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COUPLED COINCIDENCE POINT THEOREMS FOR MIXED (T,S)-MONOTONE OPERATORS

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Abstract. The purpose of this paper is to prove the existence and uniqueness of coupled coincidence points involving a new-contraction mapping condition for a mixed (T,S)-monotone mapping in a complete partially ordered G-metric spaces. As an application, we give an existence and uniqueness for the solution of an initial-boundary value problem. The results obtained extend and generalize some results in the literature.

Keywords: G-metric space, coupled coincidence point, partially ordered metric space, mixed (T,S)-monotone property, differential equation.

2010 AMS Subject Classification: 47H10, 54H25

1. Introduction

In 1922, S. Banach proved the famous and well known Banach contraction principle concerning the fixed point of contraction mappings defined on a complete metric space. In recent years, Gahler [9, 10] introduced the notion of 2-metric spaces while Dhage [1] introduced the concept of D-metric spaces. Later on, Mustafa and Sims [14] showed that most of the results

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concerning Dhage's D-metric spaces are invalid. Therefore, they introduced a new notion of generalized metric space, called G-metric space.

In 2006, Bhaskar and Lakshmikantham [11] introduced the notion of coupled fixed point and proved some fixed point theorem under certain condition. Later, Lakshmikantham and Ćirić in [12] extended these results by defining of g-monotone property. After that many results appeared on fixed point theory ([7, 6, 14, 15, 13, 16, 17, 18, 19, 20, 21, 22]).

Now we give preliminaries and basic definitions which are used throughout the paper.

2. Preliminaries

Definition 2.1 [14, 4] Let X be a non-empty set, $G : X \times X \times X \longrightarrow \mathbb{R}^+$ be a function satisfying the following properties:

- **(G1):** $G(x,y,z) = 0$ if $x = y = z$.
- **(G2):** $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$.
- **(G3):** $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $y \neq z$.
- **(G4):** $G(x,y,z) = G(x,z,y) = G(y,z,x) = ...$ (symmetry in all three variables).
- **(G5):** $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X,G) is called a G-metric space.

Definition 2.2 [14] Let (X,G) be a G-metric space, and let (x_n) be a sequence of points of X. We say that (x_n) is G-convergent to $x \in X$ if $\lim_{n,m \longrightarrow +\infty} G(x,x_n,x_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \varepsilon$, for all $n,m \geq N$. We call x the limit of the sequence and write $x_n \longrightarrow x$ or $\lim_{n \longrightarrow +\infty} x_n = x$.

Proposition 2.3 [14] Let (X,G) be a G-metric space. The following are equivalent:

- **(1):** (x_n) is G-convergent to x.
- **(2):** $G(x_n,x_n,x) \longrightarrow 0$ as $n \longrightarrow +\infty$.
- **(3):** $G(x_n,x,x) \longrightarrow 0$ as $n \longrightarrow +\infty$.
- **(4):** $G(x_n,x_m,x) \longrightarrow 0$ as $n,m \longrightarrow +\infty$.

Definition 2.4 [14] Let (X,G) be a G-metric space. A sequence (x_n) is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $m,n,l \geq N$, that is, $G(x_n,x_m,x_l) \longrightarrow 0$ as $n,m,l \longrightarrow +\infty$.

Proposition 2.5 [15] Let (X,G) be a G-metric space. Then the following are equivalent

(1): the sequence (x_n) is G-Cauchy

(2): for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$G(x_n,x_m,x_m) < \varepsilon$, for all $m,n \geq N$.

Proposition 2.6 [14] Let (X,G) be a G-metric space. A mapping $f : X \longrightarrow X$ is G-continuous at $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever (x_n) is G-convergent to x, $(f(x_n))$ is G-convergent to $f(x)$.

Proposition 2.7 [14] Let (X,G) be a G-metric space. Then, the function $G(x,y,z)$ is jointly continuous in all three of its variables.

Definition 2.8 [14] A G-metric space (X,G) is called G-complete if every G-Cauchy sequence is G-convergent in (X,G).

Definition 2.9 [2] Let (X,G) be a G-metric space. A mapping $F : X \times X \longrightarrow X$ is said to be continuous if for any two G-convergent sequences (x_n) and (y_n) converging to x and y respectively, $\big(F(x_n,y_n)\big)$ is G-convergent to $F(x,y)$.

A partial order is a binary relation $\preceq$ over a set X which is reflexive, antisymmetric, and transitive. Now let us recall the definition of the monotonic function $f : X \longrightarrow X$ in the partially order set $(X,\preceq)$. We say that f is non-decreasing if for $x,y \in X$, $x \preceq y$, we have $fx \preceq fy$. Similarly, we say that f is non-increasing if for $x,y \in X$, $x \preceq y$, we have $fx \succeq fy$.

The concept of a mixed monotone property has been introduced by Bhaskar and Lakshmikantham in [11].

Definition 2.10 [11] Let $(X,\preceq)$ be a partially ordered set. A mapping $F : X \times X \longrightarrow X$ is said to have mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y; that is, for any $x,y \in X$

$$x_1,x_2 \in X, \quad x_1 \preceq x_2 \Longrightarrow F(x_1,y) \preceq F(x_2,y)$$

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \Longrightarrow F(x, y_2) \preceq F(x, y_1).$$

Lakshmikantham and Ćirić in [12] introduced the concept of a g-mixed monotone mapping.

Definition 2.11 [12] Let $(X, \preceq)$ be a partially ordered set. Let us consider mappings $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$. The map F is said to have mixed g-monotone property if $F(x,y)$ is monotone g-non-decreasing in x and is monotone g-non increasing in y; that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad gx_1 \preceq gx_2 \Longrightarrow F(x_1, y) \preceq F(x_2, y),$$

$$y_1, y_2 \in X, \quad gy_1 \preceq gy_2 \in X \Longrightarrow F(x, y_2) \preceq F(x, y_1).$$

Definition 2.12 [11] An element $(x,y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \longrightarrow X$ if $F(x,y) = x$ and $F(y,x) = y$.

Definition 2.13 [12] An element $(x,y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ if $F(x,y) = gx$ and $F(y,x) = gy$.

Definition 2.14 [12] We say that mappings $F : X \times X \longrightarrow X$ and $g : X \longrightarrow X$ are commutative if

$$g(F(x,y)) = F(gx, gy) \quad \text{for all } x, y \in X.$$

Habib Yazidi in [5] introduced the concept of a (T,S)-mixed monotone mapping.

Definition 2.15 [5] Let X be a non-empty set endowed with a partial order $\preceq$.
Consider the mappings $F : X \times X \longrightarrow X$ and $T, S : X \longrightarrow X$. We say that F has the mixed (T,S)- monotone property on X if for all $x, y \in X$,

$$\begin{aligned}
&x_1, x_2 \in X, \ T(x_1) \preceq S(x_2) \Longrightarrow F(x_1, y) \preceq F(x_2, y),\\
&x_1, x_2 \in X, \ T(x_1) \succeq S(x_2) \Longrightarrow F(x_1, y) \succeq F(x_2, y),\\
&y_1, y_2 \in X, \ T(y_1) \preceq S(y_2) \Longrightarrow F(x, y_1) \succeq F(x, y_2),\\
&y_1, y_2 \in X, \ T(y_1) \succeq S(y_2) \Longrightarrow F(x, y_1) \preceq F(x, y_2).
\end{aligned} \tag{1}$$

Remark 2.16 If we take $T = S$, then F has the mixed (T, S)-monotone property implies that F has the mixed T-monotone property.

Denote by Ψ the set of functions $\psi : [0, +\infty) \longrightarrow (0, +\infty)$ which satisfy

1.: ψ is continuous and non-decreasing,

2.: $\psi(t) = 0$ if and only if $t = 0$.

The elements of Ψ are called altering distance function.

Remark 2.17 If $\psi \in \Psi$ and if $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ is a continuous function with the condition $\psi(t) > \phi(t)$ for all $t > 0$, then $\phi(0) = 0$.

Now, we are ready to state and prove our results.

3. Main results

Theorem 3.1 *Let $(X, \preceq)$ be a partially ordered set and G-metric on X such that (X, G) is a complete G-metric space. Let $T, S : X \longrightarrow X$ and $F : X \times X \longrightarrow X$ be a mapping having the mixed(T, S)-monotone property on X. Suppose that*

$$\psi\big[G\big(F(x,y), F(u,v), F(w,z)\big)\big] \leq \varphi\big(\max\big\{G(Tx, Tu, Sw), G(Ty, Tv, Sz)\big\}\big) \tag{2}$$

for all $x, y, z, u, v, w \in X$ with $Tx \preceq Tu \preceq Sw$ or $Tx \succeq Tu \succeq Sw$ and $Ty \succeq Tv \succeq Sz$ or $Ty \preceq Tv \preceq Sz$, where $\psi \in \Psi$ and $\varphi : [0, \infty[\rightarrow IR$ is a continuous function with the condition $\psi(t) > \varphi(t)$ for all $t > 0$. Assume that $F(X \times X) \subseteq T(X) \cap S(X)$ and assume also that T, S and F satisfy the following hypothesis:

(1) *F, T and S are continuous,*

(2) *F commutes respectively with T and S.*

If there exist x_0, y_0, x_1 in X and y_1 such that

$$Tx_0 \preceq Sx_1 \preceq F(x_0, y_0), \;\; Ty_0 \succeq Sy_1 \succeq F(y_0, x_0),$$

then there exist $x, y \in X$ such that $Tx = Sx = F(x, y)$ and $Ty = Sy = F(y, x)$, that is, T, S and F have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Let $x_0, y_0, x_1, y_1 \in X$ such that

$$Tx_0 \preceq Sx_1 \preceq F(x_0, y_0); \; Ty_0 \succeq Sy_1 \succeq F(y_0, x_0).$$

Since $F(X\times X)\subseteq T(X)\cap S(X)$, we can choose $x_2,y_2,x_3,y_3\in X$ such that
$\begin{cases} Tx_2=F(x_0,y_0)\\ Ty_2=F(y_0,x_0)\end{cases}$ and $\begin{cases} Sx_3=F(x_1,y_1)\\ Sy_3=F(y_1,x_1)\end{cases}$
Continuing this process we can construct sequences x_n and y_n in X such that

$$\begin{cases} Tx_{2n+2}=F(x_{2n},y_{2n})\\ Ty_{2n+2}=F(y_{2n},x_{2n})\end{cases}\tag{3}$$

$$\begin{cases} Sx_{2n+3}=F(x_{2n+1},y_{2n+1})\\ Sy_{2n+3}=F(y_{2n+1},x_{2n+1})\end{cases}\tag{4}$$

for all $n\geq 0$. We shall show that for all $n\geq 0$,

$$\begin{aligned} Tx_{2n}\preceq Sx_{2n+1}\preceq Tx_{2n+2}\\ Ty_{2n}\succeq Sy_{2n+1}\succeq Ty_{2n+2}\end{aligned}\tag{5}$$

As $Tx_0\preceq Sx_1\preceq F(x_0,y_0)=Tx_2$ and $Ty_0\succeq Sy_1\succeq F(y_0,x_0)=Ty_2$, our claim is satisfied for $n=0$. Suppose that (5) holds for some fixed $n\geq 0$. Since $Tx_{2n}\preceq Sx_{2n+1}\preceq Tx_{2n+2}$ and $Ty_{2n}\succeq Sy_{2n+1}\succeq Ty_{2n+2}$ and as F has the mixed (T,S)-monotone property, we have

$$\begin{gathered} Tx_{2n+2}=F(x_{2n},y_{2n})\preceq F(x_{2n+1},y_{2n})\preceq\\ F(x_{2n+1},y_{2n+1})\preceq F(x_{2n+2},y_{2n+1})\preceq F(x_{2n+2},y_{2n+2}),\end{gathered}$$

then

$$Tx_{2n+2}\preceq Sx_{2n+3}\preceq Tx_{2n+4}.$$

On the other hand,

$$\begin{gathered} Ty_{2n+2}=F(y_{2n},x_{2n})\succeq F(y_{2n+1},x_{2n})\succeq\\ F(y_{2n+1},x_{2n+1})\succeq F(y_{2n+2},x_{2n+1})\succeq F(y_{2n+2},x_{2n+2}),\end{gathered}$$

then

$$Ty_{2n+2}\succeq Sy_{2n+3}\succeq Ty_{2n+4}.$$

Thus by induction, we proved that (5) holds for all $n\geq 0$.

We complete the proof in the following steps

Step 1: We will prove that

$$\begin{aligned}\lim_{n\longrightarrow+\infty} G(F(x_n,y_n),F(x_n,y_n),F(x_{n+1},y_{n+1}))=0\\ \lim_{n\longrightarrow+\infty} G(F(y_n,x_n),F(y_n,x_n),F(y_{n+1},x_{n+1}))=0.\end{aligned} \tag{6}$$

From (5) and (2), we have

$$\begin{aligned}\psi\big[G\big(F(x_{2n},y_{2n}),F(x_{2n},y_{2n}),F(x_{2n+1},y_{2n+1})\big)\big]\leq\\ \varphi\big(\max\big\{G(Tx_{2n},Tx_{2n},Sx_{2n+1}),G(Ty_{2n},Ty_{2n},Sy_{2n+1})\big\}\big).\end{aligned} \tag{7}$$

Using the condition of the theorem, we have

$$\begin{aligned}G\big(F(x_{2n},y_{2n}),F(x_{2n},y_{2n}),F(x_{2n+1},y_{2n+1})\big)<\\ \max\big\{G(Tx_{2n},Tx_{2n},Sx_{2n+1}),G(Ty_{2n},Ty_{2n},Sy_{2n+1})\big\}.\end{aligned} \tag{8}$$

Again, using (5) and (2), we have

$$\begin{aligned}\psi\big[G\big(F(y_{2n},x_{2n}),F(y_{2n},x_{2n}),F(y_{2n+1},x_{2n+1})\big)\big]\leq\\ \varphi\big(\max\big\{G(Ty_{2n},Ty_{2n},Sy_{2n+1}),G(Tx_{2n},Tx_{2n},Sx_{2n+1})\big\}\big).\end{aligned} \tag{9}$$

Using the condition of the theorem, we have

$$\begin{aligned}G\big(F(y_{2n},x_{2n}),F(y_{2n},x_{2n}),F(y_{2n+1},x_{2n+1})\big)<\\ \max\big\{G(Ty_{2n},Ty_{2n},Sy_{2n+1}),G(Tx_{2n},Tx_{2n},Sx_{2n+1})\big\}.\end{aligned} \tag{10}$$

Combining (8) and (10), we obtain

$$\begin{aligned}\max\big\{G\big(F(x_{2n},y_{2n}),F(x_{2n},y_{2n}),F(x_{2n+1},y_{2n+1})\big),\\ G\big(F(y_{2n},x_{2n}),F(y_{2n},x_{2n}),F(y_{2n+1},x_{2n+1})\big)\big\}\leq\\ \max\big\{G(Tx_{2n},Tx_{2n},Sx_{2n+1}),G(Ty_{2n},Ty_{2n},Sy_{2n+1})\big\}.\end{aligned}$$

Then $\Big(\max\big\{G(Tx_{2n},Tx_{2n},Sx_{2n+1},G(Ty_{2n},Ty_{2n},Sy_{2n+1}))\big\}\Big)$ is a positive decreasing sequence. Hence there exists $r\geq 0$ such that

$$\lim_{n\longrightarrow+\infty}\max\big\{G(Tx_{2n},Tx_{2n},Sx_{2n+1}),G(Ty_{2n},Ty_{2n},Sy_{2n+1})\big\}=r.$$

Combining (7) and (9), we obtain

$$\begin{aligned}\max\big\{\psi\big(G(Tx_{2n+2},Tx_{2n+2},Sx_{2n+3})\big),\psi\big(G(Ty_{2n+2},Ty_{2n+2},Sy_{2n+3})\big)\big\}<\\ \varphi\big(\max\big\{G(Ty_{2n},Ty_{2n},Sy_{2n+1}),G(Tx_{2n},Tx_{2n},Sx_{2n+1})\big\}\big).\end{aligned}$$

Since ψ is non-decreasing, we get

$$\psi\Big(\max\big\{G(Tx_{2n+2},Tx_{2n+2},Sx_{2n+3}),G(Ty_{2n+2},Ty_{2n+2},Sy_{2n+3})\big\}\Big) < \varphi\big(\max\big\{G(Ty_{2n},Ty_{2n},Sy_{2n+1}),G(Tx_{2n},Tx_{2n},Sx_{2n+1})\big\}\big).$$

Letting $n \longrightarrow +\infty$ in the above inequality, we get $\psi(r) \leq \varphi(r)$, by using the condition of the theorem 3.1, we have $r = 0$. Consequently

$$\lim_{n\longrightarrow+\infty} \max\Big\{G\big(F(x_{2n},y_{2n}),F(x_{2n},y_{2n}),F(x_{2n+1},y_{2n+1})\big), G\big(F(y_{2n},x_{2n}),F(y_{2n},x_{2n}),F(y_{2n+1},x_{2n+1})\big)\Big\} = 0. \tag{11}$$

By the same way, we obtain

$$\lim_{n\longrightarrow+\infty} \max\Big\{G\big(F(x_{2n+2},y_{2n+2}),F(x_{2n+2},y_{2n+2}),F(x_{2n+1},y_{2n+1})\big), G\big(F(y_{2n+2},x_{2n+2}),F(y_{2n+2},x_{2n+2}),F(y_{2n+1},x_{2n+1})\big)\Big\} = 0. \tag{12}$$

Finally, (11) and (12) give the desired result, that is, (6) holds.

Step 2: We will prove that $(F(x_n,y_n))$ and $F((y_n,x_n))$ are Cauchy sequences. From (6), it is sufficient to show that $(F(x_{2n},y_{2n}))$ and $(F(y_{2n},x_{2n}))$ are Cauchy sequences. We proceed by negation and suppose that at least one of the sequences $(F(x_{2n},y_{2n}))$ or $(F(y_{2n},x_{2n}))$ is not a Cauchy sequence. This implies that

$$G\big(F(x_{2n},y_{2n}),F(x_{2m},y_{2m}),F(x_{2m},y_{2m})\big) \nrightarrow 0$$

or

$$G\big(F(y_{2n},x_{2n}),F(y_{2m},x_{2m}),F(y_{2m},x_{2m})\big) \nrightarrow 0$$

as $n,m \longrightarrow +\infty$. Consequently

$$\max\Big\{G\big(F(x_{2n},y_{2n}),F(x_{2m},y_{2m}),F(x_{2m},y_{2m})\big), G\big(F(y_{2n},x_{2n}),F(y_{2m},x_{2m}),F(y_{2m},x_{2m})\big)\Big\} \nrightarrow 0,$$

as $n,m \longrightarrow +\infty$. Then there exists $\varepsilon > 0$ for which we can find two subsequences of positive integers (m_i) and (n_i) such that n_i is the smallest index for which $n_i > m_i > i$,

$$\max\Big\{G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2n_i},y_{2n_i})\big), G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2n_i},x_{2n_i})\big)\Big\} \geq \varepsilon. \tag{13}$$

This means that

$$\begin{aligned}(14)\qquad &\max\Big\{G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2n_i-2},y_{2n_i-2})\big),\\ &\qquad G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2n_i-2},x_{2n_i-2})\big)\Big\}<\varepsilon.\end{aligned}$$

From (1), (14) and using the rectangle property of G , we get

$$\begin{aligned}\varepsilon &\le \max\Big\{G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2n_i},y_{2n_i})\big),\\ &\qquad G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2n_i},x_{2n_i})\big)\Big\}\\ &\le \max\Big\{G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2n_i-2},y_{2n_i-2})\big),\\ &\qquad G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2n_i-2},x_{2n_i-2})\big)\Big\}\\ &\quad+\max\Big\{G\big(F(x_{2n_i-2},y_{2n_i-2}),F(x_{2n_i-2},y_{2n_i-2}),F(x_{2n_i-1},y_{2n_i-1})\big),\\ &\qquad G\big(F(y_{2n_i-2},x_{2n_i-2}),F(y_{2n_i-2},x_{2n_i-2}),F(y_{2n_i-1},x_{2n_i-1})\big)\Big\}\\ &\quad+\max\Big\{G\big(F(x_{2n_i-1},y_{2n_i-1}),F(x_{2n_i-1},y_{2n_i-1}),F(x_{2n_i},y_{2n_i})\big),\\ &\qquad G\big(F(y_{2n_i-1},x_{2n_i-1}),F(y_{2n_i-1},x_{2n_i-1}),F(y_{2n_i},x_{2n_i})\big)\Big\}.\end{aligned}$$

Letting $i\longrightarrow+\infty$ in above inequality and using (6), we obtain that

$$\begin{aligned}(15)\qquad &\lim_{i\longrightarrow+\infty}\max\Big\{G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2n_i},y_{2n_i})\big),\\ &\qquad G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2n_i},x_{2n_i})\big)\Big\}=\varepsilon.\end{aligned}$$

Also, we have

$$\begin{aligned}
\varepsilon \leq \max\Big\{ & G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2n_i},y_{2n_i})\big),\\
& G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2n_i},x_{2n_i})\big)\Big\}\\
\leq \max\Big\{ & G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2m_i-1},y_{2m_i-1})\big),\\
& G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2m_i-1},x_{2m_i-1})\big)\Big\}\\
+\max\Big\{ & G\big(F(x_{2m_i-1},y_{2m_i-1}),F(x_{2m_i-1},y_{2m_i-1}),F(x_{2n_i},y_{2n_i})\big),\\
& G\big(F(y_{2m_i-1},x_{2m_i-1}),F(y_{2m_i-1},x_{2m_i-1}),F(y_{2n_i},x_{2n_i})\big)\Big\}\\
\leq \max\Big\{ & G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2m_i-1},y_{2m_i-1})\big),\\
& G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2m_i-1},x_{2m_i-1})\big)\Big\}\\
+\max\Big\{ & G\big(F(x_{2m_i-1},y_{2m_i-1}),F(x_{2m_i-1},y_{2m_i-1}),F(x_{2m_i},y_{2m_i})\big),\\
& G\big(F(y_{2m_i-1},x_{2m_i-1}),F(y_{2m_i-1},x_{2m_i-1}),F(y_{2m_i},x_{2m_i})\big)\Big\}\\
+\max\Big\{ & G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2n_i},y_{2n_i})\big),\\
& G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2n_i},x_{2n_i})\big)\Big\}.
\end{aligned}$$

Using that $G(x,x,y) \leq 2G(x,y,y)$ for any $x,y \in X$, we obtain

$$\begin{aligned}
\varepsilon \leq \max\Big\{ & G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2n_i},y_{2n_i})\big),\\
& G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2n_i},x_{2n_i})\big)\Big\}\\
\leq \max\Big\{ & G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2m_i-1},y_{2m_i-1})\big),\\
& G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2m_i-1},x_{2m_i-1})\big)\Big\}\\
+\max\Big\{ & G\big(F(x_{2m_i-1},y_{2m_i-1}),F(x_{2m_i-1},y_{2m_i-1}),F(x_{2n_i},y_{2n_i})\big),\\
& G\big(F(y_{2m_i-1},x_{2m_i-1}),F(y_{2m_i-1},x_{2m_i-1}),F(y_{2n_i},x_{2n_i})\big)\Big\}\\
\leq 3\max\Big\{ & G\big(F(x_{2m_i-1},y_{2m_i-1}),F(x_{2m_i-1},y_{2m_i-1}),F(x_{2m_i},y_{2m_i})\big),\\
& G\big(F(y_{2m_i-1},x_{2m_i-1}),F(y_{2m_i-1},x_{2m_i-1}),F(y_{2m_i},x_{2m_i})\big)\Big\}\\
+\max\Big\{ & G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2n_i},y_{2n_i})\big),\\
& G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2n_i},x_{2n_i})\big)\Big\}.
\end{aligned}$$

Using (6), (15) and letting $i \longrightarrow +\infty$ in the above inequality, we obtain

$$\begin{aligned}
\lim_{i\longrightarrow+\infty}\max\Big\{ & G\big(F(x_{2m_i-1},y_{2m_i-1}),F(x_{2m_i-1},y_{2m_i-1}),F(x_{2n_i},y_{2n_i})\big),\\
& G\big(F(y_{2m_i-1},x_{2m_i-1}),F(y_{2m_i-1},x_{2m_i-1}),F(y_{2n_i},x_{2n_i})\big)\Big\} = \varepsilon.
\end{aligned} \tag{16}$$

On other hand, we have

$$\begin{aligned}
&\max\Big\{G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2n_i},y_{2n_i})\big),\\
&\qquad G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2n_i},x_{2n_i})\big)\Big\}\\
&\le \max\Big\{G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2n_i+1},y_{2n_i+1})\big),\\
&\qquad G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2n_i+1},x_{2n_i+1})\big)\Big\}\\
&\quad+\max\Big\{G\big(F(x_{2n_i+1},y_{2n_i+1}),F(x_{2n_i+1},y_{2n_i+1}),F(x_{2n_i},y_{2n_i})\big),\\
&\qquad G\big(F(y_{2n_i+1},x_{2n_i+1}),F(y_{2n_i+1},x_{2n_i+1}),F(y_{2n_i},x_{2n_i})\big)\Big\}.
\end{aligned}$$

Since ψ is a continuous non-decreasing function, it follows from the above inequality that

$$\begin{aligned}
\psi(\varepsilon)\le\psi\Big(&\max\Big\{G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2n_i+1},y_{2n_i+1})\big),\\
&G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2n_i+1},x_{2n_i+1})\big)\Big\}\Big)
\end{aligned}\tag{17}$$

Using the contractive condition, on one hand we have

$$\begin{aligned}
&\psi\Big(G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2n_i+1},y_{2n_i+1})\big)\Big)\le\\
&\varphi\Big(\max\Big\{G(Tx_{2m_i},Tx_{2m_i},Sx_{2n_i+1}),G(Ty_{2m_i},Ty_{2m_i},Sy_{2n_i+1})\Big\}\Big)\le\\
&\varphi\Big(\max\Big\{G\big(F(x_{2m_i-2},y_{2m_i-2}),F(x_{2m_i-2},y_{2m_i-2}),F(x_{2n_i-1},y_{2n_i-1})\big),\\
&\qquad G\big(F(y_{2m_i-2},x_{2m_i-2}),F(y_{2m_i-2},x_{2m_i-2}),F(y_{2n_i-1},x_{2n_i-1})\big)\Big\}\Big).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\psi\Big(G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2n_i+1},x_{2n_i+1})\big)\Big)\le\\
&\varphi\Big(\max\Big\{G(Ty_{2m_i},Ty_{2m_i},Sy_{2n_i+1}),G(Tx_{2m_i},Tx_{2m_i},Sx_{2n_i+1})\Big\}\Big)\le\\
&\varphi\Big(\max\Big\{G\big(F(y_{2m_i-2},x_{2m_i-2}),F(y_{2m_i-2},x_{2m_i-2}),F(y_{2n_i-1},x_{2n_i-1})\big),\\
&\qquad G\big(F(x_{2m_i-2},y_{2m_i-2}),F(x_{2m_i-2},y_{2m_i-2}),F(x_{2n_i-1},y_{2n_i-1})\big)\Big\}\Big).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\max\Big\{\psi\Big(G\big(F(x_{2m_i},y_{2m_i}),F(x_{2m_i},y_{2m_i}),F(x_{2n_i+1},y_{2n_i+1})\big)\Big),\\
&\qquad\psi\Big(G\big(F(y_{2m_i},x_{2m_i}),F(y_{2m_i},x_{2m_i}),F(y_{2n_i+1},x_{2n_i+1})\big)\Big)\Big\}\le\\
&\psi\Big(\max\Big\{G\big(F(y_{2m_i-2},x_{2m_i-2}),F(y_{2m_i-2},x_{2m_i-2}),F(y_{2n_i-1},x_{2n_i-1})\big),\\
&\qquad G\big(F(x_{2m_i-2},y_{2m_i-2}),F(x_{2m_i-2},y_{2m_i-2}),F(x_{2n_i-1},y_{2n_i-1})\big)\Big\}.
\end{aligned}\tag{18}$$

We claim that

$$\lim_{i\longrightarrow+\infty}\max\Big\{G\big(F(x_{2m_i-2},y_{2m_i-2}),F(x_{2m_i-2},y_{2m_i-2}),F(x_{2n_i-1},y_{2n_i-1})\big),\\ G\big(F(y_{2m_i-2},x_{2m_i-2}),F(y_{2m_i-2},x_{2m_i-2}),F(y_{2n_i-1},x_{2n_i-1})\big)\Big\}=\varepsilon. \tag{19}$$

In fact, using the rectangle property, we have

$$\begin{gathered}G\big(F(x_{2m_i-2},y_{2m_i-2}),F(x_{2m_i-2},y_{2m_i-2}),F(x_{2n_i-1},y_{2n_i-1})\big)\leq\\ G\big(F(x_{2m_i-2},y_{2m_i-2}),F(x_{2m_i-2},y_{2m_i-2}),F(x_{2m_i-1},y_{2m_i-1})\big)\\ +G\big(F(x_{2m_i-1},y_{2m_i-1}),F(x_{2m_i-1},y_{2m_i-1}),F(x_{2n_i},y_{2n_i})\big)\\ +G\big(F(x_{2n_i},y_{2n_i}),F(x_{2n_i},y_{2n_i}),F(x_{2n_i-1},y_{2n_i-1})\big).\end{gathered}$$

Letting $i\longrightarrow+\infty$ in the above inequality and using (6) and (16), we obtain

$$\lim_{i\longrightarrow+\infty}G\big(F(x_{2m_i-2},y_{2m_i-2}),F(x_{2m_i-2},y_{2m_i-2}),F(x_{2n_i-1},y_{2n_i-1})\big)\leq\varepsilon \tag{20}$$

On the other hand, we have

$$\begin{gathered}G\big(F(x_{2m_i-1},y_{2m_i-1}),F(x_{2m_i-1},y_{2m_i-1}),F(x_{2n_i},y_{2n_i})\big)\leq\\ G\big(F(x_{2m_i-1},y_{2m_i-1}),F(x_{2m_i-1},y_{2m_i-1}),F(x_{2m_i-2},y_{2m_i-2})\big)\\ +G\big(F(x_{2m_i-2},y_{2m_i-2}),F(x_{2m_i-2},y_{2m_i-2}),F(x_{2n_i-1},y_{2n_i-1})\big)\\ +G\big(F(x_{2n_i-1},y_{2n_i-1}),F(x_{2n_i-1},y_{2n_i-1}),F(x_{2n_i},y_{2n_i})\big).\end{gathered}$$

Letting $i\longrightarrow+\infty$ in the above inequality and using (6) and (16), we obtain

$$\lim_{i\longrightarrow+\infty}G\big(F(x_{2m_i-2},y_{2m_i-2}),F(x_{2m_i-2},y_{2m_i-2}),F(x_{2n_i-1},y_{2n_i-1})\big)\geq\varepsilon. \tag{21}$$

Combining (20) and (21), we get

$$\lim_{i\longrightarrow+\infty}G\big(F(x_{2m_i-2},y_{2m_i-2}),F(x_{2m_i-2},y_{2m_i-2}),F(x_{2n_i-1},y_{2n_i-1})\big)=\varepsilon.$$

By the same way, we obtain

$$\lim_{i\longrightarrow+\infty}G\big(F(y_{2m_i-2},x_{2m_i-2}),F(y_{2m_i-2},x_{2m_i-2}),F(y_{2n_i-1},x_{2n_i-1})\big)=\varepsilon.$$

Thus we proved (19). Finally, letting $i\longrightarrow+\infty$ in (18), using (17), (19) and the continuity of ψ and ϕ, we get $\psi(\varepsilon)\leq\varphi(\varepsilon)$, using the condition of the theorem 3.1, we get $\varepsilon=0$, which is a contradiction. Thus $(F(x_{2n},y_{2n}))$ and $(F(y_{2n},x_{2n}))$ are Cauchy sequences in X, which gives us that $(F(x_n,y_n))$ and $(F(y_n,x_n))$ are also Cauchy sequences.

Step 3: Existence of a coupled coincidence point.

Since $(F(x_n,y_n))$ and $((F(y_n,x_n)))$ are Cauchy sequences in the complete metric space (X,G), there exist $\alpha,\beta \in X$ such that:

$$\lim_{n\longrightarrow+\infty} F(x_n,y_n)=\alpha \text{ and } \lim_{n\longrightarrow+\infty} F(y_n,x_n)=\beta.$$

Therefore, $\lim_{n\longrightarrow+\infty} Tx_{2n+2}=\alpha$, $\lim_{n\longrightarrow+\infty} Ty_{2n+2}=\beta$, $\lim_{n\longrightarrow+\infty} Sx_{2n+3}=\alpha$ and

$$\lim_{n\longrightarrow+\infty} Sy_{2n+3}=\beta.$$

Using the continuity and the commutativity of F and T, we have

$$T(Tx_{2n+2})=T(F(x_{2n},y_{2n}))=F(Tx_{2n},Ty_{2n})$$

and

$$T(Ty_{2n+2})=T(F(y_{2n},x_{2n}))=F(Ty_{2n},Tx_{2n}).$$

Letting $n\longrightarrow+\infty$, we get $T\alpha=F(\alpha,\beta)$ and $T\beta=F(\beta,\alpha)$.
Using also the continuity and the commutativity of F and S, by the same way, we obtain $S\alpha=F(\alpha,\beta)$ and $S\beta=F(\beta,\alpha)$. Therefore $T\alpha=F(\alpha,\beta)=S\alpha$ and $T\beta=F(\beta,\alpha)=S\beta$. Thus we proved that (α,β) is a coupled coincidence point of T,S and F. This completes the proof.

In the next result, we prove that the previous theorem is still valid if we replace the continuity of F by some conditions.

Theorem 3.2 *If we replace the continuity hypothesis of F in Theorem 3.1 by the following conditions:*
(i) if a non-decreasing sequence (x_n) is such that $x_n\longrightarrow x$, then $x_n\preceq x$ for all n,
(ii) if a non-increasing sequence (y_n) is such that $y_n\longrightarrow y$, then $y\preceq y_n$ for all n.
(iii) $x,y\in X,\ x\preceq y\Longrightarrow Tx\preceq Sy$,
(iv) $x,y\in X,\ x\succeq y\Longrightarrow Tx\succeq Sy$.
Then T,S and F have a coupled coincidence point.

Proof. Following the proof of Theorem 3.1, we have that $F(x_n,y_n)$ and $F(y_n,x_n)$ are Cauchy sequences in the complete metric space (X,G), there exist $\alpha,\beta\in X$ such that

$$\lim_{n\longrightarrow+\infty} F(x_n,y_n)=\alpha \text{ and } \lim_{n\longrightarrow+\infty} F(y_n,x_n)=\beta.$$

Therefore

$$\lim_{n\longrightarrow+\infty} F(x_{2n},y_{2n}) = \alpha \text{ and } \lim_{n\longrightarrow+\infty} F(y_{2n},x_{2n}) = \beta.$$

Hence $\lim_{n\longrightarrow+\infty} Tx_{2n+2} = \alpha$, $\lim_{n\longrightarrow+\infty} Ty_{2n+2} = \beta$, $\lim_{n\longrightarrow+\infty} Sx_{2n+3} = \alpha$ and $\lim_{n\longrightarrow+\infty} Sy_{2n+3} = \beta$. Using the commutativity of F and G and of F and S and the contractive condition, it follows from conditions (iii)-(iv) that

$$\begin{aligned}
&\psi\Big(G\big(T(F(x_{2n},y_{2n})),T(F(x_{2n},y_{2n})),S(F(x_{2n+1},y_{2n+1}))\big)\Big) = \\
&\psi\Big(G\big(F(Tx_{2n},Ty_{2n}),F(Tx_{2n},Ty_{2n}),F(Sx_{2n+1},Sy_{2n+1})\big)\Big) \leq \\
&\varphi\Big(\max\Big\{G\big(T(Tx_{2n}),T(Tx_{2n}),S(Sx_{2n+1})\big),G\big(T(Ty_{2n}),T(Ty_{2n}),S(Sy_{2n+1})\big)\Big\}\Big).
\end{aligned} \tag{22}$$

Similarly, we have

$$\begin{aligned}
&\psi\Big(G\big(T(F(y_{2n},x_{2n})),T(F(y_{2n},x_{2n})),S(F(y_{2n+1},x_{2n+1}))\big)\Big) = \\
&\varphi\Big(G\big(F(Ty_{2n},Tx_{2n}),F(Ty_{2n},Tx_{2n}),F(Sy_{2n+1},Sx_{2n+1})\big)\Big) \leq \\
&\varphi\Big(\max\Big\{G\big(T(Ty_{2n}),T(Ty_{2n}),S(Sy_{2n+1})\big),G\big(T(Tx_{2n}),T(Tx_{2n}),S(Sx_{2n+1})\big)\Big\}\Big).
\end{aligned} \tag{23}$$

Combining (22), (23) and the fact that $\max\{\varphi(a),\varphi(b)\} = \varphi(\max\{a,b\})$ for $a,b \in [0,+\infty)$, from (iii)-(iv), we obtain

$$\begin{aligned}
&\psi\Big(\max\Big\{G\big(T(F(x_{2n},y_{2n})),T(F(x_{2n},y_{2n})),S(F(x_{2n+1},y_{2n+1}))\big),\\
&\qquad G\big(T(F(y_{2n},x_{2n})),T(F(y_{2n},x_{2n})),S(F(y_{2n+1},x_{2n+1}))\big)\Big\}\Big) \leq \\
&\varphi\Big(\max\Big\{G\big(T(Tx_{2n}),T(Tx_{2n}),S(Sx_{2n+1})\big),G\big(T(Ty_{2n}),T(Ty_{2n}),S(Sy_{2n+1})\big)\Big\}\Big).
\end{aligned}$$

Letting $n \longrightarrow +\infty$ in the last expression, using the continuity of T and S, we get This implies that

$$\begin{aligned}
&\psi\Big(\max\big\{G(T\alpha,T\alpha,S\alpha),G(T\beta,T\beta,S\beta)\big\}\Big) \leq \\
&\varphi\Big(\max\big\{G(T\alpha,T\alpha,S\alpha),G(T\beta,T\beta,S\beta)\big\}\Big).
\end{aligned}$$

By using the condition of the theorem 3.2, we have $\max\big\{G(T\alpha,T\alpha,S\alpha),G(T\beta,T\beta,S\beta)\big\} = 0$ Consequently

$$T\alpha = S\alpha \text{ and } T\beta = S\beta. \tag{24}$$

To finish the proof, we claim that $F(\alpha,\beta) = T\alpha = S\alpha$ and $F(\beta,\alpha) = T\beta = S\beta$.

Indeed, using the contractive condition, it follows from (i)-(iv) that

$$\psi\Big(G\big(F(T(x_{2n},Ty_{2n})),F(T(x_{2n},Ty_{2n})),F(\alpha,\beta)\big)\Big), \leq$$
$$\varphi\Big(\max\Big\{G\big(T(Tx_{2n}),T(Tx_{2n}),S\alpha\big),G\big(T(Ty_{2n}),T(Ty_{2n}),S\beta\big)\Big\}\Big).$$

Using the condition of the theorem, we get

$$\begin{aligned} &G\big(F(T(x_{2n},Ty_{2n})),F(T(x_{2n},Ty_{2n})),F(\alpha,\beta)\big) \leq \\ &\max\Big\{G\big(T(Tx_{2n}),T(Tx_{2n}),S\alpha\big),G\big(T(Ty_{2n}),T(Ty_{2n}),S\beta\big)\Big\}. \end{aligned} \tag{25}$$

Similarly, we have

$$\psi\Big(G\big(F(T(y_{2n},Tx_{2n})),F(T(y_{2n},Tx_{2n})),F(\beta,\alpha)\big)\Big) \leq$$
$$\varphi\Big(\max\Big\{G\big(T(Ty_{2n}),T(Ty_{2n}),S\beta\big),G\big(T(Tx_{2n}),T(Tx_{2n}),S\alpha\big)\Big\}\Big).$$

Using the condition of the theorem 3.2, we see that

$$\begin{aligned} &G\big(F(T(y_{2n},Tx_{2n})),F(T(y_{2n},Tx_{2n})),F(\beta,\alpha)\big) \leq \\ &\max\Big\{G\big(T(Ty_{2n}),T(Ty_{2n}),S\beta\big),G\big(T(Tx_{2n}),T(Tx_{2n}),S\alpha\big)\Big\}. \end{aligned} \tag{26}$$

Combining (25) and (26), we get

$$\begin{aligned} \max\Big\{&G\big(F(T(x_{2n},Ty_{2n})),F(T(x_{2n},Ty_{2n})),F(\alpha,\beta)\big), \\ &G\big(F(T(y_{2n},Tx_{2n})),F(T(y_{2n},Tx_{2n})),F(\beta,\alpha)\big)\Big\} \leq \\ \max\Big\{&G\big(T(Ty_{2n}),T(Ty_{2n}),S\beta\big),G\big(T(Tx_{2n}),T(Tx_{2n}),S\alpha\big)\Big\}. \end{aligned}$$

Using the commutativity of F and T, we write

$$\begin{aligned} \max\Big\{&G\big(T(F(x_{2n},y_{2n})),T(F(x_{2n},y_{2n})),F(\alpha,\beta)\big), \\ &G\big(T(F(y_{2n},x_{2n})),T(F(y_{2n},x_{2n})),F(\beta,\alpha)\big)\Big\} \leq \\ \max\Big\{&G\big(T(Ty_{2n}),T(Ty_{2n}),S\beta\big),G\big(T(Tx_{2n}),T(Tx_{2n}),S\alpha\big)\Big\}. \end{aligned}$$

Letting $n \longrightarrow +\infty$, using the continuity of T, we obtain

$$\max\Big\{G\big(T\alpha,T\alpha,F(\alpha,\beta)\big),G\big(T\beta,T\beta,F(\beta,\alpha)\big)\Big\} \leq \max\Big\{G\big(T\beta,T\beta,S\beta\big),G\big(T\alpha,T\alpha,S\alpha\big)\Big\}.$$

In view of (24), we deduce that

$$\max\Big\{G\big(T\alpha,T\alpha,F(\alpha,\beta)\big),G\big(T\beta,T\beta,F(\beta,\alpha)\big)\Big\} = 0.$$

Therefore, $G\big(T\alpha, T\alpha, F(\alpha,\beta)\big) = 0$ and $G\big(T\beta, T\beta, F(\beta,\alpha)\big) = 0$. Consequently

$$T\alpha = F(\alpha,\beta) \text{ and } T\beta = F(\beta,\alpha). \tag{27}$$

By the same way, we get

$$S\alpha = F(\alpha,\beta) \text{ and } S\beta = F(\beta,\alpha). \tag{28}$$

Finally, combining (24), (27) and (28), we deduce that (α,β) is a coupled coincidence point of F, T and S. This completes the proof.

Now, we give a sufficient condition for the existence and the uniqueness of the coupled common fixed point. Notice that if $(X,\preceq)$ is a partially ordered set, we endow $X\times X$ with the following partial order relation: for

$$(x,y),(u,v)\in X\times X, \quad (x,y)\preceq(u,v) \Longleftrightarrow x\preceq u \text{ and } y\succeq v.$$

Theorem 3.3 *In addition to the hypotheses of Theorem 3.1 (resp. Theorem 3.2, suppose that for every* (x,y), $(x',y')\in X\times X$ *there exists a* $(u,v)\in X\times X$ *such that* $(F(u,v),F(v,u))$ *is comparable to* $(F(x,y),F(y,x))$ *and* $(F(x',y'),F(y',x'))$. *Then* F,T *and* S *have a unique coupled common fixed point, that is, there exist a unique* $(x,y)\in X\times X$ *such that*
$x = Tx = F(x,y) = Sx$ *and* $y = Ty = F(y,x) = Sy$.

Proof. We know, from Theorem 3.1 (resp. Theorem 3.2, that exists a coupled coincidence point. We suppose that exist (x,y) and (x',y') two coupled coincidence points, that is,

$$Tx = F(x,y) = Sx,\ Ty = F(y,x) = Sy,$$

$$Tx' = F(x',y') = Sx' \text{ and } Ty' = F(y',x') = Sy'.$$

We claim that

$$Tx = Tx' = Sx' = Sx \text{ and } Ty = Ty' = Sy' = Sy. \tag{29}$$

By assumption there is $(u,v)\in X\times X$ such that $(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and $(F(x',y'),F(y',x'))$. We distinguish two cases:

First case: We assume that
$(F(x,y),F(y,x))\preceq(F(u,v),F(v,u))$ and $(F(x',y'),F(y',x'))\preceq(F(u,v),F(v,u))$.

Put $u_0 = u$ and $v_0 = v$ and we choose u_1 and v_1 such that $Tu_0 \preceq Su_1 \preceq F(u_0, v_0)$, $Tv_0 \succeq Sv_1 \succeq F(v_0, u_0)$. Similarly as in the proof of Theorem **??**, we can construct sequences (u_n) and (v_n) in X such that $\begin{cases} Tu_{2n+2} = F(u_{2n}, v_{2n}) \\ Tv_{2n+2} = F(v_{2n}, u_{2n}) \end{cases}$ and $\begin{cases} Su_{2n+3} = F(u_{2n+1}, v_{2n+1}) \\ Sv_{2n+3} = F(v_{2n+1}, u_{2n+1}) \end{cases}$ for all $n \geq 0$. Looking at the proof of Theorem 3.1, precisely at (5), we see that (Tu_{2n}) is a non- decreasing sequence, $Tu_{2n} \preceq Su_{2n+1}$, and (Tv_{2n}) is a non-increasing sequence, $Tv_{2n} \succeq Sv_{2n+1}$. Therefore, we have

$$\begin{gathered} Tx = F(x,y) \preceq F(u_0, v_0) = Tu_2 \preceq Tu_{2n} \preceq Su_{2n+1} \\ \textit{and} \\ Ty = F(y,x) \succeq F(v_0, u_0) = Tv_2 \succeq Tv_{2n} \succeq Sv_{2n+1}. \end{gathered} \tag{30}$$

Similarly, we have

$$\begin{gathered} Tx' = F(x',y') \preceq F(u_0, v_0) = Tu_2 \preceq Tu_{2n} \preceq Su_{2n+1} \\ \textit{and} \\ Ty' = F(y',x') \succeq F(v_0, u_0) = Tv_2 \succeq Tv_{2n} \succeq Sv_{2n+1}. \end{gathered} \tag{31}$$

Using (30) and the contractive condition, we write

$$\begin{aligned} &\psi\Big(G\big(F(x,y), F(x,y), F(u_{2n+1}, v_{2n+1})\big)\Big) \leq \\ &\varphi\Big(\max\Big\{G(Tx, Tx, Su_{2n+1}), G(Ty, Ty, Sv_{2n+1})\Big\}\Big) \end{aligned}$$

and

$$\begin{aligned} &\psi\Big(G\big(F(y,x), F(y,x), F(v_{2n+1}, u_{2n+1})\big)\Big) \leq \\ &\varphi\Big(\max\Big\{G(Ty, Ty, Sv_{2n+1}), G(Tx, Tx, Su_{2n+1})\Big\}\Big). \end{aligned}$$

Therefore

$$\begin{aligned} &\psi\Big(\max\Big\{G\big(F(x,y), F(x,y), F(u_{2n+1}, v_{2n+1})\big), \\ &\qquad\qquad G\big(F(y,x), F(y,x), F(v_{2n+1}, u_{2n+1})\big)\Big\}\Big) \leq \\ &\varphi\Big(\max\Big\{G(Tx, Tx, Su_{2n+1}), G(Ty, Ty, Sv_{2n+1})\Big\}\Big). \end{aligned}$$

Therefore

$$\begin{aligned} &\psi\Big(\max\Big\{G(Tx, Tx, Su_{2n+3}), G\big(Ty, Ty, Sv_{2n+3}\big)\Big\}\Big) \leq \\ &\varphi\Big(\max\Big\{G(Tx, Tx, Su_{2n+1}), G(Ty, Ty, Sv_{2n+1})\Big\}\Big). \end{aligned} \tag{32}$$

We see that

$$\begin{aligned} &\psi\Big(\max\Big\{G(Tx, Tx, Su_{2n+3}), G\big(Ty, Ty, Sv_{2n+3}\big)\Big\}\Big) \leq \\ &\varphi\Big(\max\Big\{G(Tx, Tx, Su_{2n+1}), G(Ty, Ty, Sv_{2n+1})\Big\}\Big). \end{aligned}$$

Using the condition of the theorem 3.3, we get

$$\max\Big\{G(Tx,Tx,Su_{2n+3}),G\big(Ty,Ty,Sv_{2n+3})\Big\}\Big)$$
$$\leq \max\Big\{G(Tx,Tx,Su_{2n+1}),G(Ty,Ty,Sv_{2n+1})\Big\}.$$

This implies that $\Big(\max\big\{G(Tx,Tx,Su_{2n+1}),G(Ty,Ty,Sv_{2n+1})\big\}\Big)$ is a non-increasing sequence. Hence, there exists $r \geq 0$ such that

$$\lim_{n\longrightarrow+\infty}\max\big\{G(Tx,Tx,Su_{2n+1}),G(Ty,Ty,Sv_{2n+1})\big\}=r.$$

Passing to limit in (32) as $n \longrightarrow +\infty$, we obtain

$$\psi(r) \leq \varphi(r),$$

by using the condition of the theorem 3.3, we get, $r = 0$. We deduce that

$$\lim_{n\longrightarrow+\infty}\max\big\{G(Tx,Tx,Su_{2n+1}),G(Ty,Ty,Sv_{2n+1})\big\}=0. \tag{33}$$

Similarly, one can prove that

$$\lim_{n\longrightarrow+\infty}\max\big\{G(Tx',Tx',Su_{2n+1}),G(Ty',Ty',Sv_{2n+1})\big\}=0. \tag{34}$$

By uniqueness of the limit and using (33) and (34), we have $\lim_{n\longrightarrow+\infty} Su_{2n+1} = Tx = Tx'$ and $\lim_{n\longrightarrow+\infty} Sv_{2n+1} = Ty = Ty'$. This prove the claim (29) in this case.

Second case: We assume that

$(F(x,y),F(y,x)) \succeq (F(u,v),F(v,u))$ and $(F(x',y'),F(y',x')) \succeq (F(u,v),F(v,u))$.

Put $u_0 = u$ and $v_0 = v$ and we choose u_1 and v_1 such that $Tu_0 \preceq Su_1 \preceq F(u_0,v_0)$, $Tv_0 \succeq Sv_1 \succeq F(v_0,u_0)$. Similarly as in the proof of Theorem 3.1, we can construct sequences (u_n) and (v_n) in X such that $\begin{cases} Tu_{2n+2} = F(u_{2n},v_{2n}) \\ Tv_{2n+2} = F(v_{2n},u_{2n}) \end{cases}$ and $\begin{cases} Su_{2n+3} = F(u_{2n+1},v_{2n+1}) \\ Sv_{2n+3} = F(v_{2n+1},u_{2n+1}) \end{cases}$ for all $n \geq 0$.

Looking at the proof of Theorem 3.1, precisely at (5), we see that (Tu_{2n}) is a non- decreasing sequence, $Tu_{2n} \preceq Su_{2n+1}$, and (Tv_{2n}) is a non-increasing sequence, $Tv_{2n} \succeq Sv_{2n+1}$. Therefore, we have

$$Tx = F(x,y) \succeq F(u_0,v_0) = Tu_2 \succeq Tu_{2n} \succeq Su_{2n+1}$$

and

$$Ty = F(y,x) \preceq F(v_0,u_0) = Tv_2 \preceq Tv_{2n} \preceq Sv_{2n+1}.$$

Similarly, we have

$$Tx' = F(x',y') \succeq F(u_0,v_0) = Tu_2 \succeq Tu_{2n} \succeq Su_{2n+1}$$

and

$$Ty' = F(y',x') \preceq F(v_0,u_0) = Tv_2 \preceq Tv_{2n} \preceq Sv_{2n+1}.$$

From this, we complete the proof identically as in the first case and we obtain the claim (29) in this case. Since $Tx = F(x,y) = Sx$ and $Ty = F(y,x) = Sy$, by the commutativity of F,T and F,S, we have

$$T(Tx) = T(F(x,y)) = F(Tx,Ty),\ T(Ty) = T(F(y,x)) = F(Ty,Tx)$$

(35) and

$$S(Sx) = S(F(x,y)) = F(Sx,Sy),\ S(Sy) = S(F(y,x)) = F(Sy,Sx).$$

Set $Tx = a = Sx$, $Ty = b = Sy$. Then from (35),

$$Ta = F(a,b) = Sa \quad \text{and} \quad Tb = F(b,a) = Sb. \tag{36}$$

Thus (a,b) is a coupled coincidence point. Then from (29) with $x' = a$ and $y' = b$ it follows that $Ta = Tx = Sa$ and $Tb = Ty = Sb$. Therefore

$$Ta = a = Sa \quad \text{and} \quad Tb = b = Sb. \tag{37}$$

We deduce that (a,b) is a coupled common fixed point. To prove the uniqueness, assume that (c,d) is another coupled common fixed point. Then by (29) and (37) we have $c = Tc = Ta = a$ and $d = Td = Tb = b$. This complete the proof.

Corollary 3.3 *Let $(X,\preceq)$ be a partially ordered set and G-metric on X such that (X,G) is a complete G-metric space. Let $g : X \longrightarrow X$ and $F : X \times X \longrightarrow X$ be a mapping having the mixed g-monotone property on X. Suppose that*

$$\varphi\big[G\big(F(x,y),F(u,v),F(w,z)\big)\big] \quad \leq \quad \varphi\big(\max\big\{G(gx,gu,gw),G(gy,gv,gz)\big\}\big)$$

for all $x,y,z,u,v,w \in X$ with $gx \preceq gu \preceq gw$ or $gx \succeq gu \succeq gw$ and $gy \succeq gv \succeq gz$ or $gy \preceq gv \preceq gz$, where $\psi \in \Psi$ and $\varphi : [0,\infty[\to IR$ is a continuous function with the condition $\psi(t) > \varphi(t)$ for all $t > 0$. Assume that $F(X \times X) \subseteq g(X)$ and assume also the following hypotheses:

(1) *g is continuous,*

(2) *F is continuous or g is non-decreasing mapping and X satisfies the following properties:*

(i) if a non-decreasing sequence (x_n) is such that $x_n \longrightarrow x$, then $x_n \preceq x$ for all n,

(ii) if a non-increasing sequence (y_n) is such that $y_n \longrightarrow y$, then $y \preceq y_n$ for all n.

(3) *For every $(x,y),\ (x',y') \in X \times X$ there exists a $(u,v) \in X \times X$ such that $(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and $(F(x',y'),F(y',x'))$.*

(4) *F commutes with g.*

If there exist x_0, y_0, x_1 and y_1 such that

$$gx_0 \preceq gx_1 \preceq F(x_0,y_0),\ \ gy_0 \succeq gy_1 \succeq F(y_0,x_0),$$

then there exist a unique $(x,y) \in X \times X$ such that $x = gx = F(x,y)$ and $y = gy = F(y,x)$, that is, then F and g have a unique coupled common fixed point.

Corollary 3.4 *Let $(X,\preceq)$ be a partially ordered set and G-metric on X such that (X,G) is a complete G-metric space. Let $T,S : X \longrightarrow X$ and $F : X \times X \longrightarrow X$ be a mapping having the mixed(T, S)-monotone property on X. Suppose that*

$$(38)\quad \begin{aligned} G\big(F(x,y),F(u,v),F(w,z)\big) \quad &\leq \quad \max\big\{G(Tx,Tu,Sw),G(Ty,Tv,Sz)\big\} \\ -\psi\big(\max\big\{G(Tx,Tu,Sw),G(Ty,Tv,Sz)\big\}\big) & \end{aligned}$$

for all $x,y,z,u,v,w \in X$ with $Tx \preceq Tu \preceq Sw$ or $Tx \succeq Tu \succeq Sw$ and $Ty \succeq Tv \succeq Sz$ or $Ty \preceq Tv \preceq Sz$, where $\psi \in \Psi$ and $\varphi : [0,\infty[\to IR$ is a continuous function with the condition $\psi(t) > \varphi(t)$ for all $t > 0$. Assume that $F(X \times X) \subseteq T(X) \cap S(X)$ and assume also that T,S and F satisfy the following hypothesis:

(1) *F,T and S are continuous,*

(2) *F commutes respectively with T and S.*

If there exist x_0, y_0, x_1 in X and y_1 such that

$$Tx_0 \preceq Sx_1 \preceq F(x_0,y_0),\ \ Ty_0 \succeq Sy_1 \succeq F(y_0,x_0),$$

then there exist $x,y \in X$ such that $Tx = Sx = F(x,y)$ and $Ty = Sy = F(y,x)$, that is, T, S and F have a coupled coincidence point $(x,y) \in X \times X$.

Proof. We replace in Theorem 2.1, ψ by $\psi(x) = x$ and φ by $\varphi(x) = x - \psi(x)$.

Corollary 3.5 *Let $(X,\preceq)$ be a partially ordered set and G-metric on X such that (X,G) is a complete G-metric space. Let $T,S : X \longrightarrow X$ and $F : X\times X \longrightarrow X$ be a mapping having the mixed(T, S)-monotone property on X. Suppose that*

(39)
$$\psi\big(G\big(F(x,y),F(u,v),F(w,z)\big)\big) \leq \psi\big(\max\big\{G(Tx,Tu,Sw),G(Ty,Tv,Sz)\big\}\big) \\ -\varphi\big(\max\big\{G(Tx,Tu,Sw),G(Ty,Tv,Sz)\big\}\big)$$

for all $x,y,z,u,v,w \in X$ with $Tx \preceq Tu \preceq Sw$ or $Tx \succeq Tu \succeq Sw$ and $Ty \succeq Tv \succeq Sz$ or $Ty \preceq Tv \preceq Sz$, where ψ and $\varphi \in \Psi$ with the condition $\psi(t) > \varphi(t)$ for all $t > 0$. Assume that $F(X\times X) \subseteq T(X)\cap S(X)$ and assume also that T,S and F satisfy the following hypothesis:

(1) *F,T and S are continuous,*

(2) *F commutes respectively with T and S.*

If there exist x_0,y_0,x_1 in X and y_1 such that

$$Tx_0 \preceq Sx_1 \preceq F(x_0,y_0), \;\; Ty_0 \succeq Sy_1 \succeq F(y_0,x_0),$$

then there exist $x,y\in X$ such that $Tx = Sx = F(x,y)$ and $Ty = Sy = F(y,x)$, that is, T, S and F have a coupled coincidence point $(x,y) \in X\times X$.

Proof. We replace in Theorem 3.1, φ by $\varphi(x) = \psi(x)-\varphi_1(x)$ where $\varphi_1 \in \Psi$ Let us denote by S the class of continuous function $\beta : [0,\infty) \longrightarrow [0,1)$ which satisfies the condition $\beta(t_n) \to 0 \Rightarrow t_n \to 0$.

Corollary 3.6 *Let $(X,\preceq)$ be a partially ordered set and G-metric on X such that (X,G) is a complete G-metric space. Let $T,S : X \longrightarrow X$ and $F : X\times X \longrightarrow X$ be a mapping having the mixed(T, S)-monotone property on X. Suppose that*

(40)
$$G\big(F(x,y),F(u,v),F(w,z)\big) \leq \beta\big(\max\big\{G(Tx,Tu,Sw),G(Ty,Tv,Sz)\big\}\big) \\ \max\big\{G(Tx,Tu,Sw),G(Ty,Tv,Sz)\big\}$$

for all $x,y,z,u,v,w \in X$ with $Tx \preceq Tu \preceq Sw$ or $Tx \succeq Tu \succeq Sw$ and $Ty \succeq Tv \succeq Sz$ or $Ty \preceq Tv \preceq Sz$, where ψ and $\varphi \in \Psi$ with the condition $\psi(t) > \varphi(t)$ for all $t > 0$. Assume that $F(X\times X) \subseteq T(X)\cap S(X)$ and assume also that T,S and F satisfy the following hypothesis:

(1) *F,T and S are continuous,*

(2) *F commutes respectively with T and S.*

If there exist x_0, y_0, x_1 *in* X *and* y_1 *such that*

$$Tx_0 \preceq Sx_1 \preceq F(x_0, y_0), \;\; Ty_0 \succeq Sy_1 \succeq F(y_0, x_0),$$

then there exist $x, y \in X$ *such that* $Tx = Sx = F(x,y)$ *and* $Ty = Sy = F(y,x)$*, that is,* T*,* S *and* F *have a coupled coincidence point* $(x,y) \in X \times X$.

Proof. We replace in Theorem 3.1, ψ by $\psi(x) = x$ and $\varphi(x) = \beta(x)x$.

4. Applications to periodic boundary value problems

In this section, we study the existence and uniqueness of solution to a periodic boundary value problem, as an application to the fixed point theorem given by Corollary 3.6.

Let $C([0,T],\mathbb{R})$ be the set of all continuous functions $u : [0,T] \longrightarrow \mathbb{R}$ and consider a mapping $g : C([0,T],\mathbb{R}) \longrightarrow C([0,T],\mathbb{R})$. Consider the periodic boundary value problem

$$u' = f(t,u) + h(t,u), \;\; t \in (0,T) \tag{41}$$

$$u(0) = u(T), \tag{42}$$

where f, h are two continuous functions satisfying the following conditions:
There exist positive constants $\lambda_1, \lambda_2, \mu_1$ and μ_2, such that for all $u, v \in (C([0,T],\mathbb{R}),\; gv(t) \leq gu(t)$,

$$0 \leq (f(t,u(t)) + \lambda_1 u(t)) - (f(t,v(t)) + \lambda_1 v(t)) \leq \mu_1 \ln[(gu(t) - gv(t))^2 + 1] \tag{43}$$

$$-\mu_2 \ln[(gu(t) - gv(t))^2 + 1] \leq (h(t,u(t)) - \lambda_2 u(t)) - (h(t,v(t)) - \lambda_2 v(t)) \leq 0 \tag{44}$$

with

$$\frac{2\max\{\mu_1, \mu_2\}}{\lambda_1 + \lambda_2} < 1. \tag{45}$$

We firstly study the existence of a solution of the following periodic system:

$$\begin{aligned} u' + \lambda_1 u - \lambda_2 v &= f(t,u) + h(t,v) + \lambda_1 u - \lambda_2 v \\ v' + \lambda_1 v - \lambda_2 u &= f(t,v) + h(t,u) + \lambda_1 v - \lambda_2 u, \end{aligned} \tag{46}$$

with the periodicity condition

$$u(0) = u(T) \text{ and } v(0) = v(T). \tag{47}$$

This problem is equivalent to the integral equations:

$$\begin{aligned} u(t) &= \int_0^T k_1(t,s)[f(s,u)+h(s,v)+\lambda_1 u - \lambda_2 v] \\ &+ \int_0^T k_2(t,s)[f(s,v)+h(s,u)+\lambda_1 v - \lambda_2 u]ds \\ v(t) &= \int_0^T k_1(t,s)[f(s,v)+h(s,u)+\lambda_1 v - \lambda_2 u] \\ &+ \int_0^T k_2(t,s)[f(s,u)+h(s,v)+\lambda_1 u - \lambda_2 v]ds, \end{aligned}$$

where

$$k_1(t,s) = \begin{cases} \frac{1}{2}\left[\frac{e^{\sigma_1(t-s)}}{1-e^{\sigma_1 T}} + \frac{e^{\sigma_2(t-s)}}{1-e^{\sigma_2 T}}\right] & 0 \le s < t \le T \\ \frac{1}{2}\left[\frac{e^{\sigma_1(t+T-s)}}{1-e^{\sigma_1 T}} + \frac{e^{\sigma_2(t+T-s)}}{1-e^{\sigma_2 T}}\right] & 0 \le t < s \le T \end{cases}$$

$$k_2(t,s) = \begin{cases} \frac{1}{2}\left[\frac{e^{\sigma_2(t-s)}}{1-e^{\sigma_2 T}} + \frac{e^{\sigma_1(t-s)}}{1-e^{\sigma_1 T}}\right] & 0 \le s < t \le T \\ \frac{1}{2}\left[\frac{e^{\sigma_2(t+T-s)}}{1-e^{\sigma_2 T}} + \frac{e^{\sigma_1(t+T-s)}}{1-e^{\sigma_1 T}}\right] & 0 \le t < s \le T \end{cases}$$

Here, $\sigma_1 = -(\lambda_1+\lambda_2)$ and $\sigma_2 = (\lambda_2-\lambda_1)$.

From ([11], Lemma 3.2), we have

$$k_1(t,s) \le 0,\ 0 \le t,s \le T \text{ and } k_2(t,s) \ge 0,\ 0 \le t,s \le T. \tag{48}$$

We assume that there exist $\alpha, \beta \in C([0,T])$ such that

$$\begin{aligned} g(\alpha(t)) \le &\int_0^1 k_1(t,s)(f(s,\alpha(s))+h(s,\beta(s))+\lambda_1\alpha(s)-\lambda_2\beta(s))ds \\ &+\int_0^1 k_2(t,s)(f(s,\beta(s))+h(s,\alpha(s))+\lambda_1\beta(s)-\lambda_2\alpha(s))ds \end{aligned} \tag{49}$$

and

$$\begin{aligned} g(\beta(t)) \ge &\int_0^1 k_1(t,s)(f(s,\beta(s))+h(s,\alpha(s))+\lambda_1\beta(s)-\lambda_2\alpha(s))ds \\ &+\int_0^1 k_2(t,s)(f(s,\alpha(s))+h(s,\beta(s))+\lambda_1\alpha(s)-\lambda_2\beta(s))ds \end{aligned} \tag{50}$$

We endow $X = C([0,T],\mathbb{R})$ with the metric $G(x,y,z) = \max_{t\in[0,T]}\{|x(t)-y(t)|, |y(t)-z(t)|, |x(t)-z(t)|\}$ for $x,y,z \in X$.

This space can be equipped with a partial order given by

$$x,y \in C([0,T]),\ x \preceq y \Longleftrightarrow x(t) \le y(t), \qquad \text{for any } t \in [0,T].$$

In $X \times X$ we define the following partial order

$$(x,y),(u,v) \in X \times X, \ \ (x,y) \preceq (u,v) \Longleftrightarrow x \preceq u \text{ and } y \succeq v.$$

Since for any $x,y \in X$ we have that $\max(x,y)$ and $\min(x,y) \in X$, assumption 3 of Corollary 3.6 is satisfied for $(X,\preceq)$. Moreover in [3] it is proved that $(X,\preceq)$ satisfies assumption 2 of Corollary 3.6.

Now, we shall prove the following result.

Theorem 4.1 *Suppose that $g : X \times X$ is a non-decreasing continuous mapping. Suppose also that (40)-(41) and (46)-(47) hold. Then (43)-(44) has a unique solution. Therefore (38)-(39) has also a unique solution.*

Proof. We introduce the operator $F : X \times X \longrightarrow X$ defined by

$$\begin{aligned} F(u,v)(t) &= \int_0^T k_1(t,s)[f(s,u)+h(s,v)+\lambda_1 u-\lambda_2 v]ds \\ &\quad +\int_0^T k_2(t,s)[f(s,v)+h(s,u)+\lambda_1 v-\lambda_2 u]ds \end{aligned}$$

for all $u,v \in X$ and $t \in [0,T]$.

We claim that F has the mixed g-monotone property.

In fact, for $gx_1 \preceq gx_2$ and $t \in [0,T]$, we have

$$\begin{aligned} F(x_1,y)(t)-F(x_2,y)(t) &= \int_0^T k_1(t,s)(f(s,x_1(s))-f(s,x_2)+\lambda_1(x_1(s)-x_2(s))ds \\ &\quad +\int_0^T k_2(t,s)(h(s,x_1(s))-h(s,x_2)-\lambda_2(x_1-x_2))ds. \end{aligned}$$

From (40), (41) and (45), for all $t \in [0,T]$, we have

$$F(x_1,y)(t)-F(x_2,y)(t) \leq 0.$$

This implies that

$$F(x_1,y) \preceq F(x_2,y).$$

Also, for $gy_1 \preceq gy_2$ and $t \in [0,T]$, we have

$$\begin{aligned} F(x,y_1)(t)-F(x,y_2)(t) &= \int_0^T k_1(t,s)(h(s,y_1(s))-h(s,y_2)-\lambda_2(y_1(s)-y_2(s))ds \\ &\quad +\int_0^T k_2(t,s)(f(s,y_1(s))-f(s,y_2)+\lambda_1(y_1-y_2))ds. \end{aligned}$$

Looking at (40), (41) and (45), for all $t \in [0,T]$, we have

$$F(x,y_1)(t) - F(x,y_2)(t) \geq 0,$$

that is

$$F(x_1,y) \succeq F(x_2,y).$$

Thus, we proved that F has the mixed g-monotone property.

For $gx \succeq gu \succeq gz$ and $gy \preceq gv \preceq gw$, we have $F(x,y) \succeq F(u,v)$, $F(x,y) \succeq F(z,w)$, $F(u,v) \succeq F(z,w)$ and

$$\begin{aligned}
&G\big(F(x,y),F(u,v),F(z,w)\big)\\
&= \max_{t\in[0,T]} \Big\{ |F(x,y)(t) - F(u,v)(t)|, |F(x,y)(t) - F(z,w)(t)|, |F(u,v)(t) - F(z,w)(t)| \Big\}\\
&= \max_{t\in[0,T]} \Big\{ (F(x,y)(t) - F(u,v)(t)), (F(x,y)(t) - F(z,w)(t)), (F(u,v)(t) - F(z,w)(t)) \Big\}\\
&= \max_{t\in[0,T]} \Big\{ \int_0^T k_1(t,s)[(f(s,x(s)) - f(s,u(s)) + \lambda_1(x-u)) - (h(s,v(s))\\
&\quad - h(s,y(s)) - \lambda_2(y-v))]ds - \int_0^T k_2(t,s)[(f(s,v(s)) - f(s,y(s)) + \lambda_1(v-y))\\
&\quad - (h(s,u(s)) - h(s,x(s)) - \lambda_2(u-x))]ds,\\
&\int_0^T k_1(t,s)[(f(s,x(s)) - f(s,z(s)) + \lambda_1(x-z))\\
&\quad - (h(s,w(s)) - h(s,y(s)) - \lambda_2(y-w))]ds\\
&\quad - \int_0^T k_2(t,s)[(f(s,w(s)) - f(s,y(s)) + \lambda_1(w-y))]\\
&\quad - (h(s,z(s)) - h(s,x(s)) - \lambda_2(z-x))]ds,\\
&\int_0^T k_1(t,s)[(f(s,u(s)) - f(s,z(s)) + \lambda_1(u-z))\\
&\quad - (h(s,w(s)) - h(s,v(s)) - \lambda_2(v-w))]ds\\
&\quad - \int_0^T k_2(t,s)[(f(s,w(s)) - f(s,v(s)) + \lambda_1(w-v))\\
&\quad - (h(s,z(s)) - h(s,u(s)) - \lambda_2(z-u))]ds \Big\}.
\end{aligned}$$

Using (40) and (41) we get

$$
\begin{aligned}
&G\big(F(x,y),F(u,v),F(z,w)\big) \leq \\
&\max_{t\in[0,T]}\Big\{\int_0^T k_1(t,s)\Big(\mu_1 \ln\big[(gx(s)-gu(s))^2+1\big]+\mu_2 \ln\big[(gy(s)-gv(s))^2+1\big]\Big)ds \\
&\quad +\int_0^T (-k_2(t,s))\Big(\mu_1 \ln\big[(gv(s)-gy(s))^2+1\big]+\mu_2 \ln\big[(gx(s)-gu(s))^2+1\big]\Big)ds, \\
&\int_0^T k_1(t,s)\Big(\mu_1 \ln\big[(gx(s)-gz(s))^2+1\big]+\mu_2 \ln\big[(gy(s)-gw(s))^2+1\big]\Big)ds \\
&\quad +\int_0^T (-k_2(t,s))\Big(\mu_1 \ln\big[(gw(s)-gy(s))^2+1\big]+\mu_2 \ln\big[(gx(s)-gz(s))^2+1\big]\Big)ds, \\
&\int_0^T k_1(t,s)\Big(\mu_1 \ln\big[(gu(s)-gz(s))^2+1\big]+\mu_2 \ln\big[(gv(s)-gw(s))^2+1\big]\Big)ds \\
&\quad +\int_0^T (-k_2(t,s))\Big(\mu_1 \ln\big[(gw(s)-gv(s))^2+1\big]+\mu_2 \ln\big[(gu(s)-gz(s))^2+1\big]\Big)ds\Big\}
\end{aligned}
$$

then,

$$
\begin{aligned}
&G\big(F(x,y),F(u,v),F(z,w)\big) \leq \\
&\leq \max\{\mu_1,\mu_2\}\Big(\max_{t\in[0,T]}\Big\{\int_0^T (k_1(t,s)-k_2(t,s))\ln\big[(gx(s)-gu(s))^2+1\big]+ \\
&\qquad \int_0^T (k_1(t,s)-k_2(t,s))\ln\big[(gy(s)-gv(s))^2+1\big], \\
&\qquad \int_0^T (k_1(t,s)-k_2(t,s))\ln\big[(gx(s)-gz(s))^2+1\big]+ \\
&\qquad \int_0^T (k_1(t,s)-k_2(t,s))\ln\big[(gy(s)-gw(s))^2+1\big], \\
&\qquad \int_0^T (k_1(t,s)-k_2(t,s))\ln\big[(gu(s)-gz(s))^2+1\big]+ \\
&\qquad \int_0^T (k_1(t,s)-k_2(t,s))\ln\big[(gv(s)-gw(s))^2+1\big]\Big\}\Big).
\end{aligned}
$$

By property (G4) of G we have

$$
\begin{aligned}
&G\big(F(x,y),F(u,v),F(z,w)\big) \leq \\
&\Big(\max_{t\in[0,T]}\int_0^T (k_1(t,s)-k_2(t,s))\Big)\max\{\mu_1,\mu_2\}\Big(\ln\big[(G(gx,gu,gz))^2+1\big]+ \\
&\qquad \ln\big[(G(gy,gv,gw))^2+1\big]\Big) \\
&\leq 2\max\{\mu_1,\mu_2\}\max_{t\in[0,T]}\Big|\int_0^T \frac{e^{\sigma_1(t-s)}}{1-e^{\sigma_1 T}}ds+\int_0^T \frac{e^{\sigma_1(t+T-s)}}{1-e^{\sigma_1 T}}ds\Big| \\
&\qquad \ln\big[\big(\max\{G(gx,gu,gz),G(gy,gv,gw)\}\big)^2+1\big].
\end{aligned}
$$

After integrating, we get

$$
G\big(F(x,y),F(u,v),F(z,w)\big) \leq \tfrac{2\max\{\mu_1,\mu_2\}}{\lambda_1+\lambda_2}\ln\big[\big(\max\{G(gx,gu,gz),G(gy,gv,gw)\}\big)^2+1\big].
$$

From (47), we obtain

$$G\big(F(x,y),F(u,v),F(z,w)\big) \le \ln\big[(\max\{G(gx,gu,gz),G(gy,gv,gw)\})^2+1\big],$$

which implies that

$$\big(G\big(F(x,y),F(u,v),F(z,w)\big)\big)^2 \le \Big(\ln\big[(\max\{G(gx,gu,gz),G(gy,gv,gw)\})^2+1\big]\Big)^2.$$

Then,

Set $\psi(t) = t^2$ and $\varphi(t) = (\ln(t^2+1)^2$. Clearly $\psi \in \Psi$ and ϕ are altering distance functions and satisfy the condition $\psi(x) > \varphi(x)$ for $x > 0$ and from the above inequality, we obtain

$$\psi\Big(G\big(F(x,y),F(u,v),F(z,w)\big)\Big) \le \varphi\big(\max\{G(gx,gu,gz),G(gy,gv,gw)\}\big)$$

for all $x,y,u,v,z,w \in X$ such that $gx \succeq gu \succeq gz$ and $gy \preceq gv \preceq gw$.

Now, let $\alpha,\beta \in X$ be the functions given by (46) and (47).Then, we have

$$g\alpha \preceq F(\alpha,\beta) \text{ and } F(\beta,\alpha) \succeq g\beta.$$

Thus, we proved that all the required hypotheses of Corollary 3.6 are satisfied. Hence, g and F have a unique coupled fixed point $(u,v) \in X \times X$, that is, (u,v) is the unique solution of (38)-(39). This completes the proof.

Conflict of Interests

The author declares that there is no conflict of interests.

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