2},1)),I_2((x_1,y_1),(\tfrac{1}{2},1))]\\
&= [(\tfrac{1}{2x_2},1-\tfrac{y_2}{2x_2}),(\tfrac{1}{2x_1},1-\tfrac{y_1}{2x_1})].
\end{aligned}$$

(2) Define $I_1,I_2 : L\times L\to L$ as follows:

$$\begin{aligned}
I_1((x_1,y_1),(x_2,y_2)) &= (\tfrac{x_2}{x_1},y_2-2x_2+\tfrac{2x_2-2x_2y_1}{x_1})\wedge(1,0)\\
I_2((x_1,y_1),(x_2,y_2)) &= (\tfrac{x_2}{x_1},1-\tfrac{y_1+2-2y_2}{2x_1})\wedge(1,0).
\end{aligned}$$

Then it satisfies (I1)-(I4) and (I5) from:

$$\begin{aligned}
&I_1((x_1,y_1),I_2((x_2,y_2),(x_3,y_3))) = I_1((x_1,y_1),(\tfrac{x_3}{x_2},1-\tfrac{y_2+2-2y_3}{2x_2})\wedge(1,0))\\
&= (\tfrac{x_3}{x_1x_2},\tfrac{2x_1x_2-x_1y_2-2x_1+2x_1y_3-4x_3x_1+4x_3-4x_3y_1}{2x_1x_2})\wedge(1,0)\\
&I_2((x_2,y_2),I_1((x_1,y_1),(x_3,y_3))) = I_2((x_2,y_2),(\tfrac{x_3}{x_1},y_3-2x_3+\tfrac{2x_3-2x_3y_1}{x_1})\wedge(1,0))\\
&= (\tfrac{x_3}{x_1x_2},\tfrac{2x_1x_2-x_1y_2-2x_1+2x_1y_3-4x_3x_1+4x_3-4x_3y_1}{2x_1x_2})\wedge(1,0)
\end{aligned}$$

(I5) $I_2(I_1((x_1,y_1),(\frac{1}{2},1)),(\frac{1}{2},1)) = (x_1,y_1) = I_1(I_2((x_1,y_1),(\frac{1}{2},1)),(\frac{1}{2},1))$ from

$$\begin{aligned} I_1((x_1,y_1),(\tfrac{1}{2},1)) &= (\tfrac{1}{2x_1}, \tfrac{1-y_1}{x_1}) = n_1(x_1,y_1) \\ I_2((x_1,y_1),(\tfrac{1}{2},1)) &= (\tfrac{1}{2x_1}, 1-\tfrac{y_1}{2x_1}) = n_2(x_1,y_1) \end{aligned}$$

Hence (I_1,I_2) is a pair of implications and (n_1,n_2) is a pair of negations. By Theorem 3.4 (4), we obtain: $\mathbf{I}_1,\mathbf{I}_2 : L^{[2]} \times L^{[2]} \to L^{[2]}$ as follows:

$$\begin{aligned} &\mathbf{I}_1([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)]) \\ &= [I_1((x_2,y_2),(z_1,w_1),I_1((x_1,y_1),(z_2,w_2))] \\ &= [(\tfrac{z_1}{x_2}, w_1-2z_1+\tfrac{2z_1-2z_1y_2}{x_2}) \wedge (1,0), (\tfrac{x_2}{z_1}, y_2-2x_2+\tfrac{2x_2-2x_2w_1}{z_1}) \wedge (1,0)] \\ &\mathbf{I}_2([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)]) \\ &= [I_2((x_2,y_2),(z_1,w_1),I_2((x_1,y_1),(z_2,w_2))] \\ &= [(\tfrac{z_1}{x_2}, 1-\tfrac{w_1+2-2y_2}{2z_1}) \wedge (1,0), (\tfrac{z_2}{x_1}, 1-\tfrac{y_1+2-2w_2}{2x_1}) \wedge (1,0)] \end{aligned}$$

Since $\mathbf{I}_1([(x,y),(x,y)],[(z,w),(z,w)]) = [(\frac{z}{x}, w-2z+\frac{2z-2zy}{x}) \wedge (1,0), (\frac{z}{x}, w-2z+\frac{2z-2zy}{x}) \wedge (1,0)]$ and $\mathbf{I}_2([(x,y),(x,y)],[(z,w),(z,w)]) = [(\frac{z}{x}, 1-\frac{w+2-2y}{2z}) \wedge (1,0), (\frac{z}{x}, 1-\frac{w+2-2y}{2z}) \wedge (1,0)]$, $\mathbf{I}_1$ and $\mathbf{I}_2$ satisfy the condition of Theorem 2.6(4). Thus $(\underline{\mathbf{I}_1},\underline{\mathbf{I}_2})$ is a pair of implications such that

$$\begin{aligned} &\underline{\mathbf{I}_1}((x,y),(z,w)) = l(\mathbf{I}_1([(x,y),(x,y)],[(z,w),(z,w)]) \\ &= l([(\tfrac{z}{x}, w-2z+\tfrac{2z-2zy}{x}) \wedge (1,0), (\tfrac{z}{x}, w-2z+\tfrac{2z-2zy}{x}) \wedge (1,0)]) \\ &= r([(\tfrac{z}{x}, w-2z+\tfrac{2z-2zy}{x}) \wedge (1,0), (\tfrac{z}{x}, w-2z+\tfrac{2z-2zy}{x}) \wedge (1,0)]) \\ &= (\tfrac{z}{x}, w-2z+\tfrac{2z-2zy}{x}) \wedge (1,0) = \overline{\mathbf{I}_1}((x,y),(z,w)). \end{aligned}$$

$$\begin{aligned} &\underline{\mathbf{I}_2}((x,y),(z,w)) = l(\mathbf{I}_2([(x,y),(x,y)],[(z,w),(z,w)]) \\ &= l([(\tfrac{z}{x}, 1-\tfrac{w+2-2y}{2z}) \wedge (1,0), (\tfrac{z}{x}, 1-\tfrac{w+2-2y}{2z}) \wedge (1,0)]) \\ &= r([(\tfrac{z}{x}, 1-\tfrac{w+2-2y}{2z}) \wedge (1,0), (\tfrac{z}{x}, 1-\tfrac{w+2-2y}{2z}) \wedge (1,0)]) \\ &= (\tfrac{z}{x}, 1-\tfrac{w+2-2y}{2z}) \wedge (1,0) = \underline{\mathbf{I}_2}((x,y),(z,w)). \end{aligned}$$

Moreover,

$$\begin{aligned} &\mathbf{I}_i([(x_1,y_1),(x_2,y_2)],[(z_1,w_1),(z_2,w_2)]) \\ &= [\underline{\mathbf{I}_i}((x_2,y_2),(z_1,w_1)), \overline{\mathbf{I}_i}((x_1,y_1),(z_2,w_2))]. \end{aligned}$$

$\mathbf{N}_1, \mathbf{N}_2 : L^{[2]} \to L^{[2]}$ as follows:

$$\begin{aligned}
\mathbf{N}_1([(x_1,y_1),(x_2,y_2)]) &= [I_1((x_2,y_2),(\tfrac{1}{2},1)), I_1((x_1,y_1),(\tfrac{1}{2},1))] \\
&= [(\tfrac{1}{2x_2}, \tfrac{1-y_2}{x_2}), (\tfrac{1}{2x_1}, \tfrac{1-y_1}{x_1})] \\
\mathbf{N}_2([(x_1,y_1),(x_2,y_2)]) &= [I_2((x_2,y_2),(\tfrac{1}{2},1)), I_2((x_1,y_1),(\tfrac{1}{2},1))] \\
&= [(\tfrac{1}{2x_2}, 1-\tfrac{y_2}{2x_2}), (\tfrac{1}{2x_1}, 1-\tfrac{y_1}{2x_1})].
\end{aligned}$$

Conflict of Interests

The author declares that there is no conflict of interests.

References

[1] B.C. Bedregal, On interval fuzzy negations, Fuzzy Sets and Systems, 161 (2010), 2290-2313.

[2] B.C. Bedregal, On interval fuzzy S-implications, Information Sciences, 180 (2010), 1373-1389.

[3] B.C. Bedregal, R.H.N. Santiago, Interval representations, Lukasiewicz implicators and Smetz-Magrez axioms, Inform. Sci. 221 (2013), 192-200.

[4] B.C. Bedregal, A. Takahashi, The best interval representations of t-norms and automorphisms, Fuzzy Sets and Systems, 161 (2006), 3220-3230.

[5] P. Flonder, G. Georgescu, A. lorgulescu, Pseudo t-norms and pseudo-BL algebras, Soft Comput. 5 (2001), 355-371.

[6] G. Georgescu, A. Popescue, Non-commutative Galois connections, Soft Comput. 7 (2003), 458-467.

[7] G. Georgescu, A. Popescue, Non-commutative fuzzy structures and pairs of weak negations, Fuzzy Sets and Systems, 143 (2004), 129-155.

[8] G. Georgescu, A. Popescue, Non-dual fuzzy connections, Arch. Math. Log. 43 (2004), 1009-1039.

[9] U. Höhle, E. P. Klement, Non-classical logic and their applications to fuzzy subsets, Kluwer Academic Publisher, Boston, 1995.

[10] D. Li, Y. Li, Algebraic structures of interval-valued fuzzy (S,N)-implications, Int. J. Approx. Reason. 53 (2012), 892-900.

[11] Y.C. Kim, Pairs of interval negations and interval implications, Int. J. Pure Appl. Math. 88 (2013), 305-319.

[12] E. Turunen, Mathematics Behind Fuzzy Logic, A Springer-Verlag Co., 1999.