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OSCILLATION THEOREMS FOR SECOND-ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATION OF MIXED TYPE

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Abstract. In this paper, we study the oscillatory properties of solutions of second order neutral difference equation of the form

$$\Delta^2 \left(x_n + a_n x_{n-\tau_1} + b_n x_{n+\tau_2}\right)^\alpha = q_n x^\beta_{n-\sigma_1} + p_n x^\gamma_{n+\sigma_2}, \quad n \geq n_0.$$

where α, β and γ are ratio of odd positive integers. The results obtained here generalize and complement to the existing results. Examples are provided to illustrate the main results.

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1. Introduction

In this paper, we shall study the oscillatory behavior of solutions of second order nonlinear neutral difference equation of the form

$$\Delta^2 \left(x_n + a_n x_{n-\tau_1} + b_n x_{n+\tau_2}\right)^\alpha = q_n x^\beta_{n-\sigma_1} + p_n x^\gamma_{n+\sigma_2} \tag{1.1}$$

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where $n \in \mathbb{N}(n_0) = \{n_0, n_0 + 1 \cdots\}$, n_0 a nonnegative integer, subject to the following conditions:

(i) $\{a_n\}$ and $\{b_n\}$ are nonnegative real sequences with $0 \leq a_n \leq a$ and $0 \leq b_n \leq b$ where a and b are constants;

(ii) $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences and are not eventually zero for many values of n;

(iii) σ_1, σ_2, τ_1 and τ_2 are nonnegative integers and α, β and γ are ratio of odd positive integers.

Let $\theta = \max\{\sigma_1, \tau_1\}$. By a solution of equation (1.1) we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$, and satisfies equation (1.1) for all $n \geq n_0$. A nontrivial solution $\{x_n\}$ of equation (1.1) is said to be oscillatory if it's terms are neither eventually positive nor eventually negative and nonoscillatory otherwise.

Recently, there has been an increasing interest in the study of oscillatory behavior of solutions of neutral type difference equations since such equations have many applications in economics, engineering and population dynamics. For background results on the oscillation of neutral type difference equations, see for example [1,2], and the references cited therein.

Most of the results established in the literature for neutral type difference equations involves either delay or advanced type arguments, and only few results are available for mixed type linear equations, see for example [3-5]. Motivated by this observation, in this paper we establish conditions for the oscillation of all solutions of equation (1.1).

In Section 2, we present sufficient conditions for the oscillation of all solutions of equation (1.1). Examples are provided in Section 3 to illustrate the main results.

2. Oscillation Results

In this section, we present some sufficient conditions for the oscillation of all solutions of equation (1.1). To prove our main results we need the following lemma found in [8].

Lemma 2.1. *Let* $A \geq 0,\ B \geq 0.$

For $\eta \geq 1,$ *we have*

$$A^\eta + B^\eta \geq \frac{1}{2^{\eta-1}} (A+B)^\eta \,, \tag{2.1}$$

and for $0 < \eta < 1,$ *we have*

$$A^\eta + B^\eta \geq (A+B)^\eta. \tag{2.2}$$

Lemma 2.2. *Let* $\delta > 1$ *be a ratio of odd positive integers and* $\sigma \geq 2$ *be a positive integer. If*

$$\sum_{n=n_0}^{\infty} \left(\sum_{s=n-\sigma+1}^{n-1} R_s \right) = \infty, \tag{2.3}$$

then the difference inequality

$$\Delta^2 y_n - R_n \ y_{n+\sigma}^\delta \geq 0 \tag{2.4}$$

where $\{R_n\}$ *is a positive real sequence, has no eventually positive increasing solution.*

Proof. Let $\{y_n\}$ be a positive increasing solution of (2.4). Then $y_n > 0$, and $\Delta y_n > 0$ for all $n \geq n_1 \geq n_0$. From the inequality (2.4) we have

$$\Delta^2 y_n \geq R_n \ y_{n+\sigma}^\delta.$$

Summing the last inequality from $n - \sigma + 1$ to $n - 1$ we obtain,

$$\Delta y_n \geq \Delta y_n - \Delta y_{n-\sigma+1} \geq \sum_{s=n-\sigma+1}^{n-1} R_s \ y_{s+\sigma}^\delta \geq y_{n+1}^\delta \sum_{s=n-\sigma+1}^{n-1} R_s,$$

or

$$\frac{\Delta y_n}{y_{n+1}^\delta} \geq \sum_{s=n-\sigma+1}^{n-1} R_s,$$

or

$$\int_{y_n}^{y_{n+1}} \frac{ds}{s^\delta} \geq \frac{1}{y_{n+1}^\delta} \Delta y_n \geq \sum_{s=n-\sigma+1}^{n-1} R_s.$$

Summing again from n_1 to $n - 1$

$$\int_{y_{n_1}}^{y_n} \frac{ds}{s^\delta} \geq \sum_{s=n_1}^{n-1} \left(\sum_{t=n-\sigma+1}^{n-1} R_t \right),$$

$$-\frac{y_n^{1-\delta}}{\delta-1}+\frac{y_{n_1}^{1-\delta}}{\delta-1}\geq\sum_{n=n_1}^{n}\left(\sum_{t=n-\sigma+1}^{n-1}R_t\right).$$

As $n\to\infty$,we have $\sum\limits_{n=n_1}^{\infty}\left(\sum\limits_{t=n-\sigma+1}^{n-1}R_t\right)<\infty$ which contradicts condition (2.3). This completes the proof.

Lemma 2.3. *Let $\delta<1$ and $\sigma\geq1$ be a positive integer. If*

$$\sum_{n=n_0}^{\infty}\left(\sum_{s=n}^{n+\sigma}R_s\right)=\infty, \tag{2.5}$$

then the difference inequality

$$\Delta^2y_n-R_n\ y_{n-\sigma}^{\delta}\geq0 \tag{2.6}$$

has no eventually positive decreasing solution.

Proof. Let $\{y_n\}$ be a positive decreasing solution of (2.6). Then $y_n>0$ and $\Delta y_n<0$ for all $n\geq n_1\geq n_0$. From the inequality (2.6), we have

$$\Delta^2y_n\geq R_n\ y_{n-\sigma}^{\delta}.$$

Summing the last inequality from n to $n+\sigma$ we get,

$$-\Delta y_n\geq\Delta y_{n+\sigma+1}-\Delta y_n\geq\sum_{s=n}^{n+\sigma}R_s\ y_{s-\sigma}^{\delta}\geq y_n^{\delta}\sum_{s=n}^{n+\sigma}R_s,$$

or

$$-\frac{\Delta y_n}{y_n^{\delta}}\geq\sum_{s=n}^{n+\sigma}R_s,$$

or

$$\int\limits_{y_{n+1}}^{y_n}\frac{ds}{s^{\delta}}\geq\frac{y_n-y_{n+1}}{y_n^{\delta}}\geq\sum_{s=n}^{n+\sigma}R_s.$$

Summing the last inequality from n_1 to $n-1$, we have

$$\int\limits_{0}^{y_{n_1}}\frac{ds}{s^{\delta}}\geq\sum_{n=n_1}^{n-1}\left(\sum_{s=n}^{n+\sigma}R_s\right),$$

or

$$\frac{y_{n_1}^{1-\delta}}{\delta-1}\geq\sum_{n=n_1}^{n-1}\left(\sum_{t=n-\sigma+1}^{n-1}R_t\right).$$

As $n\to\infty$, we have $\sum\limits_{n=n_1}^{\infty}\left(\sum\limits_{s=n}^{n+\sigma}R_s\right)<\infty$ which contradicts condition (2.5). This completes the proof

Theorem 2.4. *Let $0 < \beta \leq 1,\ \gamma \geq 1,\ a \leq 1,\ b \leq 1,\ \sigma_1 > \tau_1$ and $\sigma_2 > \tau_2$. Assume that*

(i) the difference inequality

$$\Delta^2 y_n - \frac{Q_n}{(1 + a^\beta + b^\beta)^{\beta/\alpha}} y_{n+\tau_1-\sigma_1}^{\beta/\alpha} \geq 0 \tag{2.7}$$

has no eventually positive decreasing solution, and

(ii) the difference inequality

$$\Delta^2 y_n - \frac{P_n}{4^{\gamma-1}(1 + a^\beta + b^\beta)^{\gamma/\alpha}} y_{n-\tau_2+\sigma_2}^{\gamma/\alpha} \geq 0 \tag{2.8}$$

has no eventually positive increasing solution, where $P_n = \min\{p_n, p_{n-\tau_1}, p_{n+\tau_2}\}$ and $Q_n = \{q_n, q_{n-\tau_1}, q_{n+\tau_2}\}$. Then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists an integer $n_1 \geq n_0$ such that $x_n > 0,\ x_{n-\tau_1} > 0,\ x_{n+\tau_2} > 0,\ x_{n-\sigma_1} > 0$ and $x_{n+\sigma_2} > 0$ for all $n \geq n_1$. Setting

$$z_n = (x_n + a_n x_{n-\tau_1} + b_n x_{n+\tau_2})^\alpha,$$

and

$$y_n = z_n + a^\beta z_{n-\tau_1} + b^\beta z_{n+\tau_2}. \tag{2.9}$$

Then $z_n > 0$, $y_n > 0$, and

$$\Delta^2 z_n = q_n x_{n-\sigma_1}^\beta + p_n x_{n+\sigma_2}^\gamma \geq 0,\ n \geq n_1.$$

Then $\{\Delta z_n\}$ is of one sign eventually. From (1.1) and (2.9) we have

$$\begin{aligned}\Delta^2 y_n &= \Delta^2 z_n + a^\beta \Delta^2 z_{n-\tau_1} + b^\beta \Delta^2 z_{n+\tau_2} \\ &= q_n x_{n-\sigma_1}^\beta + p_n x_{n+\sigma_2}^\gamma + a^\beta \left(q_{n-\tau_1} x_{n-\sigma_1-\tau_1}^\beta + p_{n-\tau_1} x_{n+\sigma_2-\tau_1}^\gamma\right) \\ &+ b^\beta \left(q_{n+\tau_2} x_{n-\sigma_1+\tau_2}^\beta + p_{n+\tau_2} x_{n+\sigma_2+\tau_2}^\gamma\right).\end{aligned}$$

Using Lemma 2.1 and using the fact $a \leq 1,\ b \leq 1$ in the above equation, we obtain

$$\begin{aligned}\Delta^2 y_n &\geq Q_n (x_{n-\sigma_1} + a x_{n-\tau_1-\sigma_1} + b x_{n+\tau_2-\sigma_1})^\beta \\ &+ \frac{P_n}{4^{\gamma-1}} (x_{n+\sigma_2} + a x_{n-\tau_1+\sigma_2} + b x_{n+\tau_2+\sigma_2})^\gamma\end{aligned}$$

$$\Delta^2 y_n = Q_n z_{n-\sigma_1}^{\beta/\alpha} + \frac{P_n}{4^{\gamma-1}} z_{n+\sigma_2}^{\gamma/\alpha}. \tag{2.10}$$

Next we consider the following two cases:

Case(1). If $\Delta z_n < 0$, then $\Delta y_n < 0$ for all $n \geq n_1$. Inview of (2.10), we get

$$\Delta^2 y_n \geq Q_n z_{n-\sigma_1}^{\beta/\alpha}. \tag{2.11}$$

From the monotonicity of $\{z_n\}$, we find

$$z_n \geq \frac{y_{n+\tau_1}}{(1+a^\beta+b^\beta)}.$$

Substituting the above inequality in (2.11), we obtain

$$\Delta^2 y_n \geq \frac{Q_n}{(1+a^\beta+b^\beta)^{\beta/\alpha}} y_{n+\tau_1-\sigma_1}^{\beta/\alpha}.$$

Therefore $\{y_n\}$ is a positive decreasing solution of the difference inequality (2.7) which is a contradiction.

Case (2). If $\Delta z_n > 0$, then $\Delta y_n > 0$ for all $n \geq n_1$. Inview of (2.10), we get

$$\Delta^2 y_n \geq \frac{1}{4^{\gamma-1}} P_n z_{n+\sigma_2}^{\gamma/\alpha}. \tag{2.12}$$

Again using the monotonicity of $\{z_n\}$, we find

$$z_n \geq \frac{y_{n-\tau_2}}{(1+a^\beta+b^\beta)}.$$

Substituting the above inequality in (2.12), we obtain

$$\Delta^2 y_n \geq \frac{P_n}{4^{\gamma-1}(1+a^\beta+b^\beta)^{\gamma/\alpha}} y_{n-\tau_2+\sigma_2}^{\gamma/\alpha}.$$

Therefore $\{y_n\}$ is a positive increasing solution of the difference inequality (2.8) which is a contradiction. This completes the proof.

From the Lemmas 2.2 and 2.3 and the Theorem 2.4, we obtain the following corollary.

Corollary 2.5. *Let $\beta < \alpha < \gamma$, $a \leq 1$, $b \leq 1$, $\sigma_1 \geq \tau_1 + 1$, $\sigma_2 \geq \tau_2 + 2$. If*

$$\sum_{n=n_0}^{\infty} \sum_{s=n}^{n+\sigma_1-\tau_1} Q_s = \infty, \tag{2.13}$$

and

$$\sum_{n=n_0}^{\infty} \sum_{s=n-\sigma_2+\tau_2+1}^{n-1} P_s = \infty \tag{2.14}$$

then every solution of equation (1.1) is oscillatory.

Theorem 2.6. *Let $\beta \geq 1,\ 0 < \gamma \leq 1,\ a \geq 1,\ b \geq 1,\ \sigma_1 > \tau_1,$ and $\sigma_2 > \tau_2$. Assume that*

(i) the difference inequality

$$\Delta^2 y_n - \frac{Q_n}{4^{\beta-1}\left(1+a^\beta+b^\beta\right)^{\beta/\alpha}} y^{\beta/\alpha}_{n+\tau_1-\sigma_1} \geq 0 \tag{2.15}$$

has no eventually positive decreasing solution, and

(ii) the difference inequality

$$\Delta^2 y_n - \frac{P_n}{\left(1+a^\beta+b^\beta\right)^{\gamma/\alpha}} y^{\gamma/\alpha}_{n-\tau_2+\sigma_2} \geq 0 \tag{2.16}$$

has no eventually positive increasing solution, then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists an integer $n_1 \geq n_0$ such that $x_n > 0,\ x_{n-\tau_1} > 0,\ x_{n+\tau_2} > 0,\ x_{n-\sigma_1} > 0$ and $x_{n+\sigma_2} > 0$ for all $n \geq n_1$. Setting

$$z_n = \left(x_n + a_n x_{n-\tau_1} + b_n x_{n+\tau_2}\right)^\alpha,$$

and

$$y_n = z_n + a^\beta z_{n-\tau_1} + b^\beta z_{n+\tau_2}.$$

Then $z_n > 0$, $y_n > 0$, and

$$\Delta^2 z_n = q_n x^\beta_{n-\sigma_1} + p_n x^\gamma_{n+\sigma_2} \geq 0,\ n \geq n_1.$$

Hence $\{\Delta z_n\}$ is of one sign eventually. From equation (1.1) we have

$$\begin{aligned}\Delta^2 y_n &= \Delta^2 z_n + a^\beta \Delta^2 z_{n-\tau_1} + b^\beta \Delta^2 z_{n+\tau_2} \\ &= q_n x^\beta_{n-\sigma_1} + p_n x^\gamma_{n+\sigma_2} + a^\beta \left(q_{n-\tau_1} x^\beta_{n-\sigma_1-\tau_1} + p_{n-\tau_1} x^\gamma_{n+\sigma_2-\tau_1}\right) \\ &+ b^\beta \left(q_{n+\tau_2} x^\beta_{n-\sigma_1+\tau_2} + p_{n+\tau_2} x^\gamma_{n+\sigma_2+\tau_2}\right).\end{aligned}$$

Using Lemma 2.1 and the fact $a \geq 1,\ b \geq 1$ in the above equation, we obtain

$$\Delta^2 y_n \geq \frac{Q_n}{4^{\beta-1}} \left(x_{n-\sigma_1} + a x_{n-\tau_1-\sigma_1} + b x_{n+\tau_2-\sigma_1}\right)^\beta$$

$$P_n \left(x_{n+\sigma_2} + a x_{n-\tau_1+\sigma_2} + b x_{n+\tau_2+\sigma_2}\right)^\gamma$$

$$\Delta^2 y_n = \frac{Q_n}{4^{\beta-1}} z^{\beta/\alpha}_{n-\sigma_1} + P_n z^{\gamma/\alpha}_{n+\sigma_2} \tag{2.17}$$

Next we consider the following two cases:

Case(1). If $\Delta z_n < 0$, then $\Delta y_n < 0$ for all $n \geq n_1$. From (2.17), we get

$$\Delta^2 y_n \geq \frac{Q_n}{4^{\beta-1}} z_{n-\sigma_1}^{\beta/\alpha}. \tag{2.18}$$

From the monotonicity of $\{z_n\}$, we find

$$z_n \geq \frac{y_{n+\tau_1}}{(1+a^\beta+b^\beta)}.$$

Substituting the above inequality in (2.18), we obtain

$$\Delta^2 y_n \geq \frac{1}{4^{\beta-1}(1+a^\beta+b^\beta)^{\beta/\alpha}} Q_n y_{n+\tau_1-\sigma_1}^{\beta/\alpha}.$$

Therefore $\{y_n\}$ is a positive decreasing solution of the difference inequality (2.15) which is a contradiction.

Case (2). If $\Delta z_n > 0$, then $\Delta y_n > 0$ for all $n \geq n_1$. From (2.17), we have

$$\Delta^2 y_n \geq \ P_n z_{n+\sigma_2}^{\gamma/\alpha}. \tag{2.19}$$

Again using the monotonicity of $\{z_n\}$, we find

$$z_n \geq \frac{y_{n-\tau_2}}{(1+a^\beta+b^\beta)}.$$

Substituting the above inequality in (2.19), we obtain

$$\Delta^2 y_n \geq \frac{1}{(1+a^\beta+b^\beta)^{\gamma/\alpha}} P_n y_{n-\tau_2+\sigma_2}^{\gamma/\alpha}.$$

Therefore $\{y_n\}$ is a positive increasing solution of the difference inequality (2.16) which is a contradiction. This completes the proof.

Theorem 2.7. *Assume that $\gamma = \beta \geq 1,\ \sigma_1 \geq \tau_1,\ \sigma_2 \geq \tau_2 + 2$. If*

$$\Delta^2 y_n - \frac{P_n}{(4^{\beta-1})\left(1+a^\beta+\dfrac{b^\beta}{2^{\beta-1}}\right)^{\beta/\alpha}} y_{n-\tau_2+\sigma_2}^{\beta/\alpha} \geq 0, \tag{2.20}$$

has no eventually positive increasing solution, and

$$\Delta^2 y_n - \frac{Q_n}{(4^{\beta-1})\left(1+a^\beta+\dfrac{b^\beta}{2^{\beta-1}}\right)^{\beta/\alpha}} y_{n-\sigma_1+\tau_1}^{\beta/\alpha} \geq 0 \tag{2.21}$$

has no eventually positive decreasing solution, then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists an integer $n_1 \geq n_0$ such that $x_n > 0,\ x_{n-\tau_1} > 0,\ x_{n+\tau_2} > 0,\ x_{n-\sigma_1} > 0$ and $x_{n+\sigma_2} > 0$ for all $n \geq n_1$. Setting

$$z_n = (x_n + a_n x_{n-\tau_1} + b_n x_{n+\tau_2})^{\alpha},$$

and

$$y_n = z_n + a^{\beta} z_{n-\tau_1} + \frac{b^{\beta}}{2^{\beta-1}} z_{n+\tau_2}.$$

Then $z_n > 0$, $y_n > 0$, and

$$\Delta^2 z_n = q_n x^{\beta}_{n-\sigma_1} + p_n x^{\beta}_{n+\sigma_2} \geq 0.$$

Then $\{\Delta z_n\}$ is of one sign eventually. On the other hand

$$\Delta^2 y_n = q_n x^{\beta}_{n-\sigma_1} + p_n x^{\beta}_{n+\sigma_2} + a^{\beta} q_{n-\tau_1} x^{\beta}_{n-\sigma_1-\tau_1} + a^{\beta} p_{n-\tau_1} x^{\beta}_{n+\sigma_2-\tau_1} + \frac{b^{\beta}}{2^{\beta-1}} q_{n+\tau_2} x^{\beta}_{n-\sigma_1+\tau_2} + \frac{b^{\beta}}{2^{\beta-1}} p_{n+\tau_2} x^{\beta}_{n+\sigma_2+\tau_2}$$

Using (2.1) in the above inequality we obtain

$$\Delta^2 y_n \geq \frac{Q_n}{2^{\beta-1}} \left(x_{n-\sigma_1} + a x_{n-\sigma_1-\tau_1}\right)^{\beta} + \frac{q_{n+\tau_2}}{2^{\beta-1}} b^{\beta} x^{\beta}_{n-\sigma_1+\tau_2} + \frac{P_n}{2^{\beta-1}} \left(x_{n+\sigma_2} + a x_{n+\sigma_2-\tau_1}\right)^{\beta} + \frac{p_{n+\tau_2}}{2^{\beta-1}} b^{\beta} x^{\beta}_{n+\sigma_2+\tau_2}$$

$$\Delta^2 y_n \geq \frac{1}{(4^{\beta-1})} \left(Q_n z^{\beta/\alpha}_{n-\sigma_1} + P_n z^{\beta/\alpha}_{n+\sigma_2}\right), \quad n \geq n_1. \tag{2.22}$$

Next we consider the following two cases:

Case (1). Assume $\Delta z_n > 0$. Then $\Delta y_n > 0$ for all $n \geq n_1$. From (2.22), we have

$$\Delta^2 y_{n+\tau_2} \geq \frac{1}{(4^{\beta-1})} P_{n+\tau_2} z^{\beta/\alpha}_{n+\sigma_2+\tau_2}. \tag{2.23}$$

Applying the monotonicity of $\{z_n\}$, we find

$$y_{n+\sigma_2} = z_{n+\sigma_2} + a^{\beta} z_{n-\tau_1+\sigma_2} + \frac{b^{\beta}}{2^{\beta-1}} z_{n+\tau_2+\sigma_2} \leq \left(1 + a^{\beta} + \frac{b^{\beta}}{2^{\beta-1}}\right) z_{n+\tau_2+\sigma_2}. \tag{2.24}$$

Combining (2.23) and (2.24) we have

$$\Delta^2 y_{n+\tau_2} \geq \frac{P_{n+\tau_2}}{(4^{\beta-1})\left(1+a^\beta+\frac{b^\beta}{2^{\beta-1}}\right)^{\beta/\alpha}} y^{\beta/\alpha}_{n+\sigma_2}. \tag{2.25}$$

Thus

$$\Delta^2 y_n - \frac{P_n}{(4^{\beta-1})\left(1+a^\beta+\frac{b^\beta}{2^{\beta-1}}\right)^{\beta/\alpha}} y^{\beta/\alpha}_{n-\tau_2+\sigma_2} \geq 0.$$

Therefore $\{y_n\}$ is a positive increasing solution of the difference inequality (2.20), a contradiction.

Case (2). Assume $\Delta z_n < 0$. Then $\Delta y_n < 0$ for all $n \geq n_1$. From (2.22) we see that

$$\Delta^2 y_{n-\tau_1} \geq \frac{1}{(4^{\beta-1})} Q_{n-\tau_1} z^{\beta/\alpha}_{n-\tau_1-\sigma_1}.$$

From the monotonicity of $\{z_n\}$, we find

$$y_{n-\sigma_1} \leq \left(1+a^\beta+\frac{b^\beta}{2^{\beta-1}}\right) z_{n-\tau_1-\sigma_1}.$$

Combining the last two inequalities, we obtain

$$\Delta^2 y_{n-\tau_1} \geq \frac{Q_{n-\tau_1}}{(4^{\beta-1})\left(1+a^\beta+\frac{b^\beta}{2^{\beta-1}}\right)^{\beta/\alpha}} y^{\beta/\alpha}_{n-\sigma_1} \tag{2.26}$$

or

$$\Delta^2 y_n - \frac{Q_n}{(4^{\beta-1})\left(1+a^\beta+\frac{b^\beta}{2^{\beta-1}}\right)^{\beta/\alpha}} y^{\beta/\alpha}_{n-\sigma_1+\tau_1} \geq 0.$$

Therefore $\{y_n\}$ is a positive decreasing solution of the difference inequality (2.21), a contradiction. This completes the proof.

Theorem 2.8. *Assume that $\gamma = \beta = \alpha \geq 1,\ \sigma_1 \geq \tau_1,\ \sigma_2 \geq \tau_2 + 2$. If*

$$\limsup_{n\to\infty} \sum_{s=n}^{n+\sigma_2-\tau_2-2} (\ n+\sigma_2-\tau_2-s-1)\, P_s > (4^{\alpha-1})\left(1+a^\alpha+\frac{b^\alpha}{2^{\alpha-1}}\right), \tag{2.27}$$

and

$$\limsup_{n\to\infty} \sum_{s=n-\sigma_1+\tau_1}^{n} (\ n-s+1)\, Q_s > (4^{\alpha-1})\left(1+a^\alpha+\frac{b^\alpha}{2^{\alpha-1}}\right), \tag{2.28}$$

hold, then every solution of equation (1.1) is oscillatory.

Proof. Conditions (2.27) and (2.28) imply that the difference inequalities (2.20) and (2.21) has no positive increasing solution and no positive decreasing solutions respectively, see [2, Lemma 7.6.15]. The result now follows from Theorem 2.7.

Theorem 2.9. *Assume that* $0 < \gamma = \beta \leq 1,\ \sigma_1 \geq \tau_1,\ \sigma_2 \geq \tau_2 + 2$. *If*

$$\Delta^2 y_n - \frac{P_n}{(1 + a^\beta + b^\beta)^{\beta/\alpha}} y^{\beta/\alpha}_{n-\tau_2+\sigma_2} \geq 0 \tag{2.29}$$

has no eventually positive increasing solution, and

$$\Delta^2 y_n - \frac{Q_n}{(1 + a^\beta + b^\beta)^{\beta/\alpha}} y^{\beta/\alpha}_{n-\sigma_1+\tau_1} \geq 0 \tag{2.30}$$

has no eventually positive decreasing solution, then every solution of equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists an integer $n_1 \geq n_0$ such that $x_n > 0,\ x_{n-\tau_1} > 0,\ x_{n+\tau_2} > 0,\ x_{n-\sigma_1} > 0$ and $x_{n+\sigma_2} > 0$ for all $n \geq n_1$. Setting

$$z_n = (x_n + a_n x_{n-\tau_1} + b_n x_{n+\tau_2})^\alpha,$$

and

$$y_n = z_n + a^\beta z_{n-\tau_1} + b^\beta z_{n+\tau_2}.$$

Then $z_n > 0$, $y_n > 0$, and

$$\Delta^2 z_n = q_n x^\beta_{n-\sigma_1} + p_n x^\beta_{n+\sigma_2} \geq 0.$$

Then $\{\Delta z_n\}$ is of one sign eventually. On the other hand

$$\begin{aligned}\Delta^2 y_n =& q_n x^\beta_{n-\sigma_1} + p_n x^\beta_{n+\sigma_2} + a^\beta q_{n-\tau_1} x^\beta_{n-\sigma_1-\tau_1} + a^\beta p_{n-\tau_1} x^\beta_{n+\sigma_2-\tau_1} \\ &+ b^\beta q_{n+\tau_2} x^\beta_{n-\sigma_1+\tau_2} + b^\beta p_{n+\tau_2} x^\beta_{n+\sigma_2+\tau_2}.\end{aligned}$$

Using (2.2) in the above inequality we obtain

$$\Delta^2 y_n \geq \ Q_n z^{\beta/\alpha}_{n-\sigma_1} + P_n z^{\beta/\alpha}_{n+\sigma_2}, \quad n \geq n_1. \tag{2.31}$$

Next we consider the following two cases:

Case (1). Assume $\Delta z_n > 0$. Then $\Delta y_n > 0$ for all $n \geq n_1$. From (2.31), we have

$$\Delta^2 y_n \geq \; P_n z_{n+\sigma_2}^{\beta/\alpha}. \tag{2.32}$$

Applying the monotonicity of $\{z_n\}$, we find

$$y_{n+\sigma_2} = z_{n+\sigma_2} + a^\beta z_{n-\tau_1+\sigma_2} + b^\beta z_{n+\tau_2+\sigma_2} \leq \left(1 + a^\beta + b^\beta\right) z_{n+\tau_2+\sigma_2}. \tag{2.33}$$

Combining (2.32) and (2.33), we have

$$\Delta^2 y_{n+\tau_2} \geq \frac{P_{n+\tau_2}}{(1 + a^\beta + b^\beta)^{\beta/\alpha}} y_{n+\sigma_2}^{\beta/\alpha}. \tag{2.34}$$

Thus

$$\Delta^2 y_n - \frac{P_n}{(1 + a^\beta + b^\beta)^{\beta/\alpha}} y_{n-\tau_2+\sigma_2}^{\beta/\alpha} \geq 0.$$

Therefore $\{y_n\}$ is a positive increasing solution of the difference inequality (2.29), a contradiction.

Case (2). Assume $\Delta z_n < 0$ then $\Delta y_n < 0$ for all $n \geq n_1$. From (2.31) we see that

$$\Delta^2 y_{n-\tau_1} \geq \; Q_{n-\tau_1} z_{n-\tau_1-\sigma_1}^{\beta/\alpha}.$$

Using the monotonicity of $\{z_n\}$ we find

$$y_{n-\sigma_1} \leq \left(1 + a^\beta + b^\beta\right) z_{n-\tau_1-\sigma_1}.$$

Combining the last two inequalities, we obtain

$$\Delta^2 y_{n-\tau_1} \geq \frac{Q_{n-\tau_1}}{(1 + a^\beta + b^\beta)^{\beta/\alpha}} y_{n-\sigma_1}^{\beta/\alpha}. \tag{2.35}$$

or

$$\Delta^2 y_n - \frac{Q_n}{(1 + a^\beta + b^\beta)^{\beta/\alpha}} y_{n-\sigma_1+\tau_1}^{\beta/\alpha} \geq 0.$$

Therefore $\{y_n\}$ is a positive decreasing solution of the difference inequality (2.30), a contradiction. This completes the proof.

Remark 2.1. *If $\alpha = \beta = \gamma = 1$, Theorem 2.9 reduces to Theorem 7.6.6 of [2]. Further the results established in this paper extend and complement to the results obtained in [2,3,4,5].*

3. Examples

In this section we present some examples to illustrate the main results.

Examples 3.1. *Consider the difference equation*

$$\Delta^2\left(x_n+\frac{1}{2}x_{n-1}+\frac{1}{3}x_{n+2}\right)=\frac{1}{n+1}x_{n-3}^{1/3}+\frac{n}{n+1}x_{n+4}^{3},\ n\geq 2. \tag{3.1}$$

Here $a_n=\frac{1}{2},\ b_n=\frac{1}{3},\ \tau_1=1,\ \tau_2=2,\ \sigma_1=3,\ \sigma_2=4,\ p_n=\frac{n}{n+1},$ $q_n=\frac{1}{n+1},\ \alpha=1,\ \beta=\frac{1}{3},$ *and* $\gamma=3$. *It is easy to see that all conditions of Corollary 2.5 are satisfied and hence all solutions of equation (3.1) are oscillatory.*

Example 3.2. *Consider the difference equation*

$$\Delta^2\left(x_n+\frac{1}{2}x_{n-1}+\frac{1}{3}x_{n+2}\right)^3=\frac{1}{n}x_{n-3}+\left(\frac{1}{n}+\frac{125}{54}\right)x_{n+4}^{5},\ n\geq 2. \tag{3.2}$$

Here $a_n=2,\ b_n=3,\ \tau_1=1,\ \tau_2=2,\ \sigma_1=3,\ \sigma_2=4,\ p_n=\left(\frac{1}{n}+\frac{125}{54}\right),$ $q_n=\frac{1}{n},\ \alpha=3,\ \beta=1,$ *and* $\gamma=5$. *It is easy to see that all conditions of Corollary 2.5 are satisfied and hence all solutions of equation (3.2) are oscillatory. In fact* $\{x_n\}=\{(-1)^n\}$ *is one such solution of equation (3.2).*

Example 3.3. *Consider the difference equation*

$$\Delta^2\left(x_n+3x_{n-1}+4x_{n+2}\right)^3=\frac{240n}{n+1}x_{n-2}^{3}+n\ x_{n+4}^{3},\ n\geq 1. \tag{3.3}$$

Here $a_n=3,\ b_n=4,\ \tau_1=1,\ \tau_2=2,\ \sigma_1=2,\ \sigma_2=4,\ p_n=n,\ q_n=\frac{240n}{n+1},$ $\alpha=\beta=\gamma=3$. *It is easy to see that all conditions of Theorem 2.8 are satisfied and hence all solutions of equation (3.3) are oscillatory.*

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