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CONTINUOUS HYBRID BLOCK STOMER COWELL METHODS FOR SOLUTIONS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we present continuous hybrid stomer cowell methods. Using the method of collocation and interpolation of power series approximate solution to derive a continuous linear multistep method. Block method was later used to generate the non overlapping solutions at selected grid points. The method developed, is consistent, zero-stable and convergent. The performance of the new block method was tested with some second order initial value problems and it was found to compare favourably with the existing methods.

Keywords: collocation, differential system, interpolation, zero stable, convergent.

2000 AMS Subject Classification: 65L05, 65L06, 65D30

1. Introduction

Consider the second order initial value problem of the form

$$y'' = f(x,y,y'),\ y(x_0) = y_0,\ y'(x_0) = y'_0, \tag{1.1}$$

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where x_0 is the initial point and f is continuous within the interval of integration and satisfies the existence and uniqueness condition.

Conventionally equation (1.1) is usually reduced to systems of first order differential equations before an approximate method is then applied to solve it. Some of the disadvantages of method of reduction include; the writing of complicated computer code which always consume longer time and more human effort [1].

Direct method of solving higher order ordinary differential equations in predictor-corrector mode have been studied by many scholars including [2], [3], [4], [5] to mention but a few. Although their methods yielded good results but the implementation is too costly, because the predictors are developed in the same way as the correctors and subroutines are very complicated to write since they require special techniques to supply the starting values. This eventually leads to longer computer time and human effort. In order to circumvent the set backs of the predictor-corrector methods, [11] among others proposed block method for solving general second order initial value problems of ordinary differential equation. The method is capable of giving evaluations at different grids points without overlapping as in the predictor-corrector method, hence it does not require the development of separate predictors.

Schoars later developed one step method so as to improve on the accuracy of Linear Multistep Method. It was discovered that as the step length is reducing the efficiency of the method increases as well. Among the authors that worked in this area are [7], [10] and [11].

In this article, we propose a method which combines the properties of one step method implemented in block method for the solution of second order initial value problems.

2. Methodology

We consider a power series approximate solution in the form:

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j, \tag{2.1}$$

where r and s are the numbers of interpolation and collocation points respectively. The second derivative of (2.1) gives

$$y^{''}(x) = \sum_{j=2}^{r+s-1} j(j-1)a_j x^{j-2}. \tag{2.2}$$

Substituting (2.2) into (1.1) gives

$$f\left(x,y,y'\right) = \sum_{j=2}^{r+s-1} j(j-1)a_j x^{j-2}. \tag{2.3}$$

Interpolating (2.1) at the $x_{n+r}, r = \frac{4}{6}, \frac{5}{6}$ and collocating (2.3) at $x_{n+s}, s = 0, \frac{2}{6}, \frac{4}{6}, 1$ gives a system of non linear equation of the form

$$AX = U, \tag{2.4}$$

where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}^T,$$

$$U = \begin{bmatrix} y_{n+\frac{4}{6}} & y_{n+\frac{5}{6}} & f_n & f_{n+\frac{1}{4}} & f_{n+\frac{2}{6}} & f_{n+\frac{4}{6}} & f_{n+1} \end{bmatrix}^T,$$

and

$$X = \begin{bmatrix} 1 & x_{n+\frac{2}{3}} & x^2_{n+\frac{2}{3}} & x^3_{n+\frac{2}{3}} & x^4_{n+\frac{2}{3}} & x^5_{n+\frac{2}{3}} \\ 1 & x_{n+\frac{5}{6}} & x^2_{n+\frac{5}{6}} & x^3_{n+\frac{5}{6}} & x^4_{n+\frac{5}{6}} & x^5_{n+\frac{5}{6}} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{3}} & 12x^2_{n+\frac{1}{3}} & 20x^3_{n+\frac{1}{3}} \\ 0 & 0 & 2 & 6x_{n+\frac{2}{3}} & 12x^2_{n+\frac{2}{3}} & 20x^2_{n+\frac{2}{3}} \\ 0 & 0 & 2 & 6x_{n+1} & 12x^2_{n+1} & 20x^2_{n+1} \end{bmatrix}.$$

Solving (2,4) for $a_j's$ using Guassian elimination method and substituting back into (2,1) gives a continuous linear multistep method which when solved for the independent solution at the grid points gives a continuous block formula of the form

$$y_{n+j} = y(x) = \sum_{m=0}^{1} \frac{(jh)^{(m)}}{m!} y_n^{(m)} + h^2 \sum_{j=0}^{1} \sigma_j f_{n+j} + \sigma_{\frac{1}{3}} f_{n+\frac{1}{3}} + \sigma_{\frac{2}{3}} f_{n+\frac{2}{3}} \tag{2.5}$$

the coefficient of f_{n+j} are given by

$$\sigma_0 = \frac{-1}{120}\left(27t^5 - 90t^4 + 110t^3 - 60t^2\right),$$

$$\sigma_{\frac{1}{3}} = \frac{1}{40}\left(27t^5 - 75t^4 + 60t^3\right),$$

$$\sigma_{\frac{2}{3}} = \frac{-1}{40}\left(27t^5 - 60t^4 + 30t^3\right),$$

$$\sigma_1 = \frac{1}{120}\left(27t^5 - 45t^4 + 20t^3\right),$$

evaluating (2,5) a $t = 0.\frac{1}{3}, \frac{2}{3}, 1$ gives a discrete block formula of the form

$$\mathbf{A}^{(0)}\mathbf{Y}_m^{(i)} = \sum_i^1 e_i y_n^{(i)} + h^{2-i}\left[df(y_n) + bF(Y_m)\right], \tag{2.6}$$

where i is the power of the derivative and $A^0 = 6\times 6$ identical matrix.

$$\begin{aligned}
\mathbf{Y}_m &= \begin{bmatrix} y_{n+\frac{1}{6}} & y_{n+\frac{1}{3}} & y_{n+\frac{1}{2}} & y_{n+\frac{2}{3}} & y_{n+\frac{5}{6}} & y_{n+1} \end{bmatrix}^T,\\
\mathbf{y}_n^{(i)} &= \begin{bmatrix} y_{n-\frac{1}{6}} & y_{n-\frac{1}{3}} & y_{n-\frac{1}{2}} & y_{n-\frac{2}{3}} & y_{n-\frac{5}{6}} & y_n \end{bmatrix}^T,\\
F(Y_m) &= \begin{bmatrix} f_n & f_{n+\frac{1}{6}} & f_{n+\frac{1}{3}} & f_{n+\frac{1}{2}} & f_{n+\frac{2}{3}} & f_{n+\frac{5}{6}} & f_{n+1} \end{bmatrix}^T,\\
f(y_n) &= \begin{bmatrix} f_n & f_{n-\frac{1}{6}} & f_{n-\frac{1}{3}} & f_{n-\frac{1}{2}} & f_{n-\frac{2}{3}} & f_{n-\frac{5}{6}} & f_{n-1} \end{bmatrix}^T.
\end{aligned}$$

When $i = 0$,

$$e_0 = \begin{bmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1 \end{bmatrix}, e_1 = \begin{bmatrix} 0&0&0&0&0&\frac{1}{6}\\ 0&0&0&0&0&\frac{1}{3}\\ 0&0&0&0&0&\frac{1}{2}\\ 0&0&0&0&0&\frac{2}{3}\\ 0&0&0&0&0&\frac{5}{6}\\ 0&0&0&0&0&1 \end{bmatrix},$$

$$d_0 = \begin{bmatrix} 0&0&0&0&0&\frac{1057}{103680}\\ 0&0&0&0&0&\frac{97}{3240}\\ 0&0&0&0&0&\frac{193}{3840}\\ 0&0&0&0&0&\frac{28}{405}\\ 0&0&0&0&0&\frac{1825}{20736}\\ 0&0&0&0&0&\frac{13}{120} \end{bmatrix}, \qquad b_0 = \begin{bmatrix} 0&\frac{193}{34560}&0&\frac{-83}{34560}&0&\frac{53}{106680}\\ 0&\frac{19}{540}&0&\frac{-13}{1080}&0&\frac{1}{405}\\ 0&\frac{117}{1280}&0&\frac{-27}{1280}&0&\frac{17}{3840}\\ 0&\frac{22}{135}&0&\frac{-2}{135}&0&\frac{-2}{405}\\ 0&\frac{1625}{6912}&0&\frac{125}{6912}&0&\frac{125}{20736}\\ 0&\frac{3}{10}&0&\frac{3}{40}&0&\frac{1}{60} \end{bmatrix}.$$

When $i = 1$,

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad d_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{119}{1152} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{128} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{9} \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{128} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \end{bmatrix},$$

$$b_1 = \begin{bmatrix} 0 & \frac{107}{1152} & 0 & \frac{-43}{1152} & 0 & \frac{1}{128} \\ 0 & \frac{19}{72} & 0 & \frac{-5}{72} & 0 & \frac{1}{72} \\ 0 & \frac{51}{128} & 0 & \frac{-3}{128} & 0 & \frac{1}{128} \\ 0 & \frac{4}{9} & 0 & \frac{1}{9} & 0 & 0 \\ 0 & \frac{475}{1152} & 0 & \frac{325}{1152} & 0 & \frac{25}{1152} \\ 0 & \frac{3}{8} & 0 & \frac{3}{8} & 0 & \frac{1}{8} \end{bmatrix}.$$

3. Analysis of our new Schemes

Let the linear operator $\mathfrak{L}\{y(x):h\}$ on (2.6) as

$$\mathfrak{L}\{y(x):h\} = \mathbf{A}^0 y_m^{(i)} - \sum_{i=0}^{1-i} h^i e_i y_n^{(i)} - h^{2-i}\ [df(y_n) + bF(y_m)] \tag{2.7}$$

Expanding y_{n+j} and f_{n+j} in Taylor series and comparing the coefficients of h gives

$$\mathfrak{L}\{y(x):h\} = C_0 y(x) + C_1 y^1(x) + ... + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + ...$$

The linear operator $\mathfrak{L}$ and associated block method are said to be of order p if $C_0 = C_1 = ... = C_p = C_{p+1} = 0$ $C_{p+2} \neq 0.C_{p+2}$ is called the error constant and implies that the truncation error is given by $t_{n+k} = C_{p+2} h^{p+2} y^{p+2}(x) + O\left(h^{p+3}\right)$. Comparing the coefficient of h , the order of the method is five with error constant

$$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = 0$$

$$c_6 = \left[\frac{-49}{11197440}, \frac{-7}{349920}, \frac{-1}{27640}, \frac{-1}{21870}, \frac{-125}{2239488}, \frac{-1}{12960}\right].$$

4. Consistency

A method is said to be consistent, if it has order greater than one. From the above analysis, it is obvious that our method is consistent.

4.1. Zero stability

A block method is said to be zero stable as $h \to 0$ the $r_j,\ j = 1(1)\ k$ of the first characteristics polynomial $\rho(r) = 0$ that is $\left|\left[\sum A^0 R^{k-1}\right]\right| \le 1$, for those root with $|R| = 1$ must be simple

For our method

$$\rho(r) = \left| r \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0,$$

$r^5(r-1) = 0$. Hence our method gives roots $[0,0,0,0,0,1]$ therefore it is zero stable.

4.2. Convergence

The necessary and sufficient conditions for a linear multistep method to be convergent is that it must be consistent and zero stable. Hence our method is convergent.

5. Numerical Examples

In this section, we test the efficiency of our method on some numerical examples.

Problem 1. $y'' + y = 0,\ 0 \le x \le 0.2,$

$y(0) = 1,\ y'(0) = 1 \quad h = 0.1$

Exact Solution: $y(x) = \cos x + \sin x$

Source: [3]

Problem 2. $y'' - 100y = 0,\ \ 0 \le x \le 0.02,$

$y(0) = 1,\ \ y'(0) = -10 \quad h = 0.01$

Exact Solution: $y(x) = \exp(-10x)$

Source: [9]

Problem 3. $y'' - x(y')^2 = 0,$

$$y(0) = 1,\ y^{'}(0) = \frac{1}{2} \quad h = 0.01$$

Exact Solution: $y(x) = 1 + \frac{1}{2}\log(\frac{2+x}{2-x})$

Source: [9]

Error$=$ |Exact result $-$ Computed result|

NM= Error in New Method

Table 1: Comparison of absolute errors for Problem I

X	Error in [3]	Error in [9]	NM
0.1	6.92×10^{-09}	0.00	8.071810×10^{-11}
0.2	1.76×10^{-08}	6.00×10^{-09}	3.317995×10^{-10}
0.3	1.62×10^{-08}	1.80×10^{-08}	7.638832×10^{-11}
0.4	4.37×10^{-08}	2.50×10^{-08}	1.383957×10^{-11}
0.5	1.20×10^{-07}	3.40×10^{-08}	2.195177×10^{-11}
0.6	1.87×10^{-07}	4.70×10^{-08}	3.195674×10^{-11}
0.7	3.07×10^{-07}	5.30×10^{-08}	4.383823×10^{-11}
0.8	4.19×10^{-07}	6.40×10^{-08}	5.747571×10^{-11}
0.9	5.79×10^{-07}	7.50×10^{-08}	7.275167×10^{-11}
1.0	7.27×10^{-07}	8.80×10^{-09}	8.949942×10^{-11}
1.1	9.20×10^{-07}	9.80×10^{-08}	1.075156×10^{-11}
1.2	1.10×10^{-06}	1.11×10^{-07}	1.265626×10^{-11}

Table 2: Comparison of absolute errors for Problem 2

X	Error in [3]	Error in [9]	NM
0.01	4.96×10^{-09}	3.00×10^{-10}	7.360845×10^{-11}
0.02	1.49×10^{-08}	1.00×10^{-09}	2.854670×10^{-10}
0.03	6.68×10^{-09}	2.80×10^{-09}	6.246097×10^{-10}
0.04	2.40×10^{-08}	3.00×10^{-09}	1.082590×10^{-09}
0.05	6.77×10^{-07}	5.10×10^{-09}	1.551924×10^{-09}
0.06	1.08×10^{-07}	6.40×10^{-09}	2.332691×10^{-09}
0.07	1.77×10^{-07}	9.40×10^{-09}	3.118856×10^{-09}
0.08	2.32×10^{-07}	1.15×10^{-08}	4.011705×10^{-09}
0.09	3.13×10^{-07}	1.49×10^{-08}	5.012991×10^{-09}
0.10	3.95×10^{-07}	1.77×10^{-08}	6.126237×10^{-09}
0.11	4.96×10^{-07}	2.30×10^{-08}	7.356706×10^{-09}
0.12	6.00×10^{-06}	2.70×10^{-08}	8.711392×10^{-09}

Table 3: Comparison of absolute errors for Problem 3

X	Error in [5]	Error in [9]	NM
0.1	0.26075×10^{-09}	1.603160×10^{-13}	1.332268×10^{-10}
0.2	1.98167×10^{-09}	4.296563×10^{-13}	1.154632×10^{-10}
0.3	6.50741×10^{-09}	2.575717×10^{-13}	4.107825×10^{-11}
0.4	15.5924×10^{-09}	1.424638×10^{-12}	1.039169×10^{-11}
0.5	31.5045×10^{-09}	3.539835×10^{-12}	2.193801×10^{-11}
0.6	56.3746×10^{-09}	6.332490×10^{-12}	4.176659×10^{-11}
0.7	96.1640×10^{-09}	9.671490×10^{-12}	7.476242×10^{-11}
0.8	156.868×10^{-09}	1.475506×10^{-11}	1.291411×10^{-11}
0.9	248.698×10^{-09}	2.297873×10^{-11}	2.189582×10^{-11}
1.0	387.984×10^{-09}	3.483280×10^{-11}	3.700151×10^{-11}

6. Discussion of result

We have considered three numerical examples to test the efficiency of our method. Problem $[1-3]$ was solved by [9]. They proposed a self starting Linear Multistep Method for direct solution of initial value problem of second order ordinary differential equation to solve the three problems we considered . Our method gave better approximation as shown in Tables $(1-3)$ despite the higher order methods they proposed.

7. Conclusion

We proposed a continuous hybrid stomer cowell method for the solution of second order initial value problems which was implemented in continuous block method. Continuous block method has advantage of evaluation at all selected points within the interval of integration. The results show that our method gives better approximation than the existing methods.

Conflict of Interests

The authors declare that there is no conflict of interests.

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