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ASYMPTOTIC BEHAVIOR IN NEUTRAL DIFFERENCE EQUATIONS WITH NEGATIVE COEFFICIENTS AND SEVERAL DELAY ARGUMENTS

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Abstract. In this paper, we study the asymptotic behavior of the solutions of a neutral type difference equation of the form

$$\Delta\left[x(n)+\sum_{j=1}^{w} q_j(n)x(\tau_j(n))\right]-p(n)x(\sigma(n))=0, \qquad n\geq 0$$

where $(-p(n))_{n\geq 0}$ is a sequence of negative real numbers such that $p(n)\geq p$, $p\in\mathbb{R}_+$, $\tau_j(n)$, $j=1,...,w$ are general retarded arguments, $\sigma(n)$ is a general deviated argument, $(q_j(n))_{n\geq 0}$, $j=1,...,w$ are sequences of real numbers, and Δ denotes the forward difference operator $\Delta x(n)=x(n+1)-x(n)$.

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1. Introduction

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A neutral difference equation is a difference equation in which the higher order difference of the unknown sequence appears in the equation both with and without delays or advances. See, for example, [1−3, 9] and the references cited therein. We should note that, the theory of neutral difference equations presents complexities, and results which are true for non-neutral difference equations may not be true for neutral equations [16].

The study of the asymptotic and oscillatory behavior of the solutions of neutral difference equations has a strong theoretical interest. Moreover, results on those equations can be applied in several disciplines/fields of science and mathematics, including circuit theory, bifurcation analysis, population dynamics, stability theory, the dynamics of delayed network systems and others. As a result of the wide range of applications, neutral difference equations have attracted a great interest in the literature.

Consider the neutral difference equation in which the difference of the unknown sequence appears in the equation both with and without more than one delays

$$\Delta\left[x(n)+\sum_{j=1}^{w} q_j(n)x(\tau_j(n))\right]-p(n)x(\sigma(n))=0, \quad n\geq 0, \tag{E}$$

where $(-p(n))_{n\geq 0}$ is a sequence of negative real numbers such that $p(n)\geq p$, $p\in\mathbb{R}_+$, $(q_j(n))_{n\geq 0}$, $j=1,...,w$ are sequences of real numbers, $(\tau_j(n))_{n\geq 0}$, $j=1,...,w$ are increasing sequences of integers that satisfy

$$\begin{gathered}\tau_j(n)\leq n-1,\ j=1,...,w \qquad \forall n\geq 0, \qquad \lim_{n\to\infty}\tau_j(n)=+\infty\\ \text{and}\\ \tau_\ell(n)<\tau_m(n+1), \qquad \forall \ell,m\in[1,w]\cap\mathbb{N}\end{gathered} \tag{1.1}$$

and $(\sigma(n))_{n\geq 0}$ is an increasing sequence of integers such that

$$\begin{gathered}\sigma(n)\leq n-1 \qquad \forall n\geq 0, \qquad \lim_{n\to\infty}\sigma(n)=+\infty,\\ \text{or}\\ \sigma(n)\geq n+1 \qquad \forall n\geq 0.\end{gathered} \tag{1.2}$$

Define

$$k_1=-\min_{\substack{n\geq 0\\ 1\leq j\leq w}}\tau_j(n)\ , \qquad k_2=-\min_{n\geq 0}\sigma(n)$$

and

$$k = \max\{k_1, k_2\}.$$

(Clearly, k is a positive integer.)

By a *solution* of the neutral difference equation (E), we mean a sequence of real numbers $(x(n))_{n\geq -k}$ which satisfies (E) for all $n \geq 0$. It is clear that, for each choice of real numbers c_{-k}, c_{-k+1}, ..., c_{-1}, c_0, there exists a unique solution $(x(n))_{n\geq -k}$ of (E) which satisfies the initial conditions $x(-k) = c_{-k}$, $x(-k+1) = c_{-k+1}$, ..., $x(-1) = c_{-1}$, $x(0) = c_0$.

A solution $(x(n))_{n\geq -k}$ of the neutral difference equation (E) is called *oscillatory* if for every positive integer n_0 there exist $n_1, n_2 \geq n_0$ such that $x(n_1)x(n_2) \leq 0$. In other words, a solution $(x(n))_{n\geq -k}$ is *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

In the special case where $\tau_j(n) = n - a_j$ and $\sigma(n) = n \pm b$, $a_j, b \in \mathbb{N}$, equation (E) takes the form

$$(\mathrm{E}') \qquad \Delta\left[x(n) + \sum_{j=1}^{w} q_j(n)x(n-a_j)\right] - p(n)x(n \pm b) = 0, \quad n \geq 0.$$

In the last few decades, our insight in the asymptotic behavior of neutral difference equations has been significantly advanced. A large number of papers have contributed to the research on this subject. See [4−8, 10−24] and the references cited therein.

The objective in this paper is to investigate the asymptotic behavior of the solutions of Eq. (E). Equation (E) formally describes an extended neutral difference equation, involving several retarded arguments $\tau_j(n)$, $j = 1, 2, ..., w$. In the following sections, we (first) establish some preliminary results that will serve as a useful tool in examining the asymptotic behavior of the solutions of Eq. (E), depending on sequences of real numbers $(q_j(n))$, $j = 1, 2, ..., w$. Then we postulate and prove a theorem setting convergence and divergence conditions on the solutions of Eq. (E).

2. Preliminaries

Assume that $(x(n))_{n\geq -k}$ is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As $(-x(n))_{n\geq -k}$ is also a solution of (E), we can restrict ourselves to the case where $x(n) > 0$ for all large n. Let $n_1 \geq -k$ be an integer such that $x(n) > 0$, $\forall n \geq n_1$. Then, there exists $n_0 \geq n_1$ such that

$$x(\sigma(n)) > 0,\, x(\tau_j(n)) > 0, \quad j = 1, 2, ..., w \qquad \forall n \geq n_0.$$

Set

$$z(n) = x(n) + \sum_{j=1}^{w} q_j(n) x(\tau_j(n)). \tag{2.1}$$

In view of (2.1), Eq. (E) becomes

$$\Delta z(n) - p(n) x(\sigma(n)) = 0. \tag{2.2}$$

Taking into account that $p(n) \geq p > 0$, we have

$$\Delta z(n) = p(n) x(\sigma(n)) \geq p x(\sigma(n)) > 0 \qquad \forall n \geq n_0,$$

which means that the sequence $(z(n))$ is eventually strictly increasing, regardless of the values of the terms $q_j(n)$.

Let the domain of τ_j be the set $D(\tau_j) = \mathbb{N}_{n_j^*} = \left\{ n_j^*, n_j^* + 1, n_j^* + 2, ... \right\}$, where n_j^* is the smallest natural number that τ_j is defined. Set

$$n_* = \max_{1 \leq j \leq w} n_j^*.$$

Then τ_j, $j = 1, 2, ..., w$ are defined in the set $\mathbb{N}_{n_*} = \{n_*, n_* + 1, n_* + 2, ...\}$.

Set

$$x(\tau_{\rho(n)}(n)) = \max\{x(\tau_1(n)), x(\tau_2(n)), ..., x(\tau_w(n))\} \tag{2.3}$$

where $\rho(n)$ is a sequence that takes values in the set $\{1, 2, ..., w\}$. Clearly, condition (1.1) guarantees that $\left(x(\tau_{\rho(n)}(n))\right)$ is a subsequence of $(x(n))$.

Notice that

$$\tau_{j_1}(\tau_{j_2}(\cdots \tau_{j_\ell}(n))) = \tau_{j_1}(n_s) \qquad \text{where} \qquad n_s = \tau_{j_2}(\cdots \tau_{j_\ell}(n)). \tag{2.4}$$

The following lemma provides us with some useful tools for establishing the main results:

Lemma 2.1. *Assume that the sequence* $(x(n))_{n\geq -k}$ *is a positive solution of* (E). *Then the following statements hold:*

(i) *If*

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty,$$

then

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} q_j(\sigma(n))x(\tau_j(\sigma(n))), \qquad A \in \mathbb{R}. \tag{2.5}$$

(ii) *If*

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty,$$

then

$$z(n) > 0, \qquad \text{eventually.} \tag{2.6}$$

Proof. Summing up (2.2) from n_0 to n, $n \geq n_0$, we obtain

$$z(n+1) = z(n_0) + \sum_{i=n_0}^{n} p(i)x(\sigma(i)). \tag{2.7}$$

For the above relation, exactly one of the following can be true:

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = S_0 < +\infty, \tag{2.7.a}$$

or

$$\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty. \tag{2.7.b}$$

Assume that (2.7.a) holds. Since $p(n) \geq p > 0$, we have

$$+\infty > S_0 = \sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) \geq p \sum_{i=n_0}^{\infty} x(\sigma(i)).$$

The last inequality guarantees that

$$\sum_{i=n_0}^{\infty} x(\sigma(i)) < +\infty$$

and, consequently

$$\lim_{n\to\infty} x(\sigma(n)) = 0. \tag{2.8}$$

Also, (2.7.a) guarantess that $\lim_{n\to\infty} z(n)$ exists as a real number. Set

$$\lim_{n\to\infty} z(n) = A \in \mathbb{R}.$$

Since $(z(\sigma(n)))$ is a subsequence of $(z(n))$, we have

$$\lim_{n\to\infty} z(\sigma(n)) = A,$$

or

$$\lim_{n\to\infty} \left[x(\sigma(n)) + \sum_{j=1}^{w} q_j(\sigma(n)) x(\tau_j(\sigma(n))) \right] = A.$$

Using (2.8), we obtain

$$\lim_{n\to\infty} \sum_{j=1}^{w} q_j(\sigma(n)) x(\tau_j(\sigma(n))) = A.$$

Thus

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} q_j(\sigma(n)) x(\tau_j(\sigma(n))).$$

The proof of Part (i) of the lemma is complete.

Assume that (2.7.b) holds. Then, by taking limits on both sides of (2.7) we obtain

$$\lim_{n\to\infty} z(n) = +\infty,$$

which in conjunction with the fact that the sequence $(z(n))$ is eventually strictly increasing, means that

$$z(n) > 0 \qquad \text{eventually.}$$

The proof of Part (ii) of the lemma is complete.

The proof of the lemma is complete.

3. Main results

Throughout this section, we are going to use the following conditions:

$$q(n) \le -1 \tag{C$_1$}$$

$$-1 < q(n) < 0, \qquad \liminf q(n) > -1 \tag{C$_2$}$$

$$0 \le q(n) \le 1, \qquad \limsup q(n) < 1 \tag{C$_3$}$$

$$q(n) \ge 1, \tag{C$_4$}$$

where

$$q(n) = \sum_{j=1}^{w} q_j(n). \tag{3.1}$$

The asymptotic behavior of the solutions of the neutral difference equation (E) is described by the following theorem:

Theorem 3.1. *For Eq.* (E) *the following statements hold:*

(I) *Every nonoscillatory solution tends to infinity or it has more than one real accumulation point which is zero, if the terms $q_j(n)$ are all nonpositive and condition* (C$_1$) *holds.*

(II) *Every nonoscillatory solution tends to zero or to infinity, if the terms $q_j(n)$ are all nonpositive and condition* (C$_2$) *holds.*

(III) *Every nonoscillatory solution is unbounded, if the terms $q_j(n)$ are all nonnegative and condition* (C$_3$) *holds.*

(IV) *Every nonoscillatory solution does not converge in $\mathbb{R}$, if the terms $q_j(n)$ are all nonnegative and condition* (C$_4$) *holds.*

Proof. Assume that $(x(n))_{n\ge -k}$ is a nonoscillatory solution of (E). Then it is either eventually positive or eventually negative. As $(-x(n))_{n\ge -k}$ is also a solution of (E), we can restrict ourselves to the case where $x(n) > 0$ for all large n. Let $n_1 \ge -k$ be an integer such that $x(n) > 0$, $\forall n \ge n_1$. Then, there exists $n_0 \ge n_1$ such that

$$x(\sigma(n)) > 0,\ x(\tau_j(n)) > 0, \quad j = 1, 2, ..., w \qquad \forall n \ge n_0.$$

Set

$$z(n) = x(n) + \sum_{j=1}^{w} q_j(n) x(\tau_j(n)). \tag{2.1}$$

In view of (2.1), Eq. (E) becomes

$$\Delta z(n) - p(n) x(\sigma(n)) = 0. \tag{2.2}$$

Taking into account that $p(n) \geq p > 0$, we have

$$\Delta z(n) = p(n)x(\sigma(n)) \geq px(\sigma(n)) > 0 \qquad \forall n \geq n_0,$$

which means that the sequence $(z(n))$ is eventually strictly increasing, regardless of the values of the terms $q_j(n)$.

Assume that the terms $q_j(n)$ are all nonpositive and (C_1) holds.

If (2.7.a) holds then, in view of Part (i) of Lemma 2.1, we have

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} q_j(\sigma(n)) x(\tau_j(\sigma(n))), \qquad A \in \mathbb{R}$$

which means that the sequence $(x(n))$ has at least one real accumulation point which is zero.

If (2.7.b) holds, then, in view of (2.7), we have

$$\lim_{n\to\infty} z(n) = +\infty,$$

which guarantees that

$$\lim_{n\to\infty} x(n) = +\infty.$$

The proof of the Part (I) of the theorem is complete.

Assume that the terms $q_j(n)$ are all nonpositive and (C_2) holds.

If (2.7.a) holds then, in view of Part (i) of Lemma 2.1, we have

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} q_j\left(\sigma(n)\right) x(\tau_j(\sigma(n))), \qquad A \in \mathbb{R}$$

which guarantees that $A \leq 0$.

Since $(z(n))$ is eventually strictly increasing, we have

$$z(n) = x(n) + \sum_{j=1}^{w} q_j(n)\, x(\tau_j(n)) < A \leq 0.$$

Using (3.1) and (2.3), the last inequality becomes

$$x(n) + \left(\sum_{j=1}^{w} q_j(n) \right) x(\tau_{\rho_1(n)}(n)) < 0,$$

or

$$x(n) < -q(n)x(\tau_{\rho_1(n)}(n)). \tag{3.2}$$

Set

$$\liminf q(n) = q > -1. \tag{3.3}$$

For every $\varepsilon > 0$ with $\varepsilon < 1 + q$ there exists n_3 such that

$$q(n) > q - \varepsilon > -1, \qquad \forall n \geq n_3. \tag{3.4}$$

Using (2.3), (3.4) and (2.4), inequality (3.2) becomes

$$\begin{aligned} x(n) &< -q(n)x(\tau_{\rho_1(n)}(n)) < (-q+\varepsilon)x(\tau_{\rho_1(n)}(n)) \\ &< (-q+\varepsilon)^2 x(\tau_{\rho_2(n)}(\tau_{\rho_1(n)}(n))) \\ &< \cdots < (-q+\varepsilon)^{m(n_s)} x(\tau_{\rho_{m(n_s)}}(n_\lambda)) \to 0 \ \text{ as } n \to \infty \end{aligned}$$

and consequently

$$\lim_{n\to\infty} x(n) = 0.$$

If (2.7.b) holds, then, in view of (2.7), we have

$$\lim_{n\to\infty} z(n) = +\infty,$$

which guarantees that

$$\lim_{n\to\infty} x(n) = +\infty.$$

The proof of the Part (II) of the theorem is complete.

Assume that the terms $q_j(n)$ are all nonnegative.

Then $q(n) \geq 0$. By (2.7) we have

$$z(n+1) = z(n_0) + \sum_{i=n_0}^{n} p(i)x(\sigma(i)) > 0,$$

which in conjunction with the fact that the sequence $(z(n))$ is eventually strictly increasing, means that

$$\lim_{n\to\infty} z(n) > 0.$$

Assume that (C_3) holds. If (2.7.a) holds then, in view of Part (i) of Lemma 2.1, we have

$$\lim_{n\to\infty} z(n) = A = \lim_{n\to\infty} \sum_{j=1}^{w} q_j\left(\sigma(n)\right) x(\tau_j(\sigma(n))), \qquad A \in \mathbb{R}.$$

Clearly $(z(n))$ is bounded and therefore $(x(n))$ is bounded. Set

$$\limsup x(n) = M.$$

Then there exists a subsequence $(x(\theta(n)))$ of $(x(n))$ such that

$$\lim_{n\to\infty} x(\theta(n)) = M.$$

Therefore

$$\lim_{n\to\infty} \left[x(\theta(n)) + \sum_{j=1}^{w} q_j\left(\theta(n)\right) x(\tau_j(\theta(n))) \right] = A,$$

or

$$\lim_{n\to\infty} \left[\sum_{j=1}^{w} q_j\left(\theta(n)\right) x(\tau_j(\theta(n))) \right] = A - M \geq 0,$$

i.e.,

$$M \leq A. \tag{3.5}$$

On the other hand

$$\lim_{n\to\infty} z(\sigma(n)) = A,$$

or

$$\lim_{n\to\infty} \left[x(\sigma(n)) + \sum_{j=1}^{w} q_j\left(\sigma(n)\right) x(\tau_j(\sigma(n))) \right] = A. \tag{3.6}$$

Set

$$\limsup q(n) = d < 1. \tag{3.7}$$

Using (2.8) and (3.7), relation (3.6) becomes

$$\lim_{n\to\infty} \sum_{j=1}^{w} q_j\left(\sigma(n)\right) x(\tau_j(\sigma(n))) = A$$

and consequently

$$\limsup \left[\sum_{j=1}^{w} q_j\left(\sigma(n)\right) x(\tau_j(\sigma(n))) \right] = A.$$

Thus

$$\limsup \left[\left(\sum_{j=1}^{w} q_j\left(\sigma(n)\right) \right) x(\tau_{\rho(\sigma(n))}(\sigma(n))) \right] \geq A,$$

or

$$\limsup \left[q\left(\sigma(n)\right) x(\tau_{\rho(\sigma(n))}(\sigma(n))) \right] \geq A,$$

or

$$\limsup q\left(\sigma(n)\right) \limsup x(\tau_{\rho(\sigma(n))}(\sigma(n))) \geq A.$$

Therefore

$$M > dM \geq \limsup q\left(\sigma(n)\right) \limsup x(\tau_{\rho(\sigma(n))}(\sigma(n))) \geq A,$$

or

$$M > A,$$

which contradicts (3.5). This means that $(x(n))$ is unbounded, and consequently (2.7.a) is not satisfied. Therefore (2.7.b) holds. Now,by (2.7), we have

$$\lim_{n\to\infty} z(n) = +\infty,$$

which guarantees that $(x(n))$ is unbounded.

The proof of the Part (III) of the theorem is complete.

Assume that the terms $q_j(n)$ are all nonnegative and (C_4) holds.

If (2.7.a) holds, then

$$\lim_{n\to\infty} z(n) = A \in \mathbb{R}$$

Since $q(n) > 0$, in view of Part (III), we have

$$\lim_{n\to\infty} z(n) > 0,$$

which means that $A > 0$. Combined with the fact that $\lim_{n\to\infty} x(\sigma(n)) = 0$, we conclude that $(x(n))$ has more than one real accumulation points. Therefore $(x(n))$ does not converge in $\mathbb{R}$.

If (2.7.b) holds, clearly $\lim_{n\to\infty} z(n) = +\infty$, which means that $(x(n))$ is unbounded, and therefore $(x(n))$ does not converge in $\mathbb{R}$.

The proof of Part (IV) of the theorem is complete.

The proof of the theorem is complete.

As a consequence of Theorem 3.1, we postulate the following corollary:

Corollary 3.2. *For Eq.* (E$'$) *the following statements hold:*

(i) *Every nonoscillatory solution either tends to zero or tends to infinity, if the terms $q_j(n)$ are all nonpositive and $q(n) < 0$.*

(ii) *Every nonoscillatory solution is unbounded, if the terms $q_j(n)$ are all nonnegative and $q(n) \geq 0$.*

References

[1] R. P. Agarwal, M. Bohner, S. R. Grace and D. O'Regan, *Discrete Oscillation Theory*, Hindawi Publishing Corporation, New York, 2005.

[2] R. Bellman and K. L. Cooke, *Differential-Difference Equations*, Academic Press, New York-london, 1963.

[3] R. K. Brayton and R. A. Willoughby, On the numerical integration of a symmetric system of difference-differential equations of neutral type, *J. Math. Anal. Appl.* **18** (1967), 182–189.

[4] W. E. Brumley, On the asymptotic behavior of solutions of differential-difference equations of neutral type, *J. Differential Equations* **7** (1970), 175–188.

[5] G. E. Chatzarakis, G. L. Karakostas and I. P. Stavroulakis, Convergence of the positive solutions of a nonlinear neutral difference equation, *Nonlinear Oscillations,* **14** (2011), 407-418.

[6] G. E. Chatzarakis and G. N. Miliaras, Convergence and divergence of the solutions of a neutral difference equation, *J. Appl. Math.,* vol. 2011, Article ID 262316, 18 pages, 2011.

[7] D. A. Georgiou, E. A. Grove and G. Ladas, Oscillation of neutral difference equations with variable coefficients, *Lecture notes in pure and Appl. Math.,* **127**, Dekker, New York (1991), 165–173.

[8] I. Gyori and L. Horvat, Asymptotic constancy in linear difference equations: limit formulae and sharp conditions, *Adv. Difference Equ.,* Vol. 2010, doi: 10.1155/2010/789302.

[9] I. Gyori and G. Ladas, *Oscillation theory of delay differential equations with applications*, Clarendon Press, Oxford, 1991.

[10] D. C. Hung, Oscillation and Convergence for a neutral difference equation, *Journal of Science, Mathematics-Physics* **24** (2008), 133–143.

[11] J. Jia, X. Zhong, X. Gong, R. Quyang and H. Han, Nonoscillation of first-order neutral difference equation, *Modern Applied Science* **3** (2009), 30–33.

[12] B. S. Lalli and B. G. Zhang, On existence of positive solutions and bounded oscillations for neutral difference equations, *J. Math. Anal. Appl.* **166** (1992), 272–287.

[13] B. S. Lalli, B. G. Zhang and J. Z. Li, On the oscillation of solutions of neutral difference equations, *J. Math. Anal. Appl.* **158** (1991), 213–233.

[14] M. Migda and G. Zhang, On unstable neutral Difference equations with maxima, *Math. Slovaca,* **56** (2006), 451–463.

[15] H. Peics, Positive solutions of neutral delay difference equation, *Novi Sad J. Math.* **35** (2005), 111–122.

[16] W. Snow, Existence, uniqueness and stability for nonlinear differential-difference equations in the neutral case, N. Y. U. Courant Inst. Math. Sci., IMM-NYU **328** (February 1965).

[17] X.H. Tang, Asymptotic behavior of solutions for neutral difference equations, *Comput. Math. Appl.* **44** (2002), 301–315.

[18] X.H. Tang and S. S. Cheng, Positive solutions of a neutral difference equation with positive and negative coefficients, *Georgian Math. J.* **11** (2004), 177–185.

[19] E. Thandapani and P. M. Kumar, Oscillation and nonoscillation of nonlinear neutral delay difference equations, *Tamkang J. Math.* **38** (2007), 323–333.

[20] E. Thandapani, R. Arul and P. S. Raja, The asymptotic behavior of nonoscillatory solutions of nonlinear neutral type difference equations, *Math. Comput. Modelling.* **39** (2004), 1457–1465.

[21] E. Thandapani, S. L. Marian and J. R. Graef, Asymptotic behavior of nonoscillatory solutions of neutral difference equations, *Comput. Math. Appl.* **45** (2003), 1461–1468.

[22] X. Wang, Asymptotic behavior of solutions for neutral difference equations, *Comput. Math. Appl.* **52** (2006), 1595–1602.

[23] J. Wei, Asymptotic behavior results for nonlinear neutral delay difference equations, *Appl. Math. Comput.* **217** (2011), 7184–7190.

[24] J. S. Yu and Z. C. Wang, Asymptotic behavior and oscillation in neutral delay difference equations, *Funkcial. Ekvac.* **37** (1994), 241–248.