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MULTI-TEAM HAWK-DOVE GAME

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Abstract. In this paper, we will propose and study the evolutionary dynamics of a two-team hawk-dove (HD) game, each team consists of two players. The replicator dynamics equations for the two team HD game will be set. Then, we will find the equilibrium solutions and the conditions of their locally asymptotic stability. Numerical simulations will be used to illustrate the behavior of the proposed game.

Keywords: multi-team games, hawk-dove game, replicator dynamics equations.

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1. Introduction

Evolutionary Game theory [1, 2] is an important approach to study complex adaptive systems. However, standard game theory has shown some differences from observations and experiments i.e., in the prisoner's dilemma game (PD), ultimatum game, etc. We propose that a reason for such discrepancies is that some realistic features (e.g. dynamics, repetitions, memory, local effects and mistakes) are not included in the standard game theory. Repeated game has been shown [1] to explain the PD paradox. Memory

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games [3, 4, 5] have been shown to solve both the PD and ultimatum game paradoxes. The importance of allowing mistakes has been realized long ago [6, 7] but recently it has been included in evolutionary games [8, 9, 10]. It explains why in real life, some bad or dominated strategies persist. An interesting approach to dynamic games has been pioneered by Puu [11] and expanded later on. This approach combines dynamic, repetition and mistakes features by turning the game into a dynamical system and by assuming bounded rationality.

Recently multi-team games have been introduced [12, 13]. In these games there are several teams, each team consists of some players. An example of multi-team games is M branches of McDonald fast-food shops competing against L branches of Burger King fast-food ones.

In section 2, we study the dynamics of the standard hawk-dove game. In section 3, we study the dynamic of multi-team hawk-dove game. We proposed two ways to overcome the singularity of the HD game. The equilibrium points and their stability are discussed in this section. Some numerical results are given.

2. Dynamic Hawk-Dove Game

Conflicts among animals (of the same kind) for food, habitat and mates are often settled by displays rather than all-out fitting. The escalating fights leading to injury or death are relatively rare. John Maynard Smith [14] used game theory to explain this phenomena. He suppose that there are only two possible behavioral types: one escalates the fight until injury or his opponent retreats, the other behavior sticks to display and retreats if his opponent escalates. These two types of behavior are called hawks and doves.

The hawk-dove game [15] is a good example of how game theory is applied in population dynamics. Hawk-dove game is a two player game, such that two strategies, hawk (H) and dove (D) are allowed with the payoff matrix;

$$\Pi = \begin{bmatrix} \dfrac{V-C}{2} & V \\ 0 & \dfrac{V}{2} \end{bmatrix},$$

where V is the value of the contested resource (food, habitat, mates,..) and C is the cost of an escalated fight. It is always assumed that the value of the resource is less than the cost of a fight, i.e., $0 < V < C$.

The exact value of the dove vs. dove payoff varies between model formulations. Sometimes the players are assumed to split the payoff equally ($V/2$ for each), other times the payoff is assumed to be zero (since this is the expected payoff to a war of attrition game, which is the presumed models for a contest decided by display duration). While the hawk-dove game is typically taught and discussed with the payoffs in terms of V and C, the solutions hold true for any matrix with the payoffs matrix [16];

$$\Pi = \begin{bmatrix} X & W \\ L & T \end{bmatrix},$$

where $X < L < T < W$.

Biologists have explored modified versions of classic hawk-dove game to investigate a number of biologically relevant factors. These include adding variation in resource holding potential, and differences in the value of winning to the different players [17], allowing the players to threaten each other before choosing moves in the game [18], and extending the interaction to two plays of the game [19].

The replicator dynamic equation for the hawk-dove game is given by;

$$\frac{d\,x_i}{d\,t} = x_i\left((\Pi\,x)_i - x\,\Pi\,x\right),\ i = 1, 2,$$

where x_1, x_2 are the fractions of hawks and doves respectively and Π is the above payoff matrix. These equations can be written in the following form;

$$\frac{d\,x_1}{d\,t} = \frac{x_1}{2}\left((V - C)\,x_1\,(1 - x_1) + 2\,V\,x_2\,(1 - x_1) - V\,x_2^2\right),$$

provided that $x_1 + x_2 = 1$.

In a population consisting mostly of doves, hawks will spread, for they are likely to meet only doves and gain V, while a dove will only get $V/2$. But in a population of mostly hawks, the dove avoids every fight and keeps its fitness unchanged, while hawks meet hawks with loss in their fitness by $(C - V)/2$. Neither type of behaviors are better than the other. So, the two behaviors are unstable.

The above equations admits three fixed points (equilibrium states), the unstable pure states $(1,0)$ with the eigenvalues $\lambda_{1,2} = \frac{C-V}{2}, \frac{C-V}{2} > 0$ and $(0,1)$ with the eigenvalues $\lambda_{1,2} = \frac{V}{2} > 0, -\frac{V}{2} < 0$ and the interior asymptotically stable state $(\frac{V}{C}, \frac{C-V}{C})$ with the eigenvalues $\lambda_{1,2} = -\frac{V\,(C-V)}{2\,C} < 0$.

3. Dynamic Two-Team Hawk-Dove Game

Now, consider two teams playing HD game among them and every player plays HD game with his team-mate. Thus each player has 4 possible strategies, H$\grave{\text{H}}$ which means that he (she) plays hawk with his (her) team-mates and with the other team as well. The second strategy is H$\grave{\text{D}}$ which means that a player that adopts hawk with his team-mates and adopts dove against the other team. Similarly one can define D$\grave{\text{H}}$ and D$\grave{\text{D}}$. The payoff matrix Π of this game is given by,

$$\Pi = \begin{bmatrix} \frac{v-c+v'-c'}{2} & \frac{v-c+2\,v'}{2} & \frac{2\,v+v'-c'}{2} & v+v' \\ \frac{v-c}{2} & \frac{v-c+v'}{2} & v & v+\frac{v'}{2} \\ \frac{v'-c'}{2} & v' & \frac{v+v'-c'}{2} & \frac{v}{2}+v' \\ 0 & \frac{v'}{2} & \frac{v}{2} & \frac{v+v'}{2} \end{bmatrix}, \tag{1}$$

where $0 < v < c$ and $0 < v' < c'$.

When we tried to get the equilibrium points to this game, we noticed that the solutions eventually tends to a fixed point which depends on the initial conditions not just the parameter values v, v', c, c'. The reason is that replicator equations are not sufficient to uniquely determine all the fractions since the given payoff matrix in equation (1) is singular ($det(\Pi) = 0$).

This singularity is lifted if mistakes are allowed into the game. Recently the papers [8, 9, 10] have studied the dynamics of learning in multi-agent systems, where the agents use reinforcement learning [20]. They showed that, although the agents are not directly interacting with each other, a collective game between them arises through their interaction with the environment. Such interactions can be modelled via a modified replicator type equations:

$$\frac{dx_i}{dt} = x_i\,[(\Pi x)_i - x\Pi x] + \gamma_i\; x_i\; \sum_{j=1}^{n} x_j\, ln(\frac{x_j}{x_i}) \tag{2}$$

where $\gamma_i,\ i = 1, 2, .., n$ are nonnegative constants measuring the average rate of mistakes (lose of memory) done by the player adopting strategy i. Note that equation (2) implies that non of x_i can be equal 0 or 1. Applying (2) to the values $v = 1, c = 3, v' = 1, c' = 2$, we obtained the solution $x_1 = 0.173,\ x_2 = 0.184,\ x_3 = 0.328,\ x_4 = 0.315$ for $\gamma_i = 0.05,\ i = 1, 2, 3, 4$ which is independent of the initial conditions.

An alternative way to avoid the singularity of the payoff matrix (1) is to modify it according to our game. So, the following payoff matrix of the hawk-dove game is proposed:

$$\Pi\,(HH',\ HH') = \frac{1}{2}\left(\min(v, v') - c - c'\right), \tag{3}$$

where $\min\,(v,\ v')$ is the minimum value of $v,\ v'$ and $v \neq v'$. Since going into two fights typically gains less than the sum of going into each of the fights. Thus the following payoff matrix is the modification proposed for the (1);

$$\Pi = \begin{bmatrix} \frac{\min(v,v')-c-c'}{2} & \frac{v-c+2\,v'}{2} & \frac{2\,v+v'-c'}{2} & v+v' \\ \frac{v-c}{2} & \frac{v-c+v'}{2} & v & v+\frac{v'}{2} \\ \frac{v'-c'}{2} & v' & \frac{v+v'-c'}{2} & v'+\frac{v}{2} \\ 0 & \frac{v'}{2} & \frac{v}{2} & \frac{v+v'}{2} \end{bmatrix}, \tag{4}$$

which is not singular (since $v \neq v'$). The replicator equation of of the two team hawk-dove game is given in the form;

$$\frac{dx_i}{dt} \;=\; x_i\,[\,(\Pi\,x)_i - x\,\Pi\,x\,]\,,\ i = 1, 2, 3, 4, \tag{5}$$

where x_1, x_2, x_3, x_4 are the fractions of the total population adopting HH', HD', DH' and DD' respectively. Using that $\min\,(v, v') = v$ and the above proposed payoff matrix in equation (4). The replicator dynamics equation (5) can be written in details in the following form;

$$
\begin{aligned}
\frac{dx_1}{dt} = \frac{x_1}{2} \quad [\quad & (v+v')\,(x_1+x_2+x_3+x_4)\,(1-x_1-x_2-x_3-x_4) \\
& -c\,(x_1+x_2)\,(1-x_1-x_2) - c'\,(x_1+x_3)\,(1-x_1-x_3) \\
& +(v-v')\,x_1\,(1-x_1) + v\,(x_3+x_4) + v'\,(x_2+x_4) \quad],
\end{aligned}
$$

$$
\begin{aligned}
\frac{dx_2}{dt} = \frac{x_2}{2} \quad [\quad & (v+v')\,(x_1+x_2+x_3+x_4)\,(1-x_1-x_2-x_3-x_4) \\
& -c\,(x_1+x_2)\,(1-x_1-x_2) + c'\,(x_1+x_3)^2 - (v-v')\,x_1^2 \\
& +v\,(x_3+x_4) - v'\,(x_1+x_3) \quad],
\end{aligned}
$$

$$
\begin{aligned}
\frac{dx_3}{dt} = \frac{x_3}{2} \quad [\quad & (v+v')\,(x_1+x_2+x_3+x_4)\,(1-x_1-x_2-x_3-x_4) \\
& +c\,(x_1+x_2)^2 - c'\,(x_1+x_3)\,(1-x_1-x_3) - (v-v')\,x_1^2 \\
& -v\,(x_1+x_2) + v'\,(x_2+x_4) \quad],
\end{aligned}
$$

$$
\begin{aligned}
\frac{dx_4}{dt} = \frac{x_4}{2} \quad [\quad & (v+v')\,(x_1+x_2+x_3+x_4)\,(1-x_1-x_2-x_3-x_4)] \\
& +c\,(x_1+x_2)^2 + c'\,(x_1+x_3)^2 - (v-v')\,x_1^2 \\
& -v\,(x_1+x_2) - v'\,(x_1+x_3) \quad].
\end{aligned}
$$

Proposition 3.1. *Pure strategies are not asymptotically stable for the system (5).*

Proof. We have four pure strategies:

$$
\begin{aligned}
S_1 &= (\,1,\,0,\,0,\,0), \quad S_2 = (\,0,\,1,\,0,\,0), \\
S_3 &= (\,0,\,0,\,1,\,0), \quad S_4 = (\,0,\,0,\,0,\,1).
\end{aligned}
$$

The first one $(1,0,0,0)$ is not asymptotically stable since its eigenvalues $\frac{c+c'-2\,v}{2}$, $\frac{c+c'-2\,v}{2}$, $\frac{c-2\,v+v'}{2}$, $\frac{c'-v}{2}$ are positive. It is clear from the values of the first column of the payoff matrix (4), that the fitness of the individual that adopts the strategy HH'

is less that the others

$$E(HH', HH') < E(HD', HH'), E(DH', HH'), E(DD', HH'),$$

so it is evolutionary unstable [1].

The second one $(0, 1, 0, 0)$ is not stable since there are three positive eigenvalues of its four eigenvalues $\frac{c-v-v'}{2}$, $\frac{c-v+v'}{2}$, $\frac{c-v}{2}$, $\frac{v'}{2}$. The same thing, it is clear from the values of the second column of the payoff matrix (4), that the fitness of the individual that adopts the strategy HD' is less that the others

$$E(HD', HD') < E(HH', HD'), E(DH', HD'), E(DD', HD'),$$

so it is evolutionary unstable.

The third one $(0, 0, 1, 0)$ is not stable since there are three positive eigenvalues of its four eigenvalues $\frac{c'-v'-v}{2}$, $\frac{c'-v'+v}{2}$, $\frac{c'-v'}{2}$, $\frac{v}{2}$. From the values of the third column of the payoff matrix (4), we find that the fitness of the individual that adopts the strategy DH' is less that the others

$$E(DH', DH') < E(HH', DH'), E(HD', DH'), E(DD', DH'),$$

so it is evolutionary unstable.

Finally, the fourth one $(0, 0, 0, 1)$ is not stable since there are three positive eigenvalues of its four eigenvalues $\frac{v+v'}{2}$, $-\frac{v+v'}{2}$, $\frac{v}{2}$, $\frac{v'}{2}$. The same thing, it is clear from the values of the fourth column of the payoff matrix (4), that the fitness of the individual that adopts the strategy DD' is less that the others

$$E(DD', DD') < E(HH', DD'), E(HD', DD'), E(DH', DD'),$$

so it is evolutionary unstable. This completes the proof.

The system (5) has the following boundary equilibrium points. The first one is $S_5 = \left(\frac{v'}{c'+v'-v}, \frac{c'-v}{c'+v'-v}, 0, 0\right)$, which is unstable because there are at least two positive eigenvalues of its four eigenvalue $\left(\lambda = \frac{c-v}{2}, \lambda = \frac{(c-v)(c'+v'-v)+v'(v'-v)}{2(c'+v'-v)}\right)$.

The second one is $S_6 = \left(\frac{v}{c+v'-v},\, 0\,, \frac{c+v'-2\,v}{c+v'-v},\, 0\right)$, which is unstable because there are at least two positive eigenvalues of its four eigenvalue $\left(\lambda = \frac{c'-v'}{2}, \lambda = \frac{(c'-v')\,(c+v'-v)+v\,(v'-v)}{2\,(c+v'-v)}\right)$.

The third one is $S_7 = \left(\frac{v+v'}{c-v+c'+v'},\, 0\,,\, 0\,, \frac{c-2\,v+c'}{c-v+c'+v'}\right)$, which is unstable because if one of the following two eigenvalues is negative the other is positive $\left(\lambda = \frac{-v'\,(c-v)+v\,(c'-v)}{2\,(c-v+c'+v')}, \lambda = \frac{v'\,(c+v')-v\,(c'+v;)}{2\,(c-v+c'+v')}\right)$.

The fourth one $S_8 = \left(0,\, \frac{c'-v'+v}{c+c'},\, \frac{c-v'+v}{c+c'},\, 0\right)$, which is not stable since at least one of the following two eigenvalues is positive and the other is negative $\left(\lambda = \frac{c\,(c'-v')-c'\,v}{2\,(c+c')}, \lambda = \frac{-c\,(c'-v')+c'\,v}{2\,(c+c')}\right)$.

The fifth one $S_9 = \left(0,\, \frac{v}{c},\, 0,\, \frac{c-v}{c}\right)$, is not stable since there is two positive eigenvalues of its eigenvalues $\left(\lambda = \frac{v'}{2}, \lambda = \frac{v'}{2}\right)$.

The sixth one $S_{10} = \left(0,\, 0\,, \frac{v'}{c'}, \frac{c'-v'}{c'}\right)$, is not stable since there is two positive eigenvalues of its eigenvalues $\left(\lambda = \frac{v}{2}, \lambda = \frac{v}{2}\right)$.

The seventh one $S_{11} = \left(\frac{v'}{c'+v'-v}, \frac{v}{c}-\frac{v'}{c'+v'-v},\, 0\,,\, 1-\frac{v}{c}\right)$, provided that $\frac{v'}{c'+v'-v} < \frac{v}{c}$, is unstable since it has the following positive eigenvalue $\lambda = \frac{v'\,(v'-v)}{2\,(c'+v'-v)}$.

The eighth one $S_{12} = \left(\frac{v}{c+v'-v},\, 0\,, \frac{v'}{c'}-\frac{v}{c+v'-v},\, 1-\frac{v'}{c'}\right)$, provided that $\frac{v}{c+v'-v} < \frac{v'}{c'}$, is unstable since it has the following positive eigenvalue $\lambda = \frac{v\,(v'-v)}{2\,(c+v'-v)}$.

The ninth one $S_{13} = \left(x_1,\, \frac{v}{c} - \frac{c+v'-v}{c}\,x_1,\, \frac{v'}{c'} - \frac{c'+v'-v}{c'}\,x_1,\, 0\right)$, where $x_1 = \frac{c'\,v-c\,(c'-v')}{c\,c'+(c+c')\,(v'-v)}$ provided that $\frac{v}{c+v'-v} < \frac{v'}{c'}$, is unstable since it has a positive eigenvalue.

The tenth one $S_{14} = \left(0,\, \frac{v}{c}, \frac{v'}{c'}, 1-\frac{v}{c}-\frac{v'}{c'}\right)$, provided that $\frac{v}{c}+\frac{v'}{c'} < 1$, is stable since there is three negative eigenvalues $\lambda = -\frac{v\,c'\,(c-v)+c\,v'\,(c'-v')}{2\,c\,c'}$, $\lambda^2+\frac{v\,c'\,(c-v)+c\,v'\,(c'-v')}{2\,c\,c'}\,\lambda+$

$\dfrac{v\,v'\,(c\,c' - v\,c' - c\,v')}{4\,c\,c'} = 0$, which has negative eigenvalues since the coefficient of λ and the constant term are positive [21], and the fourth one is 0. Figure (1) shows the stability diagram of the individuals that adopt this mixed strategy with the parameters values are $v = 1$, $c = 3$, $v' = 1.5$ and $c' = 3$. So, we get the stable steady state (0, 0.3334, 0.4999, 0.1667)

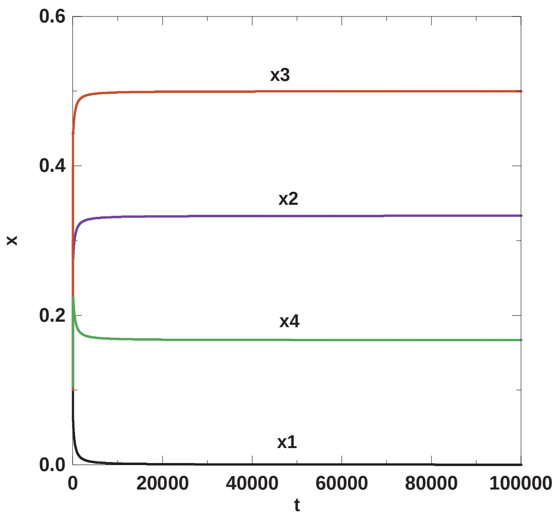

FIGURE 1

4. Conclusion *This hawk-dove game with the payoff matrix (5) has no internal equilibrium point. The only stable equilibrium point is S_{14}, which mean that the fraction x_1 that adopt the strategy HH' will vanish. Note that the other fractions has a strategy D or D' in their choices because if they do not gain they will not loss. So, this game has only one stable mixed strategy.*

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