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WAVELET ANALYSIS OF ONE ORDER LINEAR STOCHASTIC SYSTEM

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Abstract: Time series model is a sort of important stochastic system, and is a useful stochastic process in practices. In this paper, we use wavelet alternation to study one order linear stochastic system. We investigate its relative properties, stationary properties under the Haar wavelet, density and wavelet expansion.

Keywords: one order linear stochastic system; wavelet alternation; wavelet density degree.

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1. Introduction

The statistical characteristics of stochastic process under wavelet transform are important applications of wavelet analysis. Linear stochastic system is an important stochastic process. We study the one order linear stochastic system, get some statistics properties and make the wavelet transform under Haar wavelet.

Definition 1: Suppose $X(t)$ is a stochastic process. Its wavelet transform is:

$$w(s,x)=\frac{1}{s}\int_R x(t)\varphi\left(\frac{x-t}{s}\right)dt \qquad (1.1)$$

ϕ is a continuous wavelet.

Definition 2: Set $\{X(t);t\in R\}$ is the following equation:

$$\begin{cases} \dfrac{dX(t)}{dt}=-X(t)+N(t), & t\geq 0 \\ X_0=X(0) \end{cases} \qquad (1.2)$$

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And $\{N(t);t\in R\}$ is Smooth noise. We called $\{X(t);t\in R\}$ one order linear stochastic system.

So

$$\begin{cases} EN(t)=0 \\ \operatorname{cov}(N(t),N(t+\tau))=e^{-|\tau|} \end{cases} \tag{1.3}$$

As we known, (1.2) is asymptotically stable.

And the solution of (1.2) is:

$$X(t)=e^{-(t-t_0)}X_0+\int_{t_0}^{t}e^{-(t-\tau)}N(\tau)d\tau=e^{-t}X_0+\int_{t_0}^{t}e^{-(t-\tau)}N(\tau)d\tau \tag{1.4}$$

2. The statistics properties of the linear stochastic system

From the above, we know: $E(X(t))=E[e^{-t}X_0+\int_{t_0}^{t}e^{-(t-\tau)}N(\tau)d\tau]=e^{-t}X_0$

So : $$\lim_{t\to\infty}E(X(t))=0 \tag{2.1}$$

$$\begin{aligned}
\Gamma(u,v)&\triangleq\operatorname{cov}(X(u)X(v))\\
&=E[X(u)-EX(u)][X(v)-EX(v)]\\
&=E\int_0^u e^{-(u-\tau_1)}N(\tau_1)d\tau_1\cdot\int_0^v e^{-(v-\tau_2)}N(\tau_2)d\tau_2\\
&=e^{-(u+v)}\int_0^u\int_0^v e^{\tau_1+\tau_2}EN(\tau_1)\cdot N(\tau_2)d\tau_1 d\tau_2\\
&=e^{-(u+v)}\int_0^u\int_0^v e^{\tau_1+\tau_2}e^{-|\tau_1-\tau_2|}d\tau_1 d\tau_2\\
&=e^{-(u+v)}\left[\iint_{\tau_1\ge\tau_2}e^{2\tau_2}d\tau_1 d\tau_2+\iint_{\tau_1<\tau_2}e^{2\tau_1}d\tau_1 d\tau_2\right]\\
&=e^{-(u+v)}\left[\int_0^v\int_{\tau_2}^u e^{2\tau_2}d\tau_1 d\tau_2+\int_0^v\int_0^{\tau_2}e^{2\tau_1}d\tau_1 d\tau_2\right]\\
&=e^{-(u+v)}\left[\frac{1}{2}(u-v)(e^{2v}-1)+\frac{1}{2}e^{2v}-\frac{1}{2}-v\right]
\end{aligned}$$

Then:

$$\begin{aligned}
R(u,v)&=E(X(u)X(v))\\
&=E(e^{-u}X_0+\int_0^u e^{-(u-\tau_1)}N(\tau_1)d\tau_1)\cdot(e^{-v}X_0+\int_0^v e^{-(v-\tau_2)}N(\tau_2)d\tau_2)\\
&=e^{-(u+v)}X_0+\Gamma(u,v)=e^{-(u+v)}\left[\frac{1}{2}(u-v)(e^{2v}-1)+\frac{1}{2}e^{2v}-\frac{1}{2}-v+X_0\right]
\end{aligned} \tag{2.2}$$

Set $u=v$, Obviously $\lim_{u\to\infty}Var(X(u))=\frac{1}{2}$

We can see, the one order linear stochastic system is an asymptotically wide stable second moment stochastic process.

3. Correlation under wavelet transform

Consider the one order linear stochastic system of the statistical characteristics of wavelet transform.

$$Ew(s,x)=\frac{1}{s}\int_R EX(t)\varphi(\frac{x-t}{s})dt$$

$$R(\tau)=E(w(s,x)w(s,x+\tau))$$
$$=\frac{1}{s^2}\iint_{R^2}E(x(u)x(v))\varphi(\frac{x-u}{s})\varphi(\frac{x+\tau-v}{s})du\,dv$$

For example, we discuss the Haar wavelet transform.

Definition 3[4] **:** Set $$\phi(x)=\begin{cases}1, & 0\le x<1/2\\ -1, & 1/2\le x\le 1\\ 0, & else\end{cases} \qquad (3.1)$$

We call it Haar wavelet.

Then, we have

$$\phi(\frac{x-t}{s})=\begin{cases}1, & x-\frac{s}{2}\le t<x\\ -1, & x-s\le t\le x-\frac{s}{2}\end{cases}$$

$$\phi(\frac{x+\tau-t}{s})=\begin{cases}1, & x+\tau-\frac{s}{2}\le t<x+\tau\\ -1, & x+\tau-s\le t\le x+\tau-\frac{s}{2}\end{cases}$$

Set $I_1=(x-\frac{s}{2},x)$, $I_2=(x-s,x-\frac{s}{2})$

$I_3=(x+\tau-\frac{s}{2},x+\tau)$, $I_4=(x+\tau-s,x+\tau-\frac{s}{2})$

We have [5] (see (1.1)):

$$Ew(s,x)=\frac{1}{s}\int_R EX(t)\phi(\frac{x-t}{s})dt$$
$$=\frac{1}{s}\int_R e^{-t}X_0\phi(\frac{x-t}{s})dt$$
$$=\frac{1}{s}\int_{x-\frac{s}{2}}^{x}e^{-t}X_0dt-\frac{1}{s}\int_{x-s}^{x-\frac{s}{2}}e^{-t}X_0dt$$
$$=-\frac{e^{-x}X_0}{s}(e^{\frac{s}{2}}-1)^2$$

So $$\lim_{x\to\infty}Ew(s,x)=0 \qquad (3.2)$$

$$
\begin{aligned}
R(\tau) &= E(w(s,x))(w(s,x+\tau)) \\
&= \frac{1}{s^2}\iint_{R^2} EX(u)X(v)\phi(\frac{x-u}{s})\phi(\frac{x+\tau-v}{s})dudv \\
&= \frac{1}{s^2}(\int_{I_1}\int_{I_3} - \int_{I_1}\int_{I_4} - \int_{I_2}\int_{I_3} + \int_{I_2}\int_{I_4} \ )EX(u)X(v)dudv \\
&= \frac{1}{s^2}\left\{R_1(\tau) - R_2(\tau) - R_3(\tau) + R_4(\tau)\right\}
\end{aligned}
$$

And:

$$
\begin{aligned}
R_1(\tau) &= \int_{I_1}\int_{I_3} EX(u)X(v)dudv, \\
R_2(\tau) &= \int_{I_1}\int_{I_4} EX(u)X(v)dudv \\
R_3(\tau) &= \int_{I_2}\int_{I_3} EX(u)X(v)dudv, \\
R_4(\tau) &= \int_{I_2}\int_{I_4} EX(u)X(v)dudv
\end{aligned}
$$

Consider:

$$
\begin{aligned}
R_1(\tau) &= \int_{I_1}\int_{I_3} e^{-(u+v)}\left[\frac{1}{2}(e^{2v}-1)(u-v) + \frac{1}{2}e^{2v} - v - \frac{1}{2} + X_0\right]dudv \\
&= e^{-u}\Big|_{I_1}\left[e^{v}\Big|_{I_3}\cdot(-\frac{u}{2}\Big|_{I_1} - \frac{3}{2}) + \cdot e^{-v}\Big|_{I_3}\cdot(-\frac{u}{2}\Big|_{I_1} - \frac{3}{2} + X_0)\right] \\
&= (-\frac{s}{4} - \frac{3}{2})e^{\tau}(1-e^{\frac{s}{2}})(1-e^{-\frac{s}{2}}) + (-\frac{s}{4} - \frac{3}{2} + X_0)e^{-2x-\tau}(1-e^{\frac{s}{2}})^2
\end{aligned}
$$

Similar:

$$
R(\tau) = \frac{1}{s^2}(-\frac{s}{4} - \frac{3}{2})e^{\tau}(1-e^{\frac{s}{2}})^2(1-e^{-\frac{s}{2}})^2 + \frac{1}{s^2}(-\frac{s}{4} - \frac{3}{2} + X_0)e^{-2x-\tau}(1-e^{\frac{s}{2}})^4 \qquad (3.3)
$$

So
$$
\lim_{x\to\infty} R(\tau) = \frac{1}{s^2}(-\frac{s}{4} - \frac{3}{2})e^{\tau}(1-e^{\frac{s}{2}})^2(1-e^{-\frac{s}{2}})^2 \qquad (3.4)
$$

From above all, we can get these following conclusions.

Theorem3.1: $w(s,x)$ is an asymptotically wide stable second moment stochastic process. Its statistics property is related with τ :

Proof:
$$
\lim_{x\to\infty} Ew(s,x) =)
$$

$$
\lim_{x\to\infty} R(\tau) = \frac{1}{s^2}(-\frac{s}{4} - \frac{3}{2})e^{\tau}(1-e^{\frac{s}{2}})^2(1-e^{-\frac{s}{2}})^2
$$

Theorem3.2: The zero density under ultimate status k_1 of $w(s,x)$ is: $k_1 = \frac{1}{\pi}$,The average density under ultimate status k_2 is: $k_2 = \frac{1}{\pi}$.

Proof:
$$
\tilde{R}(\tau) \triangleq \lim_{x\to\infty} R(\tau) = \frac{1}{s^2}(-\frac{s}{4} - \frac{3}{2})(1-e^{\frac{s}{2}})^2(1-e^{-\frac{s}{2}})^2 e^{\tau}
$$

$$\tilde{R}(0)=\frac{1}{s^2}\left(-\frac{s}{4}-\frac{3}{2}\right)(1-e^{\frac{s}{2}})^2(1-e^{-\frac{s}{2}})^2$$

Because: $\tilde{R}'(\tau)=\tilde{R}(\tau),\therefore \tilde{R}''(\tau)=\tilde{R}(\tau)\ \therefore \tilde{R}''(0)=\tilde{R}(0)$

$$\tilde{R}^{(4)}(\tau)=\tilde{R}(\tau),\therefore \tilde{R}^{(4)}(0)=\tilde{R}(0)$$

So the zero density $k_1=\sqrt{\left|\frac{\tilde{R}''(0)}{\pi^2\tilde{R}(0)}\right|}=\frac{}{\pi}$

the average density $k_2=\sqrt{\left|\frac{\tilde{R}^{(4)}(0)}{\pi^2\tilde{R}''(0)}\right|}=\frac{1}{\pi}$.

4. Wavelet expansion of system

In order to use the idea of multi-resolution, we will start by defining the scaling function and then the wavelet in terms of it.

Let real function φ is standard orthogonal element of multi-resolution analysis $\{V_j\}_{j\in Z}$, then exist $h_k\in l^2$, we have:

$$\varphi(t)=\sqrt{2}\sum_{k\in Z}h_k\varphi(2t-k)$$

Then $$\phi(t)=\sqrt{2}\sum_{k\in Z}(-1)^k\overline{h_{1-k}}\varphi(2t-k)$$

Then the wavelet express of $y(t)$ in mean square is:

$$y(t)=2^{-\frac{j}{2}}\sum_K c_n^j\varphi(2^{-j}t-n)+2^{-\frac{j}{2}}\sum_K d_n^j\phi(2^{-j}t-n) \tag{4.1}$$

Where: $$c_n^j=2^{-\frac{j}{2}}\int_R y(t)\varphi(2^{-j}t-n)dt$$

$$d_n^j=2^{-\frac{j}{2}}\int_R y(t)\phi(2^{-j}t-n)dt$$

Then we have:

$$E[d_n^jd_m^k]=2^{-\frac{j+k}{2}}\iint_{R^2}E[y(t)y(s)]\phi(2^{-j}t-n)\phi(2^{-k}s-m)dtds$$
$$E[c_n^jc_m^k]=2^{-\frac{j+k}{2}}\iint_{R^2}E[y(t)y(s)]\varphi(2^{-j}t-n)\varphi(2^{-k}s-m)dtds$$

Where

$$\phi(2^{-j}t-n)=\begin{cases}1, & n2^{-j}\le t<(n+1/2)2^{-j}\\ -1, & (n+1/2)2^{-j}\le t<(n+1)2^{-j}\end{cases} \tag{4.2}$$

So we can obtain value of $E[d_n^jd_m^k]$.

If we let normalized scaling function to compact support over [0,1], then a solution is a scaling function that is a simple rectangle function;

$$\varphi(t)=\begin{cases}1\ ,\ 0\le t\le \\ 0,\quad else\end{cases}$$

So we get:

$$\varphi(2^j t-n)=\begin{cases}1,\ n2^j\ \le t\le (n+1)2^j 1 \\ 0,\qquad\quad else\end{cases} \tag{4.3}$$

Then we can obtain value of $E[c_n^j c_m^k]$.

Conflict of Interests

The author declares that there is no conflict of interests.

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