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ON THE EXPANSION PROBLEM OF A FUNCTION ASSOCIATED WITH A SYSTEM OF SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract: Consider the system of second order differential equations

$$Y''(x) + (\lambda^2 R(x) + Q(x))y(x) = 0$$

where $x \in (a, b)$, a, b finite or infinite ; λ, a complex parameter and $y(x) = (y_1(x), y_2(x))^T$.

$$Q(x) = \begin{pmatrix} p(x) & r(x) \\ r(x) & q(x) \end{pmatrix}, \qquad R(x) = \begin{pmatrix} s(x) & 0 \\ 0 & t(x) \end{pmatrix}$$

P(x), q(x), r(x), s(x), t(x) are all assumed to be real-valued functions summable over (a, b).

In the present paper we generalize the Parseval theorem and the expansion theorem for a function $f(x) = (f_1(x), f_2(x))^T$ satisfying $f^T(x)R(x)f(x) \in L(-\infty, \infty)$ associated with the above system of second order differential equations under the general boundary conditions, where the elements of R(x) i.e. s(x), t(x) are assumed to be greater than zero for $x \in (a, b)$, a, b being finite or infinite.

Keywords : mean convergence, functions of bounded variation, Spectral matrices, Gronwall inequality, Helly's selection principle, the Parseval theorem, the expansion theorem.

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1. Introduction

Consider the system of second order differential equations

$Y''(x) + (\lambda^2 R(x) + Q(x))y(x) = 0$ . . . . (1)

where $y(x) = (y_1(x), y_2(x))^T$

$$Q(x) = \begin{pmatrix} p(x) & r(x) \\ r(x) & q(x) \end{pmatrix}, \qquad R(x) = \begin{pmatrix} s(x) & 0 \\ 0 & t(x) \end{pmatrix},$$

P(x), q(x), r(x), s(x), t(x) are all assumed to be real-valued functions summable over (a, b), a, b being finite or infinite and λ is a complex parameter.

Here, equation (1) may be put in the form $Ly(x) = -\lambda^2 R(x)y(x)$ . . . . (2)

where L is the matrix differential operator given by

$$L \equiv \begin{pmatrix} D^2 + p(x) & r(x) \\ r(x) & D^2 + q(x) \end{pmatrix}, \qquad D^2 \equiv \frac{d^2}{dx^2}$$

The boundary conditions at a, b satisfied by a solution $U(x, \lambda) = (U_1(x, \lambda), U_2(x, \lambda))^T$

of (1) are

$[U(x, \lambda), \phi_i](a) = 0, \ [U(x, \lambda), \phi_j](b) = 0$ . . . . (3)

i = 1, 2; j = 3, 4 , where $\phi_l = \phi_l(x, \lambda)$, i = 1, 2, 3, 4 (called boundary condition vectors' after Chakravarty[3]) are the solution of (1) which together with their first derivatives take some prescribed values at x = a or b and $[.,.](\alpha)$ is the value at $x = \alpha$ of the belinearconcomitant [., .]as given in Section – 5 of Sengupta[12]. The boundary condition vectors ϕ_1, ϕ_2 at x = a andϕ_3, ϕ_4 at x = b are linearly independent of each other.

The expression

$W(a, b, \lambda) = -[\phi_1, \phi_2][\phi_3, \phi_4] + [\phi_1, \phi_3][\phi_2, \phi_4] - [\phi_1, \phi_4][\phi_2, \phi_3]$ is the wronskian with usual properties of the boundary condition vectors ϕ_l, i = 1, 2, 3, 4.

Moreover , if the boundary condition vectors ϕ_l, i = 1,2, 3, 4 satisfy

$$[\phi_1, \phi_2](a) = [\phi_3, \phi_4](b) = 0, \quad \quad . . . . (4)$$

the boundary value problem (1) –(2) leads to a self-adjoint eigenvalue problem over the finite interval

(a, b) (seeBhagat[1] and Chakravarty[3]).

For the boundary value problem (1) –(2), where s(x), t(x) > 0 Bhagat[1] constructed interalia the usual Green's matrix and the Parseval formula for an arbitrary square integrable function over the finite interval (a, b). He also obtained the Green's matrix over the interval$(0, \infty)$.

For the boundary value problem (1) – (2) with s(x) = t(x) =1, Chakravarty and Roy Palodhi[7] obtained the Green's matrix and the Parseval formula over the interval $(-\infty, \infty)$.

In the present paper we consider the boundary value problem (1)- (2) with (4), where s(x), t(x) > 0 and generalize the Parseval theorem and the expansion problem for a function f(x) = $(f_1(x), f_2(x))^T$ satisfying $f^T(x)R(x)f(x) \in L(-\infty, \infty)$.

We adopt the usual technique i.e.; to assume the results valid for the finite interval (a, b) and then to pass on to the case $(-\infty, \infty)$ by making $a \to -\infty$, $b \to \infty$, by using mostly the methods of Levitan and Sargsjan [10] and Titchmash [12] for the second order differential equations.

2. The Extension Process

Let $\phi_r \equiv \phi_r(x, \lambda) = (\phi_{r1}(x, \lambda), \phi_{r2}(x, \lambda))^T$, r = 1, 2 be the vectors which are solutions of (1) and satisfy at x = 0, the conditions

$\phi_{rj}(0, \lambda) = \delta_{rj}$, $\phi'_{rj}(0, \lambda) = 0$, r, j = 1, 2 .. . . . (5)

where δ_{rj}'s are the 'Kronecker delta'.

Corresponding to ϕ_r let us introduce $\theta_r \equiv \theta_r(x, \lambda) = (\theta_{r1}(x, \lambda), \theta_{r2}(x, \lambda))^T$, r = 1, 2, the solutions of (1) satisfy at x = 0, the conditions

$\theta_{rj}(0, \lambda) = 0$, $\theta'_{rj}(0, \lambda) = -2\delta_{rj}$, r, j = 1, 2 . . . . . (6)

where θ_r are related to ϕ_r by means of the relations

$[\phi_r , \theta_k] = -2\delta_{rk}$, r, k = 1, 2 . . . . . (7)

and $[\theta_1, \theta_2] = 0$ . . . . . (8)

In what follows ϕ_1, ϕ_2, θ_1, θ_2 form a fundamental set of solutions, their wronskian being non-zero.

It is well-known (ref. Chakravarty [5], Bhagat [1]) that there exists in the interval (0, b) a symmetric matrix $(l_{rs}(\lambda))$, r,s =1, 2 depending on λ, b and the coefficients of the boundary conditions(2) at x = b, $l_{rs}(\lambda)$ have infinite number of simple poles on the real axis and for fixed b,

$l_{rs}(\lambda) = 0(1/|\gamma|)$, as $\gamma \to 0$, where $\gamma = \text{Im. } \lambda$. . . . . . (9)

Hence there exists a pair of solutions $\Psi_r \equiv \Psi_r(b, x, \lambda) = (\Psi_{r1}(b, x, \lambda), \Psi_{r2}(b, x, \lambda))^T$, such that $\Psi_r = l_{r1}\phi_1 + l_{r2}\phi_2 + \theta_r$, r = 1, 2. . . . . . (10)

Similarly, there exists the symmetric matrix $(L_{rs}(\lambda))$ r, s = 1, 2 in (a, 0) and solutions $\chi_r \equiv \chi_r(a, x, \lambda)$

$= (\chi_{r1}(a, x, \lambda), \chi_{r2}(a, x, \lambda))^T$, such that $\chi_r = L_{r1}\phi_1 + L_{r2}\phi_2 + \theta_r$, r, = 1, 2 . . . . . (11)

where $L_{rs}(\lambda) = 0(1/|\gamma|)$, as $\gamma \rightarrow 0$, $\gamma = \text{Im}.\lambda$. . . . . . (12)

Hence$[\chi_1,\chi_2] = [\Psi_1, \Psi_2] = 0$, ... (13)

$[\chi_r, \chi_s] = 2(l_{rs} - L_{rs})$, r, s = 1, 2 . . . . . (14)

and the wronskian $W(a, b, \lambda)$ of Ψ_r, χ_r, r = 1, 2 is given by

$W(a, b, \lambda) = 4(l_{11} - L_{11})(l_{22} - L_{22}) - 4(l_{12} - L_{12})^2 \neq 0$ . . . . . (15)

Put $\tilde{\Psi}(a, b, x, \lambda) = (\tilde{\Psi}_{ij}(a, b, x, \lambda))$, i, j = 1, 2

where $\tilde{\Psi}_r \equiv \tilde{\Psi}_r(a, b, x, \lambda) = (\tilde{\Psi}_{r1}(a, b, x, \lambda), \tilde{\Psi}_{r2}(a, b, x, \lambda))^T = ([\chi_s, \Psi_s]\Psi_r - [\chi_s, \Psi_r]\Psi_s)$ $/ W(a, b, \lambda)$ . . . . . (16)

for r = 1, s = 2 and r = 2, s=1.

Also, put, $\tilde{\chi} \equiv \tilde{\chi}(a, x, \lambda) = (\chi_{ij}(a, x, \lambda))$, i, j = 1, 2

The choice of θ_k satisfying (7) – (8) is not unique and the following three independent relations uniquely determine θ_k, θ'_k, k = 1,2 ;

$W(\phi_1, \phi_2, \theta_r, \tilde{\Psi}_r) = 0$, r = 1, 2,

and $[\tilde{\Psi}_1, \theta_2] = [\tilde{\Psi}_2, \theta_1]$ . . . . . (17)

It now followsthat

$\tilde{\Psi}_r(a, b, x, y, \lambda) = [l_{r1}\phi_1 + l_{r2}\phi_2 + \theta_r] / 2(l_{11} - L_{11})$, r = 1, 2 . . . . . (18)

where $l_{11} - L_{11} = l_{22} - L_{22}$ and $l_{12} = L_{12}$. . . . . . (19)

Now we construct the matrix $G(a, b, x, y, \lambda) = (G_{ij}(a, b, x, y, \lambda))$ for i, j = 1, 2 in the following way :

Put $G(a, b, x, y, \lambda) = \tilde{\Psi}(a, b, x, \lambda)\, \tilde{\chi}^T(a, y, \lambda))$, $y \leq x$

$= \tilde{\chi}(a, x, \lambda)\, \tilde{\Psi}^T(a, b, y, \lambda)$, $y > x$ . . . . (20)

Then $G(a, b, x, y, \lambda)$ is the Green's matrix for the system (1) over (a, b) and

$G_r(a, b, x, y, \lambda) = (G_{r1}(a, b, x, y, \lambda), G_{r2}(a, b, x, y, \lambda))^T$, r = 1, 2 [called Green's vectors] satisfy

$G_r^T(a, b, x, y, \lambda)\, R(y)\, G_r(a, b, x, y, \bar{\lambda}) \in L(a, b)$ . . . . (21)

where $\bar{\lambda}$ is the conjugate of λ.

Let $f(x) = (f_1(x), f_2(x))^T$ be a real-valued function satisfying $f^T(x)R(x)f(x) \in L(a, b)$, then the vector $\Phi(a, b, x, \lambda, f) = (\Phi_1(a, b, x, \lambda, f), \Phi_2(a, b, x, \lambda, f))^T = \int_a^b G(a, b, x, y, \lambda) R(y) f(y) dy$ . . . . (22)

[called the 'resolvent of f(x)'] satisfies the non-homogeneous system.

$Y''(x) + (\lambda^2 R(x) + Q(x))y(x) = R(x)f(x)$ . . . . (23)

where R(x), Q(x) are those given in (1).

Also, $\int_a^b \Phi^T(a, b, x, \lambda, f) R(x) \Phi(a, b, x, \bar{\lambda}, f) dx \le \gamma^{-2} \cdot \int_a^b f^T(x) R(x) f(x) dx$, $\gamma = \text{Im}.\lambda$

.. . . . (24)

Let f(x) be the function continuously differentiable twice over (a, b). If λ is not an eigenvalue then it follows that

$\lambda\Phi(a, b, x, \lambda, f) = f(x) + \Phi(a, b, x, \lambda, \tilde{f})$ . . . . (25)

where $\tilde{f} \equiv \tilde{f}(x) = (\tilde{f}_1(x), \tilde{f}_2(x))^T = Lf(x)$. . . . . (26)

L being defined in (2) and $\tilde{f}^T(x) R(x) \tilde{f}(x) \in L(a, b)$.

Following Chakravarty-Acharyya [6] and Chakravarty [4], it follows that as

$a \to -\infty$, $b \to \infty$,

$\Phi(a, b, x, \lambda, f) \to \Phi(x, \lambda, f)$.

$\Phi(a, b, x, \lambda, \tilde{f}) \to \Phi(x, \lambda, \tilde{f})$ and

$G(a, b, x, y, \lambda) \to G(x, y, \lambda)$ .. . . (27)

where $f^T(x)R(x)f(x)$, $\tilde{f}^T(x)R(x)\tilde{f}(x) \in L(-\infty, \infty)$ and the matrix $G(x, y, \lambda) = (G_{ij}(x, y, \lambda))$, i, j = 1, 2 is the Green's matrix (not necessarily uniquely determined) in the singular case $(-\infty, \infty)$.

As function of x, the vector $\Phi(x, \lambda, f)$, satisfying the non-homogeneous system (23), is the resolvent of f(x), $f^T(x)R(x)f(x) \in L(-\infty, \infty)$. It is easy to prove that $\Phi(x, \lambda, f)$, $\Phi(x, \lambda, \tilde{f})$ and the Green's vectors $G_l(x, y, \lambda) = (G_{l1}(x, y, \lambda), G_{l2}(x, y, \lambda))^T$, i= 1, 2 satisfy

$\int_{-\infty}^{\infty} \Phi^T(x, \lambda, f)\, R(x)\Phi(x, \bar{\lambda}, f) \leq \gamma^{-2}\int_{-\infty}^{\infty} f^T(x)R(x)f(x)dx$,

$\lambda\phi(x, \lambda, f) = f(x) + \phi(x, \lambda, \tilde{f})$, ... (28)

$\lambda\Phi(x, \lambda, f) = f(x) + \Phi(x, \lambda, \tilde{f})$ . . . . (29)

and $G_l^T(x, y, \lambda)R(y)\, G_l(x, y, \bar{\lambda}) \in L(-\infty, \infty)$, ... (30)

where $f^T(x)R(x)f(x)$, $\tilde{f}^T(x)R(x)\tilde{f}(x) \in L(-\infty, \infty)$.

[See Chakravarty and Roy Palodhi [7], Bhagat[1]]

As $b\to\infty$, let $l_{rs}(\lambda) \to m_{rs}(\lambda)$ and $\Psi_r(b, x, \lambda) \to \Psi_r(x, \lambda)$, r, s = 1, 2 . thenfrom (10) as $b\to\infty$, we have

$\Psi_r(x, \lambda) = m_{r1}\phi_1 + m_{r2}\phi_2 + \theta_r$, r= 1, 2 . . . . (31)

Similarly, from (11), as $a\to -\infty$,

$\chi_r(x, \lambda) = M_{r1}\phi_1 + M_{r2}\phi_2 + \theta_r$, r = 1, 2 . . . . (32)

where $L_{rs}(\lambda) \to M_{rs}(\lambda)$ and $\chi_r(a, x, \lambda) \to \chi_r(x, \lambda)$, as $a \to -\infty$, for r, s = 1, 2.

Also it follows that (see Bhagat[1])

$\int_0^{\infty} \psi_r^T(x, \lambda)R(x)\Psi_r(x, \bar{\lambda})dx = -\gamma^{-1}.\ \mathrm{Im}.(m_{rr}(\lambda))$ and

$\int_{-\infty}^{0} \chi_r^T(x, \lambda)R(x)\chi_r(x, \bar{\lambda})dx = \gamma^{-1}\ .\ I_m(M_{rr}(\lambda))$, r = 1, 2.

. . . . (33)

From (19) we have

$$m_{11} - M_{11} = m_{22} - M_{22}\ ,\ m_{12} = M_{12} \qquad \ldots . (34)$$

and $\widetilde{\Psi}_r \equiv \widetilde{\Psi}_r(x, \lambda) = (m_{r1}\phi_1 + m_{r2}\phi_2 + \theta_r)/2(m_{11} - M_{11})$, r = 1, 2. $\qquad \ldots . (35)$

Thus from (20) the explicit form of the Green's matrix G(x, y, λ) over the interval (-∞, ∞) is

$$G(x, y, \lambda) = \widetilde{\Psi}(x, \lambda)\, \chi^T(y, \lambda)\ ,\ y \le x\ = \tilde{\chi}(x, \lambda)\, \widetilde{\Psi}^T(y, \lambda)\ ,\ y > x \qquad \ldots . (36)$$

3. Certain generalization of the Parseval formula

Let $Y_n(x) = (y_{1n}(x), y_{2n}(x))^T$ be an eigenvector of the boundary value problem (1) – (2) with (4) corresponding to an eigenvalue λ_n. Then

$$y_n(x) = a_{1n}\phi_1(x, \lambda_n) + a_{2n}\phi_2(x, \lambda_n) + b_{1n}\theta_1(x, \lambda_n) + b_{2n}\theta_2(x, \lambda_n) \qquad \ldots . (37)$$

where $\phi_1(x, \lambda_n)$, $\phi_2(x, \lambda_n)$, $\theta_1(x, \lambda_n)$ and $\theta_2(x, \lambda_n)$ are linearly independent solutions of (1) and a_{in}, b_{in},

i = 1, 2 are scalars (assumed all greater than zero) which are independent of one another and are so chosen that $\Psi_n(x) = A^{-1/2}y_n(x)$ is normalized in the sense that

$$\int_a^b \Psi_n{}^T(x)R(x)\Psi_n(x)dx = 1, \qquad \ldots . (38)$$

A is called the normalizing constant and can be expressed in terms of a_{in}, b_{in}, i = 1, 2 (see Sengupta [11])

Let $f(x) = (f_1(x), f_2(x))^T$, $f^T(x)R(x)f(x) \in L(a, b)$, then the Parseval formula for the function f(x) may be written in the form

$$\int_a^b f^T(x)R(x)f(x)dx = \sum_{n=-\infty}^{\infty} \frac{1}{A}\left\{\int_a^b y_n{}^T(x)R(x)f(x)dx\right\}^2 \qquad \ldots . (39)$$

(seeBhagat [1]).

For the boundary value problem (1) – (2) with (4) over the finite interval (a, b), we introduce monotonically non-decreasing functions $\alpha_{ij}(a, b, \lambda)$, $\beta_{ij}(a, b, \lambda)$, $\gamma_{ij}(a, b, \lambda)$, i, j = 1, 2 defined by

$$\alpha_{ij}(a, b, \lambda) = \frac{1}{A}.\sum_{0<\lambda_n\leq\lambda} a_{in}\, a_{jn},\ \lambda>0,$$

$$= -\frac{1}{A}\sum_{0\geq\lambda_n>\lambda} a_{in}a_{jn},\ \lambda\leq 0; \qquad \text{. . . . (40)}$$

$$\beta_{ij}(a, b, \lambda) = \frac{1}{A}\sum_{0<\lambda_n\leq\lambda} b_{in}b_{jn},\ \lambda> 0,$$

$$= -\frac{1}{A}\sum_{0\geq\lambda_n>\lambda} b_{in}b_{jn},\ \lambda\leq 0; \qquad \text{. . . . (41)}$$

$$\gamma_{ij}(a, b, \lambda) = \frac{1}{A}.\sum_{0<\lambda_n\leq\lambda} a_{in}\, b_{jn},\ \lambda>0,$$

$$= -\frac{1}{A}\sum_{0\geq\lambda_n>\lambda} a_{in}b_{jn},\ \lambda\leq 0. \qquad \text{. . . . (42)}$$

where$\gamma_{ij}(a, b, \lambda) = \gamma_{ji}(a, b, \lambda)$, i, j = 1, 2 .

Let $\phi(x, \lambda) = (\phi_{ij}(x, \lambda))$, $\theta(x, \lambda) = (\theta_{ij}(x, \lambda))$,

$\alpha(a, b, \lambda) = (\alpha_{ij}(a, b, \lambda))$, $\beta(a, b, \lambda) = (\beta_{ij}(a, b, \lambda))$,$\gamma(a, b, \lambda) = (\gamma_{ij}(a, b, \lambda))$, i, j = 1, 2

. . . . (43)

be 2x2 matrices which are all positive for positive λ and each is continued to the negative λ - axis as odd function.

$$\text{Let } E(a, b, \lambda) \equiv (E_1(a, b, \lambda), E_2(a, b, \lambda))^T = \int_a^b \phi(x,\lambda)R(x)f(x)dx,$$

$$F(a, b, \lambda) \equiv (F_1(a, b, \lambda), F_2(a, b, \lambda))^T = \int_a^b \theta(x,\lambda)R(x)f(x)dx \qquad \text{. . . . (44)}$$

With these notations the Parseval formula (39) can be written as

$$\int_a^b f^T(x)R(x)f(x)dx = \int_{-\infty}^{\infty}[E^T(a, b, \lambda)d\alpha(a, b, \lambda)E(a, b, \lambda)$$

$$+ F^T(a, b, \lambda)d\beta(a, b, \lambda)F(a, b, \lambda) + E^T(a, b, \lambda)d\gamma(a, b, \lambda)F(a, b, \lambda)$$

$+ F^T(a, b, \lambda)d\gamma(a, b, \lambda)E(a, b, \lambda)]$. . . . . (45)

We establish the following theorem

Theorem :1 *For any positive integer N, there exists a constant A= A(N) independent of a, b such that* $\bigvee_{-N}^{N}\{\alpha_{ij}(a, b, \lambda)\}$, $\bigvee_{-N}^{N}\{\beta_{ij}(a, b, \lambda)\}$, $\bigvee_{-N}^{N}\{\gamma_{ij}(a, b, \lambda)\} < A$,

i, j = 1, 2 . . . . (46)

where $\bigvee_{-N}^{N}\{.\}$ *denotes the total variation of {.} over (-N, N).*

Proof : The solutions $\phi_i(x, \lambda)$, $\theta_i(x, \lambda)$, i = 1, 2 alongwith their first derivatives, satisfying the initial conditions (5) – (6), are continuous in x, λ and for any positive $\in < 1$ and given N there exists an h, $0 \le x \le h$ and $|\lambda| \le N$ for which

$$\left|\phi_{ij}(x,\lambda) - \delta_{ij}\right| < \frac{\in}{2}, \left|\theta'_{ij}(x,\lambda) + 2\delta_{ij}\right| < \frac{\in}{2}, \left|\phi'_{ij}(x,\lambda)\right| < \frac{\in}{2} \text{ and } \left|\theta_{ij}(x,\lambda)\right| < \frac{\in}{2}$$

. . . . (47)

where δ_{ij} are the Kronecker delta, i, j = 1, 2.

Let $\int_0^h s^2(x)dx = \frac{1}{k_1}$, $\int_0^h t^2(x)dx = \frac{1}{k_2}$ . . . . (48)

where k_1 , k_2 are some constants. Let us define a vector $g_h(x) = (g_{1h}(x), g_{2h}(x))^T$ in the following way :

$g_h(x) = (k_1 s(x), k_2 t(x))^T$, $0<x<h$, $= 0$, otherwise. . . . . (49)

Then $\int_0^h g_{1h}(x)s(x)dx = 1$,

$\int_0^h g_{2h}(x)t(x)dx = 1$ and

$s'(x)g_{1h}(x) = s(x)g_{1h}'(x)$, $t'(x)g_{2h}(x) = t(x)g_{2h}'(x)$, $0<x<h$. . . . . (50)

Put $E(h, \lambda) \equiv (E_1(h, \lambda), E_2(h, \lambda))^T = \int_0^h \phi(x,\lambda)R(x)g_h(x)dx$,

$\Phi(h, \lambda) \equiv (\Phi_1(h, \lambda), \Phi_2(x, \lambda))^T = \int_0^h \theta(x,\lambda) R(x) g_h(x)dx$. . . . .

(51)

Since, $E_1(h, \lambda) - 1 = \int_0^h \phi_{11}(x, \lambda)s(x)\, g_{1h}(x)dx + \int_0^h \phi_{12}(x)\, t(x)\, g_{2h}(x)dx - \int_0^h g_{1h}(x)s(x)dx$ .... (52)

Therefore from (52) by using (47) and (50) we obtain $|E_1(h,\lambda) - 1| < \in$

. . . . (53)

Similarly, $|E_2(h,\lambda) - 1| < \in$, $|\Phi_1(h,\lambda)| < \in$ and $|\Phi_2(h,\lambda)| < \in$. . . . . (54)

By integrating by parts and using (49) we have $\int_0^h \phi_{11}(x,\lambda)s(x)g'_{1h}(x)dx$

$$= -\int_0^h \phi'_{11}(x,\lambda)s(x)g_{1h}(x)dx - \int_0^h \phi_{11}(x)s'(x)g_{1h}(x)dx.$$

Hence by using (50) it follows that $\int_0^h \phi_{11}(x,\lambda)\, s(x)g'_{1h}(x)dx$

$$= -\frac{1}{2}\int_0^h \phi'_{11}(x, \lambda)s(x)g_{1h}(x)dx \quad \text{....(55)}$$

Similarly, $\int_0^h \phi_{12}(x,\lambda)t(x)g'_{2h}(x)dx \equiv -\frac{1}{2}\int_0^h \phi'_{12}(x,\lambda)t(x)g_{2h}(x)dx$ ....(56)

Therefore putting

$$\widetilde{E}(h,\lambda) \equiv (\widetilde{E_1}(h, \lambda), \widetilde{E_2}(h, \lambda))^T = \int_0^h \phi(x,\lambda)\, R(x) g'_h(x)dx,$$

$$\widetilde{\Phi}(h, \lambda) \equiv (\widetilde{\Phi_1}(h, \lambda), \widetilde{\Phi_2}(h, \lambda))^T = \int_0^h \theta(x,\lambda)\, R(x) g'_h(x)dx, \quad \text{. . . . (57)}$$

and using (47) it follows as before that

$$\left|\widetilde{E_1}(h,\lambda)\right| < \in,\ \left|\widetilde{E_2}(h,\lambda)\right| < \in,\ \left|\widetilde{\Phi_1}(h,\lambda) - 1\right| < \in, \text{ and } \left|\widetilde{\Phi_2}(h,\lambda) - 1\right| < \in.$$

. . . . (58)

By the Parseval formula (45) applied to $g_h(x)$ defined in (49) we have

$$\int_a^b g_h^T(x)R(x)g_h(x)dx = \int_{-N}^N [E^T(h,\lambda)d\alpha(a,b,\lambda)E(h,h)$$

$+ \Phi^T(h, \lambda) d\beta(a, b, \lambda) \Phi(h, \lambda) + E^T(h, \lambda) d\gamma(a, b, \lambda) \Phi(h, \lambda)$

$+ \Phi^T(h, \lambda) d\gamma(a, b, \lambda) E(h, \lambda)]$ . . . . (59)

Then on using (53), (54) we have

$$\int_0^h |g_h^T(x) R(x) g_h(x)| dx \geq (1 - \in)^2 \{\alpha_{11}(N) - \alpha_{11}(-N)\}$$

$$+ (1 - \in)^2 \{\alpha_{22}(N) - \alpha_{22}(-N)\} - 2(1 + \in)^2 V_{-N}^{N}\{\alpha_{12}(a, b, \lambda)\}$$

$$- 2\in^2 V_{-N}^{N}\{\beta_{12}(a, b, \lambda)\} - 2\in(1 + \in) V_{-N}^{N}\{\gamma_{11}(a, b, \lambda)\}$$

$$- 2\in(1 + \in) V_{-N}^{N}\{\gamma_{22}(a, b, \lambda)\} - 2\in(1 + \in) V_{-N}^{N}\{\gamma_{12}(a, b, \lambda)\}$$

$$- 2\in(1 + \in) V_{-N}^{N}\{\gamma_{21}(a, b, \lambda)\} \quad \text{. . . . (60)}$$

By using the Hardy-Littlewood-Polya inequality [(9), Pp – 33] it follows that if

$\rho_{ij} = \alpha_{ij}(a, b, \lambda)$

or $\beta_{ij}(a, b, \lambda)$ or $\gamma_{ij}(a, b, \lambda)$, i, j = 1, 2 as defined in (40) – (42), then

$\rho_{12}^2 \leq \rho_{11}\rho_{22}$ i. e ; $\rho_{12}^2 \leq \frac{1}{4}(\rho_{11} + \rho_{22})^2 - \frac{1}{4}(\rho_{11} - \rho_{22})^2$

i.e.; $\rho_{12} \leq \frac{1}{2}(\rho_{11} + \rho_{22})$. . . . . (61)

From this we have

$$V_{-N}^{N}\{\alpha_{12}(a, b, \lambda)\} \leq \frac{1}{2}[V_{-N}^{N}\{\alpha_{11}(a, b, \lambda)\} + V_{-N}^{N}\{\alpha_{22}(a, b, \lambda)\}],$$

$$V_{-N}^{N}\{\beta_{12}(a, b, \lambda)\} \leq \frac{1}{2}[V_{-N}^{N}\{\beta_{11}(a, b, \lambda)\} + V_{-N}^{N}\{\beta_{22}(a, b, \lambda)\}],$$

$$V_{-N}^{N}\{\gamma_{ij}(a, b, \lambda)\} \leq \frac{1}{2}[V_{-N}^{N}\{\alpha_{ij}(a, b, \lambda)\} + V_{-N}^{N}\{\beta_{ij}(a, b, \lambda)\}], \text{ i, j = 1, 2} \quad \text{. . . . (62)}$$

Therefore from (60),

$$\int_0^h |g_h^T(x) R(x) g_h(x)| dx \geq (1 - 3\in - \in^2)[\alpha_{11}(N) - \alpha_{11}(-N)$$

$$+ \alpha_{22}(N) - \alpha_{22}(-N) + (-2\in - 3\in^2)[\beta_{11}(N) - \beta_{11}(-N) + \beta_{22}(N) - \beta_{22}(-N)]$$

. . . . (63)

Applying the Parseval formula (45) to the vector g_h'(x) as before we obtain

$\int_0^h |g_h' T(x) R(x) g_h'(x)|$dx ≥ (-2∈ - 3∈^{2}) [$\alpha_{11}$(N) - α_{11}(-N)

+ α_{22}(N) - α_{22}(-N) + (1 - 3∈ - ∈^{2}) [β_{11}(N) - β_{11}(-N)

+ β_{22}(N) - β_{22}(-N)] . . . . (64)

Adding (63) and (64) it follows that

(1 - 5∈ - 4∈^{2}) [α_{11}(N) -α_{11}(-N) + α_{22}(N) - α_{22}(-N) + β_{11}(N) - β_{11}(-N) + β_{22}(N) - β_{22}(-N)]

≤$\int_0^h |g_h^T(x) R(x) g_h(x)|$dx + $\int_0^h |g_h' T(x) R(x) g_h'(x)|$ dx . . . . (65)

From this the theorem is proved for the function $\alpha_{ii}((a,b,\lambda)$, $\beta_{ii}(a,b,\lambda)$, I = 1, 2. For the other functions we use the relations (62).

The theorem is therefore proved completely.

Thus the set of functions {$\alpha_{ij}(a,b,\lambda)$}, {$\beta_{ij}(a,b,\lambda)$}and {$\gamma_{ij}(a,b,\lambda)$}, i, j = 1, 2 have uniformly bounded variation in every finite interval of λ, -μ≤λ≤μ, say. Therefore by Helly's selection principle (see Titchmarsh [97[c)], art.22.19) there exist sequences {a_k}, {b_k} with $a_k \to -\infty$, $b_k \to \infty$ and the functions $\alpha_{ij}(\lambda)$, $\beta_{ij}(\lambda)$ and $\gamma_{ij}(\lambda)$, i, j = 1, 2 having bounded variations for

-μ≤λ≤μ such that $\alpha_{ij}(a_k, b_k, \lambda)$, $\beta_{ij}(a_k, b_k, \lambda)$, and $\gamma_{ij}(a_k, b_k, \lambda)$ converges to $\alpha_{ij}(\lambda)$, $\beta_{ij}(\lambda)$ and $\gamma_{ij}(\lambda)$ respectively, for i, j = 1, 2, as $a_k \to -\infty$, $b_k \to \infty$.

Hence, as in Titchmarsh ([13)], Pp. 131), applying the method of diagonals it follows that α_{ij}(a, b, λ),

β_{ij}(a, b, λ) and γ_{ij}(a, b, λ) tend respectively to $\alpha_{ij}(\lambda)$, $\beta_{ij}(\lambda)$ and $\gamma_{ij}(\lambda)$ i, j = 1, 2 as $a_k \to -\infty$, $b_k \to \infty$.

The matrices $\alpha(\lambda) = (\alpha_{ij}(\lambda))$, $\beta(\lambda) = (\beta_{ij}(\lambda))$, $\gamma(\lambda) = (\gamma_{ij}(\lambda))$, i, j = 1, 2 . . . . (66)

are called the Spectral matrices associated with the given system(1).

In the following we define 'mean convergence' of vectors as modified by Bhagat[1] and state certain results in this connection in the form of Lemmas : (1) – (4), which will be used in the subsequent discussions.

Let H(x) be a positive definite hermitian matrix of order two and $\{f_n(x)\} \equiv \{(f_{1n}(x), f_{2n}(x))^T\}$ be a sequence of vectors real or complex, satisfying $f_n^T(x)H(x)\overline{f_n}(x) \in L(a, b)$, (a, b finite or infinite). for each n, $\overline{f_n}(x)$ being the conjugate of $f_n(x)$.

Let $f(x) = (f_1(x), f_2(x))^T$ be a given vector real or complex such that $f^T(x)H(x)\bar{f}(x) \in L(a, b)$. Then if the integral

$$\int_a^b (f_n(x) - f(x))^T H(x)(\overline{f_n}(x) - \bar{f}(x))dx \to 0, \text{ as } n \to \infty \qquad \text{. . . . (67)}$$

the function $f_n(x)$ is said to converge in mean to f(x).

The following fundamental results, analogous to those in connection with the ordinary mean convergence theorems, hold.

Lemma – 1

If asequence $\{f_n(x)\}$ of vectors $f_n(x) = (f_{1n}(x), f_{2n}(x))^T$ is given, then in order that there exists an element (vector) $f(x) = (f_1(x), f_2(x))^T$ towards which it converges in mean, it is necessary and sufficient that

$$\int (f_m(x) - f_n(x))^T H(x)(\overline{f_m}(x) - \overline{f_n}(x))dx \to 0 \qquad \text{. . . . (68)}$$

as m, n $\to\infty$ independently of each other.

Lemma – 2

$$lim_{n\to\infty} \int f_n^T(x)H(x)\overline{f_n}(x)dx = \int f^T(x)H(x)\bar{f}(x)dx \qquad \text{. . . . (69)}$$

Lemma – 3

If $f_n(x)$ converges in mean to f(x) and if g (x) be a vector, $g^T(x)H(x)g$ (x) $\in$ L, then

$$lim_{n\to\infty}\int f_n^T(x)H(x)g(x)dx = \int f^T(x)H(x)g(x)dx \quad .... (70)$$

Lemma – 4

If $f_n(x)$ converges in mean to f(x) and $g_n(x)$ converges in mean to g (x) then

$$lim_{n\to\infty}\int f_n^T(x)H(x)g_n(x)dx = \int f^T(x)H(x)g(x)dx \quad .... (71)$$

We now obtain the Parseval theorem over the interval (-∞, ∞) involving the spectral matrices $\alpha(\lambda)$, $\beta(\lambda)$ and $\gamma(\lambda)$ defined in (66).

Theorem : 2 *Let $f(x) = (f_1(x), f_2(x))^T$ be a function, $f^T(x)R(x)f(x) \in L(-\infty, \infty)$. Let the monotinic non-decreasing functions $\alpha_{ij}(\lambda)$, $\beta_{ij}(\lambda)$ and $\gamma_{ij}(\lambda)$, i,j = 1, 2 defined in (66) have bounded variations over every finite interval and do not depend upon f(x). Then*

$$\int_{-\infty}^{\infty} f^T(x)R(x)f(x)dx = \int_{-\infty}^{\infty}[E^T(\lambda)d\alpha(\lambda)E(\lambda) + F^T(\lambda)d\beta(\lambda)F(\lambda) + E^T(\lambda)d\gamma(\lambda)F(\lambda) + F^T(\lambda)d\gamma(\lambda)E(\lambda)] \quad .... (72)$$

where $E(\lambda) \equiv (E_1(\lambda), E_2(\lambda))^T = l.i.m._{n\to\infty}\int_{-n}^{n}\phi(x,\lambda)R(x)f(x)dx,$

$$F(\lambda) \equiv (F_1(\lambda), F_2(\lambda))^T = l.i.m._{n\to\infty}\int_{-n}^{n}\theta(x,\lambda)R(x)f(x)dx. \quad .... (73)$$

are the ϕ, θ - Fourier transforms of f(x) respectively, $\phi(x, \lambda)$, $\theta(x, \lambda)$ being 2x2 matrices defined in (43)

Proof : *Let the function $f(x) = (f_{1n}(x), f_{2n}(x))^T$ vanish outside the interval $-n \le x \le n$, $n< \min\{|a|, |b|\}$ and have continuous derivatives upto the second order and also satisfy the boundary conditions (2). Evidently $f_n^T(x)R(x)f_n(x) \in L(-\infty, \infty)$.*

From (39) putting $f(x) = f_n(x)$ we obtain $\int_{-n}^{n} f_n^T(x)R(x)f_n(x)dx =$

$$\sum_{k=-\infty}^{\infty}\frac{1}{A}\{\int_a^b y_k^T(x)R(x)f_n(x)dx\}^2 \quad .... (74)$$

where$y_k(x)$ are the eigenvectors corresponding to the eigenvalues λ_k.

Now, $\int_a^b y_k^T(x)R(x)f_n(x)dx = -\lambda_k^{-2}\int_a^b f_n^T(x)Ly_k(x)dx$ . . . .(75)

where L is a matrix differential operator defined in (2).

By Green's formula applied to $f_n(x)$, $y_k(x)$ it follows from (74) that

$$\int_a^b y_k^T(x)R(x)f_n(x)dx = -\lambda_k^{-2}\int_a^b y_k^T(x)Lf_n(x)dx \quad . . . . (76)$$

$$= -\lambda_k^{-2}\int_a^b y_k^T(x)R(x)h_n(x)dx, \text{ say} \quad . . . . (77)$$

where $h_n(x) = ([f_{1n}''(x) + p(x)f_{1n}(x) + r(x)f_{2n}(x)] / s(x), [f_{2n}''(x) + q(x)f_{2n}(x) + r(x)f_{1n}(x)]/t(x))^T$ and in view of the definition of $f_n(x)$, $h_n^T(x) R(x)h_n(x) \in L(-n, n)$

. . . . (78)

Applying Bessel's inequality (see Bahat[1]) it also follows that

$$\sum_{k=-\infty}^{\infty}\frac{1}{A}\{\int_a^b y_k^T(x)R(x)h_n(x)dx\}^2 \le \int_{-n}^{n}|h_n^T(x)R(x)h_n(x)|dx \quad . . . . (79)$$

Hence, $\sum_{|\lambda_k|\ge\mu}\frac{1}{A}\{\int_a^b y_k^T(x)R(x)f_n(x)dx\}^2 \le \mu^{-4}\int_{-n}^{n}|h_n^T(x)R(x)h_n(x)|\,dx$

Therefore, $\left|\sum_{k=-\infty}^{\infty}\frac{1}{A}\{\int_a^b y_k^T(x)R(x)f_n(x)dx\}^2 - \sum_{-\mu\le\lambda_k\le\mu}\frac{1}{A}\{\int_a^b y_k^T(x)R(x)f_n(x)dx\}^2\right|$

$\le \mu^{-4}\int_{-n}^{n}|h_n^T(x)R(x)h_n(x)|dx$ . . . . (80)

From (80), using (75) and (45) we obtain

$$\Big|\int_{-n}^{n} f_n^T(x)R(x)f_n(x)dx - \int_{-\mu}^{\mu}[E_n^T(\lambda)d\alpha(a,b,\lambda)E_n(\lambda) + F_n^T(\lambda)d\beta(a,b,\lambda)F_n(\lambda) +$$

$$E_n^T(\lambda)d\gamma(a,b,\lambda)F_n(\lambda) + F_n^T(\lambda)d\gamma(a,b,\lambda)E_n(\lambda)]\Big| \le \mu^{-4}\int_{-n}^{n}|h_n^T(x)R(x)h_n(x)|dx$$

. . . . (81)

where $E_n(\lambda) = \int_{-n}^{n}\phi(x,\lambda)R(x)f_n(x)dx$,

$F_n(\lambda) = \int_{-n}^{n}\theta(x,\lambda)R(x)f_n(x)dx$ . . . . (82)

Since the functions $\alpha_{ij}(a, b, \lambda)$, $\beta_{ij}(a, b, \lambda)$, $\gamma_{ij}(a, b, \lambda)$ tend respectively to $\alpha_{ij}(\lambda),\beta_{ij}(\lambda),\gamma_{ij}(\lambda)$, i j = 1, 2 as

$a \to -\infty$, $b \to \infty$, it follows from (81) that

$$\Big|\int_{-n}^{n} f_n^T(x)R(x)f_n(x)dx - \int_{-\mu}^{\mu}[E_n^T(\lambda)d\alpha(\lambda)E_n(\lambda) + F_n^T(\lambda)d\beta(\lambda)F_n(\lambda) + E_n^T(\lambda)d\gamma(\lambda)F_n(\lambda) + F_n^T(\lambda)d\gamma(\lambda)E_n(\lambda)\Big| \le \mu^{-4}. \int_{-n}^{n}|h_n^T(x)R(x)h_n(x)|dx \quad \ldots. (83)$$

Thus the equation (68) follows for the special type of functions $f_n(x)$ as $\mu\to\infty$.

Now let $f(x) = (f_1(x), f_2(x))^T$ be an arbitrary vector, $f^T(x)R(x)f(x)\in L(-\infty, \infty)$. Then we approximate f(x) in mean square by the sequence $\{f_n(x)\}$ of vectors $f_n(x) = (f_{1n}(x), f_{2n}(x))^T$ satisfying the preceding conditions.

Hence, $$\lim_{n\to\infty}\int_{-n}^{n}(f(x) - f_n(x))^T R(x)(f(x) - f_n(x))dx = 0 \quad \ldots. (84)$$

so that as m, $n \to\infty$, we obtain

$$\int_{-\infty}^{\infty}(f_n(x) - f_m(x))^T R(x)(f_n(x) - f_m(x))dx \to 0 \quad \ldots. (85)$$

Therefore, as m, $n \to\infty$, we have

$$\int_{-\infty}^{\infty}[(E_n(\lambda) - E_m(\lambda))^T d\alpha(\lambda)(E_n(\lambda) - E_m(\lambda)) + (F_n(\lambda) - F_m(\lambda))^T \alpha\beta(\lambda)(F_n(\lambda) - F_m(\lambda)$$

$$+ (E_n(\lambda) - E_m(\lambda))^T d\gamma(\lambda)(F_n(\lambda) - F_m(\lambda)) + (F_m(\lambda) - F_m(\lambda))^T \alpha\gamma(\lambda)(E_n(\lambda) - E_m(\lambda))]$$

$$\equiv\int_{-\infty}^{\infty}(f_n(x) - f_m(x))^T R(x)(f_n(x) - f_m(x))dx \to 0 \quad \ldots. (86)$$

which implies the existence of the limit functions $E(\lambda)$, $F(\lambda)$ satisfying the equation (72).

We now show that as $n \to\infty$, the functions $E_n(\lambda)$, $F_n(\lambda)$ as defined in (82) converge in mean to $E(\lambda)$, $F(\lambda)$ respectively.

Hence we have,

$\int_{-\infty}^{-n}[(E(\lambda) - E_n(\lambda))^T d\alpha(\lambda)(E(\lambda) - (E_n(\lambda))$

$+ (F(\lambda) - F_n(\lambda))^T d\beta(\lambda)(F(\lambda) - F_n(\lambda)) - (E(\lambda) - E_n(\lambda))^T d\gamma(\lambda)(F(\lambda) - F_n(\lambda))$

$+ (F(\lambda) - F_n(\lambda))^T d\gamma(\lambda)(E(\lambda) - E_n(\lambda))$

$= \int_{-\infty}^{\infty}[(f(x) - f_n(x))^T R(x)(f(x) - f_n(x))dx$

$$= \int_{-\infty}^{-n} f^T(x)R(x)f(x)dx + \int_{n}^{\infty} f^T(x)R(x)f(x)dx \qquad .\,.\,.\,.\ (87)$$

From this as $n \to \infty$,

$\int_{-\infty}^{\infty}[(E(\lambda) - E_n(\lambda))^T d\alpha(\lambda)(E(\lambda) - (E_n(\lambda))$

$+ (F(\lambda) - F_n(\lambda))^T d\beta(\lambda)(F(\lambda) - F_n(\lambda)) + (E(\lambda) - E_n(\lambda))^T d\gamma(\lambda)(F(\lambda) - F_n(\lambda))$

$$+ (F(\lambda) - F_n(\lambda))^T d\gamma(\lambda)(E(\lambda) - E_n(\lambda)) \to 0 \qquad .\,.\,.\,.\ (88)$$

which proves the mean convergence of $E_n(\lambda)$, $F_n(\lambda)$ to $E(\lambda)$, $F(\lambda)$ respectively. Thus the theorem is proved completely.

Let f(x), g (x) be two vectors satisfying $f^T(x)R(x)f(x)$, $g^T(x)R(x)g(x) \in L(-\infty, \infty)$. Let $E(\lambda)$, $F(\lambda)$ and $\tilde{E}(\lambda)$,$\tilde{F}(\lambda)$ be the ϕ, θ Fourier transforms of f(x) and g (x) respectively. Then ϕ, θ Fourier transforms of f(x) $\pm g$ (x) are $E(\lambda) \pm \tilde{E}(\lambda)$, $F(\lambda) \pm \tilde{F}(\lambda)$ respectively. Thus the generalized Parseval theorem for f(x), g(x) follows in the usual manner and we have

$$\int_{-\infty}^{\infty} f^T(x)R(x)g(x)dx = \int_{-\infty}^{\infty}[E^T(\lambda)d\alpha(\lambda)\tilde{E}(\lambda) + F^T(\lambda)d\beta(\lambda)\tilde{F}(\lambda) + E^T(\lambda)d\gamma(\alpha)\tilde{F}(\lambda) + F^T(\lambda)d\gamma(\lambda)\tilde{E}(\lambda)] \qquad .\,.\,.\,.\ (89)$$

4. The expansion theorem

The following expansion theorem is now obtained.

Theorem –3 *Let $f(x) = (f_1(X), f_2(x))^T$ be a continuously differentiable function on $-\infty<x<\infty$. Then*

$$f(x) = \int_{-\infty}^{\infty}[\phi^T(x,\lambda)\, d\alpha(\lambda)E(\lambda) + \theta^T(x,\lambda)\, d\beta(\lambda)F(\lambda) + \phi^T(x,\lambda)\, d\gamma(\lambda)F(\lambda) + \theta^T(x,\lambda)d\gamma(\lambda)E(\lambda)] \quad \text{. . . . (90)}$$

where $E(\lambda)$, $F(\lambda)$, $\alpha(\lambda)$, $\beta(\lambda)$, $\gamma(\lambda)$ are those defined in the previous section. The integral is uniformly convergent for x in any finite interval.

Proof: Let $g(x) = (g_1(x), g_2(x))^T$ be a continuous function which satisfies

$$R(x)g(x) = \in_j, \; j = 1, 2, \text{ for } x\in(x, x+h), h > 0$$
$$= 0, \text{ otherwise,} \quad \text{. . . . (91)}$$

where $\in_1 = (1, 0)^T$, $\in_2 = (0, 1)^T$.

From (89) and (90) it follows that

$$\int_x^{x+h} f^T(x)dx = \int_{-\infty}^{\infty}[E^T(\lambda)d\alpha(\lambda)\int_x^{x+h}\phi(x,\lambda)dx + F^T(\lambda)d\beta(\lambda)\int_x^{x+h}\theta(x,\lambda)dx$$
$$+ E^T(\lambda)d\gamma(\lambda)\int_x^{x+h}\theta(x,\lambda)dx + F^T(\lambda)d\gamma(\lambda)\int_x^{x+h}\phi(x,\lambda)dx] \quad \text{. . . . (92)}$$

Applying Gronwall inequality (see Titchmarsh [14]Pp – 97], Chakravarty and Sengupta [8]) to the equation (21) of Sengupta [11] we obtain

$$\|\phi(x,\lambda)\| \le \lambda^{-1}.\, D.\, \exp.\{\int_0^x \|Q(z) - Q_0(z)\| dz\} \quad \text{. . . . (93)}$$

where D is a certain constant, $Q(z)$, $Q_0(z)$ are the 2x2 matrices as given in (1) and (9) of Sengupta[11] and for the matrix $A = (a_{ij})$, i, j = 1, 2, we mean $\|A\| = \sum_{1\le i,j\le 2}|a_{ij}|$. Thus it follows that $\phi(x, \lambda)$ is bounded uniformly for x in any finite interval, where $|\lambda|>\delta> 0$.

Therefore $\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}\phi(x,\lambda)dx$ converges to $\phi(x, \lambda)$ in $\lambda(-\infty<\lambda<\infty)$ for all x in any finite interval. Similar arguments hold for the $\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}\theta(x,\lambda)dx$.

Dividing both sides of the equality (92) by h and taking limit as h→0, the theorem follows by making use of the result $\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(x)dx$ = f(x), holds as f(x) is continuously differentiable.

Conflict of Interests

The author declares that there is no conflict of interests.

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