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NORMAL VARIANCE-MEAN MIXTURES (II) - A MULTIVARIATE MOMENT METHOD

WERNER HÜRLIMANN

Wolters Kluwer Financial Services, Seefeldstrasse 69, CH-8008 Zürich, Switzerland

Abstract: A moment method for the multivariate variance-mean mixture model is considered. Besides mean and covariance the method uses the coskewness and cokurtosis tensors. Its algorithmic implementation depends upon the solution of a sextic equation. Explicit formulas for the normal inverse Gaussian, the gamma, the inverse gamma and the classical tempered stable mixing distributions are included. An application to the statistical estimation of bivariate stock market indices is given. The models are successfully fitted to seven bivariate daily data sets over different time periods. The goodness-of-fit of the margins are optimized and compared.

Keywords: coskewness; cokurtosis; generalized hyperbolic; normal tempered stable; normal inverse Gaussian; variance-gamma; hyperbolic skew t

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1. Introduction

The benchmark theory in mathematical finance is the Black-Scholes-Merton framework. Based on Gaussian asset return distributions the mathematics of option-pricing has many advantages and remains tractable even in complex situations (e.g. Hürlimann (2012a)). Unfortunately, this model assumes symmetric returns with very thin tails and is therefore not

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consistent with real-world financial data. For example, observed sample logarithmic returns of equity market indices are often negatively skewed and have a much higher excess kurtosis than is allowed by a normal distribution. Many alternatives have been proposed, from stable models by Mandelbrot (1963) (see e.g. Mandelbrot (1997), part IV, Rachev and Mittnik (2000)) and generalized hyperbolic distributions (see e.g. Eberlein (2001)) to more recent tempered stable distribution (see e.g. Rachev et al. (2011)). A general model that encompasses many of the alternative proposals is the normal variance-mean (NVM) mixture model, which retains some of the convenience of the Gaussian model (see e.g. Barndorff-Nielsen et al. (1982), Bingham and Kiesel (2001), Hürlimann (2013a)). The NVM model includes important parametric families of distributions, namely the generalized hyperbolic (GH) distribution and the normal tempered stable (NTS) distribution. Analytically tractable members of the GH distribution are the normal-inverse Gaussian (NIG), the variance-gamma (VG) and the hyperbolic skew t (HST). The NTS family also includes the NIG distribution. The present contribution is devoted to statistical estimation of these multivariate models. Maximum likelihood estimation based on the EM-algorithm has been considered by many authors (e.g. Liu and Rubin (1995), Protassov (2004), Embrechts et al. (2005) and Hu (2005)). In particular, the latter author provides special algorithms for the multivariate normal inverse Gaussian (NIG), variance gamma (VG) and skew hyperbolic t (SHT). The EM-algorithm for the NIG has also been considered in Karlis and Papadimitriou (2003), Oeigard et al. (2005), and Chang et al. (2010) among others. Since computational implementation of the EM-algorithm is highly complex, it is justified to consider simpler alternative methods. We extend the multivariate moment method in Hürlimann (2013b) to the framework of the NVM mixture models and exemplify its statistical use for some of its most tractable members. Though moment methods are known to be statistically less efficient than the maximum likelihood method, there are numerous applications of them. A recent portfolio theoretical application of the moment method in the univariate NVM framework is Hürlimann (2013d). An application to bivariate option pricing is Hürlimann (2013c). A more detailed account of the content follows.

Section 2 recalls the multivariate NVM mixture model and briefly introduces the notions of coskewness and cokurtosis, or degree three and four central moments, which are often used in the newer portfolio selection theory. Theorem 2.1 derives formulas for them for the multivariate NVM mixture model with fixed first four cumulants of the mixing distribution. Section 3 presents our general multivariate moment method in terms of coskewness and cokurtosis, which depends upon the solution of a sextic equation. It shows that the covariance matrix of the multivariate NVM mixture distribution is functionally dependent upon coskewness and cokurtosis. It enables simultaneous estimation of the parameters given sample estimates of the mean vector, coskewness vector and cokurtosis matrix. The obtained algorithm is a generalization of the moment method for the multivariate asymmetric Laplace distribution presented in Hürlimann (2013b) (see also Hürlimann (2013c) for the multivariate variance gamma distribution). This method is worked out in Section 4 for a selection of important mixing distributions, namely the inverse Gaussian, the gamma, the inverse gamma and the classical tempered stable distributions. These mixing distributions give rise respectively to the normal inverse Gaussian (NIG), the variance gamma (VG), the hyperbolic skew t (HST) and the normal tempered stable (NTS) multivariate distributions. A real-life application is studied in Section 5. It concerns the statistical estimation of the corresponding bivariate models for the Standard & Poors 500 and NASDAQ 100 stock market indices. The models are successfully fitted to seven bivariate daily data sets over different time periods. The goodness-of-fit of the margins are optimized and compared. The numerical evaluation of the goodness-of-fit statistics encountered in the data analysis are done with the fast Fourier transform (FFT) approximation of a distribution with known characteristic function (see the Appendix 1 for a summary of the method). For the NIG and VG a direct numerical evaluation is also possible using the analytical formulas for their densities derived in Appendix 2.

2. Coskewness and cokurtosis of multivariate NVM mixtures

A random vector $X = (X_1,..., X_n)$ is called a n -dimensional *multivariate normal variance-mean* (NVM) *mixture* if it satisfies a stochastic representation

$$X = \xi + \beta \cdot W + \sqrt{W} \cdot Z, \tag{2.1}$$

where $Z \sim N(0,\Sigma)$ is a multivariate normal random variable with positive semi-definite covariance matrix $\Sigma=(\sigma_{ij}), 1\le i,j\le n$, W is a non-negative mixing random variable with cumulant generating function (cgf) $C_W(t)$, Z,W are independent, and $\xi=(\xi_i), \beta=(\beta_i), i=1,...,n,$ are real-valued parameter vectors. One knows that the cgf of the NVM model is given by

$$C_X(u)=\xi^T u + C_W(\beta^T u + \tfrac{1}{2}u^T\Sigma u) \tag{2.2}$$

for all values of $u=(u_1,...,u_n)$ for which the expression (2.2) exists. The first four moments and cumulants of W are summarized into vectors $m=(m_1,m_2,m_3,m_4)$ and $\kappa=(\kappa_1,\kappa_2,\kappa_3,\kappa_4)$ respectively. In terms of the latter parameters, a short hand notation for the random vector (2.1) is $X \sim NVM(\xi,\beta,\Sigma,\kappa)$. In a first step, we determine the *mean* vector $\mu=(\mu_1,...,\mu_n)$ of X, and the matrix of k-th order central moments $\overline{M}_k[X]$, $\mathrm{k}=2,3,4$. For $k=2$ the nxn matrix $\overline{M}_2[X]=D[X]=(V_{ij}), 1\le i,j\le n$, is the *covariance* matrix with elements $V_{ij}=E[(X_i-\mu_{\mathrm{i}})(X_j-\mu_{\mathrm{j}})]$. The nxn^2 matrix $\overline{M}_3[X]=(S_{ijk}), 1\le i,j,k\le n$, consists of the *coskewness* elements $S_{ijk}=E[(X_i-\mu_{\mathrm{i}})(X_j-\mu_{\mathrm{j}})(X_k-\mu_{\mathrm{k}})]$, and the nxn^3 matrix $\overline{M}_4[X]=(K_{ijk\ell}), 1\le i,j,k,\ell\le n,$ consists of the *cokurtosis* elements $K_{ijk\ell}=E[(X_i-\mu_{\mathrm{i}})(X_j-\mu_{\mathrm{j}})(X_k-\mu_{\mathrm{k}})(X_\ell-\mu_\ell)]$. In general, one has the relationships

$$\begin{aligned}
S_{ijk} &= E[X_iX_jX_k]-(\mu_{\mathrm{i}}V_{jk}+\mu_jV_{ik}+\mu_kV_{ij})-\mu_{\mathrm{i}}\mu_j\mu_k,\\
K_{ijk\ell} &= E[X_iX_jX_kX_\ell]-(\mu_{\mathrm{i}}S_{\mathrm{jk}\ell}+\mu_{\mathrm{j}}S_{\mathrm{ik}\ell}+\mu_{\mathrm{k}}S_{\mathrm{ij}\ell}+\mu_\ell S_{\mathrm{ijk}})\\
&-(\mu_{\mathrm{i}}\mu_{\mathrm{j}}V_{\mathrm{k}\ell}+\mu_{\mathrm{i}}\mu_{\mathrm{k}}V_{\mathrm{j}\ell}+\mu_{\mathrm{i}}\mu_\ell V_{\mathrm{jk}}+\mu_{\mathrm{j}}\mu_{\mathrm{k}}V_{\mathrm{i}\ell}+\mu_{\mathrm{j}}\mu_\ell V_{\mathrm{ik}}+\mu_{\mathrm{k}}\mu_\ell V_{\mathrm{ij}})-\mu_{\mathrm{i}}\mu_j\mu_k\mu_\ell.
\end{aligned} \tag{2.3}$$

The following result generalizes Proposition 2.1 in Hürlimann (2013a) for the multivariate asymmetric Laplace case with $W \sim Exp(1)$ the exponential random variable with mean one.

Theorem 2.1. (*Moments of the NVM model*) The mean, covariance, coskewness and cokurtosis parameters of the multivariate NVM random vector $X \sim NVM(\xi, \beta, \Sigma, \kappa)$ are given by

$$\begin{aligned}
&\mu_i = \xi_i + \kappa_1 \cdot \beta_i, \quad V_{ij} = \kappa_2 \cdot \beta_i \beta_j + \kappa_1 \cdot \sigma_{ij}, \quad 1 \le i, j \le n, \\
&S_{ijk} = \kappa_3 \cdot \beta_i \beta_j \beta_k + \kappa_2 \cdot (\beta_i \sigma_{jk} + \beta_j \sigma_{ik} + \beta_k \sigma_{ij}), \quad 1 \le i, j, k \le n, \\
&K_{ijk\ell} = (\kappa_4 + 3\kappa_2^2) \cdot \beta_i \beta_j \beta_k \beta_\ell \\
&+ (\kappa_3 + \kappa_1 \kappa_2) \cdot (\beta_i \beta_j \sigma_{k\ell} + \beta_i \beta_k \sigma_{j\ell} + \beta_i \beta_\ell \sigma_{jk} + \beta_j \beta_k \sigma_{i\ell} + \beta_j \beta_\ell \sigma_{ik} + \beta_k \beta_\ell \sigma_{ij}) \\
&+ (\kappa_2 + \kappa_1^2) \cdot (\sigma_{ij} \sigma_{k\ell} + \sigma_{ik} \sigma_{j\ell} + \sigma_{i\ell} \sigma_{jk}), \quad 1 \le i, j, k, \ell \le n.
\end{aligned} \tag{2.4}$$

Proof. In virtue of the representation (2.1) the expression for the mean vector is immediate. For the central moments it suffices to consider the case $\xi = 0$. With (2.1) the vector components of X are $X_i = \beta_i W + \sqrt{W} \cdot Z_i, \, i = 1,...,n,$ where W is independent of $Z_i \sim N(0, \sigma_{ii})$. The relationships between moments and cumulants

$$\begin{aligned}
&\kappa_1 = m_1, \quad \kappa_2 = m_2 - m_1^2, \quad \kappa_3 = m_3 - 3m_1 m_2 + 2m_1^2, \\
&\kappa_4 = m_4 - 4m_1 m_3 - 3m_2^2 + 12 m_1^2 m_2 - 6m_1^4
\end{aligned},$$

will be used repeatedly without further mention. One has

$$E[X_i X_j] = E[\beta_i \theta_j W^2 + (\beta_i Z_j + \beta_j Z_k) W^{3/2} + Z_i Z_j W] = m_2 \beta_i \beta_j + m_1 \sigma_{ij},$$

which implies the expression for the covariance. Similarly, one has

$$\begin{aligned}
&E[X_i X_j X_k] = E[\beta_i \beta_j \beta_k W^3 + (\beta_i \beta_j Z_k + \beta_i \beta_k Z_j + \beta_j \beta_k Z_i) W^{5/2} \\
&+ (\beta_i Z_j Z_k + \beta_j Z_i Z_k + \beta_k Z_i Z_j) W^2 + Z_i Z_j Z_k W^{3/2}].
\end{aligned}$$

With the fact that $E[Z_i Z_j Z_k] = 0$ (theorem of Isserlis) one sees that

$$E[X_i X_j X_k] = m_3 \cdot \beta_i \beta_j \beta_k + m_2 \cdot (\beta_i \sigma_{jk} + \beta_j \sigma_{ik} + \beta_k \sigma_{ij}).$$

Insert this and the fact that $\mu_i = m_1 \beta_i, V_{ij} = \kappa_2 \beta_i \beta_j + \kappa_1 \sigma_{ij}$ into the first relation of (2.3) to obtain the coskewness formula in (2.4). Proceeding in the same way, one shows that

$$\begin{aligned}
&E[X_i X_j X_k X_\ell] = E[\beta_i \beta_j \beta_k \beta_\ell W^4 \\
&+ (\beta_i \beta_j \beta_k Z_\ell + \beta_i \beta_j \beta_\ell Z_k + \beta_i \beta_k \beta_\ell Z_j + \beta_j \beta_k \beta_\ell Z_i) W^{7/2} \\
&+ (\beta_i \beta_j Z_k Z_\ell + \beta_i \beta_k Z_j Z_\ell + \beta_i \beta_\ell Z_j Z_k + \beta_j \beta_k Z_i Z_\ell + \beta_j \beta_\ell Z_i Z_k + \beta_k \beta_\ell Z_i Z_j) W^3 \\
&+ (\beta_i Z_j Z_k Z_\ell + \beta_j Z_i Z_k Z_\ell + \beta_k Z_i Z_j Z_\ell + \beta_\ell Z_i Z_j Z_k) W^{3/2} + Z_i Z_j Z_k Z_\ell W^2].
\end{aligned}$$

Since $E[Z_i Z_j Z_k Z_\ell] = \sigma_{ij}\sigma_{k\ell} + \sigma_{ik}\sigma_{j\ell} + \sigma_{i\ell}\sigma_{jk}$ (theorem of Isserlis) one gets

$$\begin{aligned}
&E[X_i X_j X_k X_\ell] = m_4 \cdot \beta_i \beta_j \beta_k \beta_\ell + m_3 \cdot (\beta_i \beta_j \sigma_{k\ell} + \beta_i \beta_k \sigma_{j\ell} \\
&+ \beta_i \beta_\ell \sigma_{jk} + \beta_j \beta_k \sigma_{i\ell} + \beta_j \beta_\ell \sigma_{ik} + \beta_k \beta_\ell \sigma_{ij}) + m_2 \cdot (\sigma_{ij}\sigma_{k\ell} + \sigma_{ik}\sigma_{j\ell} + \sigma_{i\ell}\sigma_{jk}).
\end{aligned}$$

Insert this, $\mu_i = m_1 \beta_i, V_{ij} = \kappa_2 \beta_i \beta_j + \kappa_1 \sigma_{ij}$ and the coskewness relation in (2.4), into the second part of (2.3) to get

$$\begin{aligned}
&K_{ijk\ell} = (m_4 - 4 m_1 \kappa_3 - 6 m_1^2 \kappa_2 - m_1^4) \cdot \beta_i \beta_j \beta_k \beta_\ell \\
&+ (m_3 - 2 m_1 \kappa_2 - m_1^3) \cdot (\beta_i \beta_j \sigma_{k\ell} + \beta_i \beta_k \sigma_{j\ell} + \beta_i \beta_\ell \sigma_{jk} + \beta_j \beta_k \sigma_{i\ell} + \beta_j \beta_\ell \sigma_{ik} + \beta_k \beta_\ell \sigma_{ij}) \\
&+ m_2 \cdot (\sigma_{ij}\sigma_{k\ell} + \sigma_{ik}\sigma_{j\ell} + \sigma_{i\ell}\sigma_{jk}).
\end{aligned}$$

Taking into account the relationships between moments and cumulants one obtains after rearrangement the cokurtosis formula in (2.4). ◊

3. A general moment method

We are ready for the generalization of the moment method in Hürlimann (2013b), Section 3. For any fixed $\kappa = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ and given the mean parameters (μ_i), the coskewness and cokurtosis parameters (S_{ijk}) and $(K_{ijk\ell})$, we determine the remaining parameters $(V_{ij}), (\sigma_{ij}), (\xi_i), (\beta_i)$ in terms of them. In particular, it is shown that the covariance matrix (V_{ij}) of the NVM distribution functionally depends upon coskewness and cokurtosis. First of all, given the mean μ and assuming β has been determined, it is clear that ξ is obtained from the mean vector equation as $\xi = \mu - \kappa_1 \beta$. Similarly, once $(V_{ij}), (\beta_i)$ have been determined, the parameter matrix $\Sigma = (\sigma_{ij})$ is obtained from the covariance equation as $\sigma_{ij} = \kappa_1^{-1} \cdot \{V_{ij} - \kappa_2 \cdot \beta_i \beta_j\}$. Next let us examine the coskewness equations. For this, consider the coskewness vector $S(X) = (S_1, ..., S_n)$ derived from the star product $S(X) = 1_{nxn} * \overline{M}_3[\mathrm{X}]$ such that

$$S_i = \sum_{j,k=1}^{n} S_{ijk}, \quad i = 1,...,n. \tag{3.1}$$

The following short hand notation for sums of covariances and parameters is used:

$$V_i = \sum_{j=1}^{n} V_{ij} = \sum_{j=1}^{n} V_{ji}, \quad V = \sum_{i,j=1}^{n} V_{ij}, \quad M = \sum_{i=1}^{n} \beta_i \ . \tag{3.2}$$

The evaluation of (3.1) based on the coskewness formula in (2.4) yields the relationships

$$\{\kappa_2 \cdot V - (3\kappa_2^2 - \kappa_1\kappa_3) \cdot M^2\} \cdot \beta_i + 2\kappa_2 \cdot MV_i = \kappa_1 \cdot S_i, \quad i = 1,...,n. \tag{3.3}$$

Set further $S = \sum_{i=1}^{n} S_i$ and add the equations in (3.3) to get the equation in (M, V):

$$3\kappa_2 \cdot VM - (3\kappa_2^2 - \kappa_1\kappa_3) \cdot M^3 - \kappa_1 \cdot S = 0. \tag{3.4}$$

Consider now the cokurtosis equations and define the cokurtosis matrix $K(X) = (K_{ij})$

using the star product $K(X) = 1_{nxn} * \overline{M}_4[X]$ such that

$$K_{ij} = \sum_{k,\ell=1}^{n} K_{ijk\ell}, \quad i, j = 1,...,n. \tag{3.5}$$

A calculation of (3.5) based on the last equations in (2.4) yields

$$\begin{aligned}
K_{ij} &= (\kappa_4 + 3\kappa_2^2) \cdot M^2 \beta_i \beta_j \\
&+ (\kappa_3 + \kappa_1\kappa_2) \cdot \{\kappa_1^{-1}(V - \kappa_2 M^2)\beta_i\beta_j + 2\kappa_1^{-1}(V_j - \kappa_2 M\beta_j)M\beta_i \\
&+ 2\kappa_1^{-1}(V_i - \kappa_2 M\beta_i)M\beta_j + \kappa_1^{-1}(V_{ij} - \kappa_2\beta_i\beta_j)M^2\} \\
&+ (\kappa_2 + \kappa_1^2) \cdot \{\kappa_1^{-2}(V_{ij} - \kappa_2\beta_i\beta_j)(V - \kappa_2 M^2) + 2\kappa_1^{-2}(V_i - \kappa_2 M\beta_i)(V_j - \kappa_2 M\beta_j)\} \\
&= \{\kappa_1^{-1}(\kappa_3 + \kappa_1\kappa_2) - \kappa_1^{-2}\kappa_2(\kappa_2 + \kappa_1^2)\} \cdot V\beta_i\beta_j \\
&+ \{\kappa_4 + 3\kappa_2^2 - 6\kappa_1^{-2}\kappa_2(\kappa_3 + \kappa_1\kappa_2) + 3\kappa_1^{-2}\kappa_2(\kappa_2 + \kappa_1^2)\} \cdot M^2\beta_i\beta_j \\
&+ \{2\kappa_1^{-2}(\kappa_3 + \kappa_1\kappa_2) - 2\kappa_1^{-2}\kappa_2(\kappa_2 + \kappa_1^2)\} \cdot M \cdot (V_j\beta_i + V_i\beta_j) + 2\kappa_1^{-2}(\kappa_2 + \kappa_1^2)V_iV_j \\
&+ \{\kappa_1^{-2}(\kappa_3 + \kappa_1\kappa_2) - \kappa_1^{-2}\kappa_2(\kappa_2 + \kappa_1^2)\} \cdot M^2 V_{ij} + \kappa_1^{-2}(\kappa_2 + \kappa_1^2) \cdot VV_{ij}.
\end{aligned}$$

Multiplying with κ_1^2 and rearranging one obtains the equations

$$\begin{aligned}
&\{(\kappa_1\kappa_3 - \kappa_2^2)V - (3\kappa_2(2\kappa_1\kappa_3 - \kappa_2^2) - \kappa_1^2\kappa_4)M^2\} \cdot \beta_i\beta_j \\
&+ 2(\kappa_1\kappa_3 - \kappa_2^2) \cdot M \cdot (V_j\beta_i + V_i\beta_j) + 2(\kappa_2 + \kappa_1^2) \cdot V_iV_j \\
&+ \{(\kappa_1\kappa_3 - \kappa_2^2)M^2 + (\kappa_2 + \kappa_1^2) \cdot V\} \cdot V_{ij} = \kappa_1^2 \cdot K_{ij}.
\end{aligned} \tag{3.6}$$

Further, summing (3.6) with the short hand notation $K_i = \sum_{j=1}^{n} K_{ij} = \sum_{j=1}^{n} K_{ji}, \ i = 1,...,n,$ one gets

$$\begin{aligned}
&\{3(\kappa_1\kappa_3 - \kappa_2^2) \cdot V - (3\kappa_2(2\kappa_1\kappa_3 - \kappa_2^2) - \kappa_1^2\kappa_4) \cdot M^2\} \cdot M\beta_i \\
&+ \{3(\kappa_1\kappa_3 - \kappa_2^2) \cdot M^2 + 3(\kappa_2 + \kappa_1^2) \cdot V\} \cdot V_i = \kappa_1^2 \cdot K_i, \quad i = 1,...,n.
\end{aligned} \tag{3.7}$$

With $K = \sum_{i=1}^{n} K_i$ one obtains through addition of (3.7) a further equation in (M, V), namely

$$\begin{aligned}&\{3\kappa_2(2\kappa_1\kappa_3-\kappa_2^2)-\kappa_1^2\kappa_4\}\cdot M^4-6(\kappa_1\kappa_3-\kappa_2^2)\cdot VM^2\\&-3(\kappa_2+\kappa_1^2)\cdot V^2+\kappa_1^2\cdot K=0.\end{aligned} \tag{3.8}$$

The resulting system of non-linear equations (3.3), (3.4), (3.6), (3.7), (3.8) in the unknowns $(\beta_i, M, V_{ij}, V_i, V)$ is solved by applying a three-stage procedure.

Step 1: solve the equations (3.4) and (3.8) for the parameters (M,V)

From (3.4) one gets

$$V=\frac{(3\kappa_2^2-\kappa_1\kappa_3)\cdot M^3+\kappa_1 S}{3\kappa_2\cdot M}. \tag{3.9}$$

Insert this expression into (3.8) and multiply with $9\kappa_2^2\cdot M^2$ to see that M satisfies the following sextic equation in the parameters (S,K):

$$\begin{aligned}&\{3\kappa_2^2(2\kappa_1\kappa_3-3\kappa_2^2-\kappa_4)+\kappa_3^2(5\kappa_2-\kappa_1^2)\}\cdot M^6\\&-2\{2\kappa_2\kappa_3+3\kappa_1\kappa_2^2-\kappa_1^2\kappa_3\}\cdot SM^3+3\kappa_2^2\cdot KM^2-(\kappa_2+\kappa_1^2)\cdot S^2=0.\end{aligned} \tag{3.10}$$

Step 2: solve the equations (3.3) and (3.7) for the parameters $(\beta_i, V_i),\ i=1,\dots,n$

From (3.3) one gets

$$\beta_i=\frac{\kappa_1 S_i-2\kappa_2\cdot MV_i}{\kappa_2\cdot V-(3\kappa_2^2-\kappa_1\kappa_3)\cdot M^2},\quad i=1,\dots,n. \tag{3.11}$$

Insert this into (3.7) to see that V_i is function of the parameters (M,V,S_i,K_i), which are determined using the values from Step 1. One obtains

$$V_i = \frac{A_i}{B_i}, \quad i = 1,...,n, \quad \text{with} \tag{3.12}$$

$$A_i = \kappa_1^2 \cdot K_i - \frac{\kappa_1\{3(\kappa_1\kappa_3 - \kappa_2^2)\cdot V - (3\kappa_2(2\kappa_1\kappa_3 - \kappa_2^2) - \kappa_1^2\kappa_4)\cdot M^2\}\cdot MS_i}{\kappa_2 \cdot V - (3\kappa_2^2 - \kappa_1\kappa_3)\cdot M^2},$$
$$B_i = 3\{(\kappa_1\kappa_3 - \kappa_2^2)\cdot M^2 + (\kappa_2 + \kappa_1^2)\cdot V\} \tag{3.13}$$
$$- \frac{2\kappa_2\{3(\kappa_1\kappa_3 - \kappa_2^2)\cdot V - (3\kappa_2(2\kappa_1\kappa_3 - \kappa_2^2) - \kappa_1^2\kappa_4)\cdot M^2\}\cdot M^2}{\kappa_2 \cdot V - (3\kappa_2^2 - \kappa_1\kappa_3)\cdot M^2}.$$

Step 3: the unknowns (V_{ij}) are obtained from the equation (3.6), where one must verify that the covariance matrix (V_{ij}) and its associated correlation matrix are positive semi-definite. The stated condition can be tested simply and efficiently (e.g. Kurowicka and Cooke (2006), Section 4.5.1, or Hürlimann (2012b), Lemma 2.1).

The described general moment method is useful for parameter estimation. Given a sample $(x_1,...,x_N)$ of size N, where each x_i is an observation of the random vector $X = (X_1,...,X_n)$, one considers the following sample estimates of the coskewness vector and cokurtosis matrix:

$$\hat{S}(X) = (\hat{S}_1,...,\hat{S}_n) = N^{-1}\cdot 1_{nxn} * \sum_{r=1}^{N}(x_r x_r^T \otimes x_r),$$
$$\hat{K}(X) = (\hat{K}_{ij}) = N^{-1}\cdot 1_{nxn} * \sum_{r=1}^{N}(x_r x_r^T \otimes x_r x_r^T). \tag{3.14}$$

Samples estimates of the quantities $S, K_i, K,$ are obtained through summation as

$$\hat{S} = \sum_{i=1}^{n}\hat{S}_i, \quad \hat{K}_i = \sum_{j=1}^{n}\hat{K}_{ij} = \sum_{j=1}^{n}\hat{K}_{ji}, \; i = 1,...,n, \quad \hat{K} = \sum_{i=1}^{n}\hat{K}_i. \tag{3.15}$$

Inserting these estimates into the derived formulas, one obtains for any fixed

$\kappa = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ estimates of the NVM parameters in terms of the sample mean vector, coskewness vector and cokurtosis matrix. The final Section illustrates with a real-world application of this procedure for a selection of mixing distributions.

4. Moment method for a selection of mixing distributions

To illustrate the moment method for multivariate NVM mixture models, we consider the generalized hyperbolic (GH) distribution with a generalized inverse Gaussian (GIG) mixing distribution, and the normal tempered stable (NTS) distribution with a classical tempered stable (CTS) mixing distribution. There is a wide and continued interest in the GH family of distributions (e.g. Eberlein and Keller (1995), Prause (1999). Eberlein (2001), Eberlein and Prause (2002), Bibby and Sorensen (2003), Eberlein and Hammerstien (2004), Embrechts et al. (2005), etc.). The GH distribution contains three important subfamilies, namely the normal inverse Gaussian (NIG) with an inverse Gaussian (IG) mixing distribution, the variance-gamma (VG) with a gamma mixing distribution, and the skew hyperbolic t (SHT) with an inverse gamma mixing distribution. The NIG has been used for financial modelling by Eberlein and Keller (1995), Barndorff-Nielsen (1997/98) and Rydberg (1998) among others. The VG has been introduced by Madan and Seneta (1990) (see also Madan and Milne (1991), Madan et al. (1998), Madan (2001), Carr et al. (2002), Geman (2002), Fu et al. (2006), etc.). The univariate version of the SHT has been considered in Frecka and Hopwood (1983), Theodossiu (1998), Aas and Haff (2006), Scott et al. (2009), Hürlimann (2009) and Ghysels and Wang (2011) among others. The NTS has been initially studied as subordinated Gaussian process by Barndorff-Nielsen and Shephard (2001) and Barndorff-Nielsen and Levendorskii (2001). More recent studies include Krause (2011) and Kim et al. (2012). The NTS also includes the NIG as special case. The presentation is divided into two parts.

4.1. Generalized inverse Gaussian mixing distribution

An important class of NVM models is the generalized hyperbolic (GH) distribution. It belongs to the generalized inverse Gaussian (GIG) mixing random variable $W \sim GIG(\lambda, \delta, \gamma)$ with cgf

$$C_W(t) = \frac{1}{2}\lambda \cdot \ln\left\{\frac{\gamma^2}{\gamma^2 - 2t}\right\} + \ln\left\{\frac{K_\lambda(\delta\sqrt{\gamma^2 - 2t})}{K_\lambda(\delta\gamma)}\right\},$$

where $K_\lambda(x)$ is the modified Bessel function of the third kind. The domain of variation of the parameters depends upon three cases.

Case 1: generic GH distribution with $-\infty < \lambda < \infty, \quad \delta > 0, \quad \gamma > 0$

Case 2: variance-gamma (VG) distribution with $\lambda > 0, \quad \delta = 0, \quad \gamma > 0$

Case 3: skew hyperbolic t (SHT) distribution with $\lambda < 0, \quad \delta > 0, \quad \gamma = 0$

In the limiting Case 2 the mixing distribution reduces to a gamma distribution (VG distribution) and in Case 3 one has an inverse gamma distribution (SHT distribution). We begin with the NIG distribution as representative of the generic case. In Case 1 it is convenient to re-parameterize the GIG by setting $\alpha = \delta\gamma > 0$, so that the cgf reads

$$C_W(t) = \frac{1}{2}\lambda \cdot \ln\left\{\frac{\alpha^2}{\alpha^2 - 2\delta^2 t}\right\} + \ln\left\{\frac{K_\lambda(\sqrt{\alpha^2 - 2\delta^2 t})}{K_\lambda(\alpha)}\right\}. \qquad (4.1)$$

Case 1: Multivariate normal inverse Gaussian (NIG)

The normal inverse Gaussian (NIG) is obtained for $\lambda = -\frac{1}{2}$ with cgf $C_W(t) = \alpha - \sqrt{\alpha^2 - 2\delta^2 t}$. We assume a mean of one unit, hence $\delta^2 = \alpha$. The first four cumulants are then given by

$$\kappa_1 = 1, \quad \kappa_2 = \frac{1}{\alpha}, \quad \kappa_3 = \frac{3}{\alpha^2}, \quad \kappa_4 = \frac{15}{\alpha^3}. \qquad (4.2)$$

The multivariate moment method for the NIG can be summarized as follows. The straightforward details of the derivation are left to the interested reader.

Step 1: parameters (M, V) as function of (α, S, K)

$$12SM^3 - 3\alpha KM^2 + \alpha^2(1+\alpha)S^2 = 0, \quad V = \frac{\alpha S}{3M}.$$

Step 2: parameters (β_i, V_i) as function of (α, M, V, S_i, K_i)

$$V_i = \frac{1}{3}\frac{\alpha^2 K_i - 6\alpha MS_i}{\alpha(1+\alpha)V - 2M^2}, \quad \beta_i = \frac{\alpha S_i}{V} - \frac{2}{3}\frac{(\alpha^2 K_i - 6\alpha MS_i)\cdot M}{(\alpha(1+\alpha)V - 2M^2)\cdot V}, \quad i = 1,...,n,$$

where one must assume that $\alpha(1+\alpha)V - 2M^2 \neq 0$.

Step 3: parameters (V_{ij}) as function of $(\alpha, M, V, \beta_i, V_i, K_{ij})$

$$V_{ij} = \frac{\alpha^2 K_{ij} - 2\alpha(1+\alpha)V_i V_j - 2V\beta_i\beta_j - 4(V_j\beta_i + V_i\beta_j)M}{\alpha(1+\alpha)V + 2M^2}, \quad i, j = 1,...,n.$$

Case 2: Multivariate variance-gamma (VG)

The variance-gamma (VG) is obtained from a gamma mixing random variable $W \sim \Gamma(1/\nu, 1/\nu)$ with cgf $C_W(t) = -\nu^{-1}\cdot\ln(1-\nu t)$. The first four cumulants are

$$\kappa_1 = 1, \quad \kappa_2 = \nu, \quad \kappa_3 = 2\nu^2, \quad \kappa_4 = 6\nu^3. \tag{4.3}$$

A summary of the moment method follows. The special case $\nu = 1$ is the multivariate asymmetric Laplace (AL) discussed in Hürlimann (2013b). For arbitrary $\nu > 0$ the method has first been applied in Hürlimann (2013c).

Step 1: parameters (M, V) as function of (ν, S, K)

$$(2\nu - 1)\nu^4 M^6 - 2(1+4\nu)\nu^2 SM^3 + 3\nu^2 KM^2 - (1+\nu)S^2 = 0, \quad V = \frac{\nu^2 M^3 + S}{3\nu M}.$$

Step 2: parameters (β_i, V_i) as function of (ν, M, V, S_i, K_i)

$$V_i = \frac{1}{3}\frac{K_i - 3\nu MS_i}{(1+\nu)V - \nu^2 M^2},$$
$$\beta_i = \frac{S_i}{\nu(V - \nu M^2)} - \frac{2}{3}\frac{(K_i - 3\nu MS_i)\cdot M}{(V - \nu M^2)\cdot((1+\nu)V - \nu^2 M^2)}, \quad i = 1,...,n,$$

where one must assume that $(V - \nu M^2)\cdot((1+\nu)V - \nu^2 M^2) \neq 0$.

Step 3: parameters (V_{ij}) as function of $(\nu, M, V, \beta_i, V_i, K_{ij})$

$$V_{ij} = \frac{K_{ij} - 2(1+\nu)V_i V_j - \nu^2(V - 3\nu M^2)\beta_i\beta_j - 2\nu^2(V_i\beta_j + \beta_i V_j)M}{(1+\nu)V + \nu^2 M^2}, \quad i, j = 1,...,n.$$

Case 3: Multivariate skew hyperbolic t (SHT)

The skew hyperbolic t (SHT) is obtained from an inverse gamma mixing random variable $W \sim I\Gamma(\alpha, \frac{1}{2}\delta^2)$ with characteristic function (chf) $\varphi_W(z) = 2(-i\frac{1}{2}\delta^2 z)^{\frac{\alpha}{2}} K_\alpha(\sqrt{-2i\delta^2 z})/\Gamma(\alpha)$. The cumulants exist only for $\alpha > 4$ and are given by

$$\kappa_1 = \frac{\delta^2}{2(\alpha-1)}, \quad \kappa_2 = \frac{\delta^4}{4(\alpha-1)^2(\alpha-2)},$$
$$\kappa_3 = \frac{\delta^6}{2(\alpha-1)^3(\alpha-2)(\alpha-3)}, \quad \kappa_4 = \frac{3(5\alpha-11)\delta^8}{8(\alpha-1)^4(\alpha-2)^2(\alpha-3)(\alpha-4)}.$$

Setting $\delta^2 = 2(\alpha-1)$ to normalize the mean to one unit, one obtains

$$\kappa_1 = 1, \quad \kappa_2 = \frac{1}{\alpha-2}, \quad \kappa_3 = \frac{4}{(\alpha-2)(\alpha-3)}, \quad \kappa_4 = \frac{6(5\alpha-11)}{(\alpha-2)^2(\alpha-3)(\alpha-4)}. \quad (4.4)$$

Omitting the calculations, the moment method summarizes as follows.

<u>Step 1</u>: parameters (M,V) as function of (α,S,K)

$$(\alpha+5)^2 M^6 - 2(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-7)SM^3 - 3(\alpha-2)(\alpha-3)^2(\alpha-4)KM^2$$
$$+(\alpha-1)(\alpha-2)^2(\alpha-3)^2(\alpha-4)S^2 = 0, \quad V = \frac{(\alpha-2)^2(\alpha-3)S-(\alpha+1)M^3}{3(\alpha-2)(\alpha-3)M}.$$

<u>Step 2</u>: parameters (β_i, V_i) as function of (α, M, V, S_i, K_i)

The equations (3.3) and (3.7) are equivalent to the following linear system of equations

$$\beta_i = \frac{(\alpha-2)(\alpha-3)\{(\alpha-2)S_i - 2MV_i\}}{(\alpha-2)(\alpha-3)V+(\alpha+1)M^2},$$

$$\begin{aligned}&\{\frac{3\alpha-5}{(\alpha-2)^2(\alpha-3)}V + \frac{3\alpha^2-\alpha-8}{(\alpha-2)^3(\alpha-3)(\alpha-4)}M^2\}\cdot M\beta_i \\ &+\{\frac{3\alpha-5}{(\alpha-2)^2(\alpha-3)}M^2 + \frac{\alpha-1}{\alpha-2}V\}\cdot V_i = \frac{1}{3}K_i\end{aligned}.$$

<u>Step 3</u>: parameters (V_{ij}) as function of $(\alpha, M, V, \beta_i, V_i, K_{ij})$

$$\{\frac{3\alpha-5}{(\alpha-2)^2(\alpha-3)}M^2+\frac{\alpha-1}{\alpha-2}V\}\cdot V_{ij}=K_{ij}-\frac{2(\alpha-1)}{\alpha-2}V_iV_j$$
$$-\{\frac{3\alpha-5}{(\alpha-2)^2(\alpha-3)}V+\frac{3(3\alpha^2-\alpha-8)}{(\alpha-2)^3(\alpha-3)(\alpha-4)}M^2\}\cdot\beta_i\beta_j$$
$$-\frac{2(3\alpha-5)}{(\alpha-2)^2(\alpha-3)}M(V_i\beta_j+\beta_iV_j)$$

4.2. Classical tempered stable mixing distribution

The *classical tempered stable* (CTS) mixing random variable $W \sim CTS(\alpha,\delta,\gamma)$ is determined by the cgf

$$C_W(t)=\alpha^{-1}\cdot\{\gamma^\alpha-(\gamma^2-2\delta^2t)^{\frac{\alpha}{2}}\},\quad \alpha\in(0,2),\,\delta,\gamma>0\,.$$

The corresponding NVM mixture is called normal tempered stable (NTS) model. A study and application of the univariate model is found in Hürlimann (2013e). A calculation shows that

$$C_W^{(k)}(t)=\delta^{2k}(\gamma^2-2\delta^2t)^{\frac{\alpha}{2}-k}\cdot\prod_{j=1}^{k-1}(2j-\alpha),\quad k\geq 1,$$

where an empty product is one. It follows that $\kappa_k=\delta^{2k}\gamma^{\alpha-2k}\cdot\prod_{j=1}^{k-1}(2j-\alpha),\,k\geq 1.$ Setting $\delta^2=\gamma^{2-\alpha}$ to normalize the mean to one unit, one obtains

$$\kappa_1=1,\quad \kappa_2=\frac{2-\alpha}{\gamma^\alpha},\quad \kappa_3=\frac{(2-\alpha)(4-\alpha)}{\gamma^{2\alpha}},\quad \kappa_4=\frac{(2-\alpha)(4-\alpha)(6-\alpha)}{\gamma^{3\alpha}}. \qquad (4.5)$$

In the special case $\alpha=1$ one recovers the NIG distribution analyzed in Case 1 of Section 4.1. In general, and in contrast to Section 4.1, the multivariate moment method offers more flexibility while depending upon two free parameters $\alpha\in(0,2),\gamma>0$. The method summarizes as follows.

Step 1: parameters (M,V) as function of (α,γ,S,K)

$$\begin{aligned}&2(1-\alpha)(2-\alpha)^2\{(2-\alpha)(4-\alpha)-2(1-\alpha)\gamma^{\alpha}\}M^6\\&-4(2-\alpha)\gamma^{2\alpha}\{(2-\alpha)(4-\alpha)+(1-\alpha)\gamma^{\alpha}\}SM^3\\&+3(2-\alpha)^2\gamma^{3\alpha}KM^2-\gamma^{4\alpha}(2-\alpha+\gamma^{\alpha})S^2=0,\end{aligned}$$

$$V=\frac{2(1-\alpha)(2-\alpha)M^3+\gamma^{2\alpha}S}{3(2-\alpha)\gamma^{\alpha}M}.$$

Step 2: parameters (β_i, V_i) as function of $(\alpha, \gamma, M, V, S_i, K_i)$

The equations (3.3) and (3.7) are equivalent to the following linear system of equations

$$\beta_i=\frac{\gamma^{2\alpha}S_i-2(2-\alpha)\gamma^{\alpha}MV_i}{(2-\alpha)\cdot\{\gamma^{\alpha}V-2(1-\alpha)M\}},$$

$$\begin{aligned}&2(2-\alpha)\{3\gamma^{\alpha}V-(1-\alpha)(6-\alpha)M^2\}\cdot M\beta_i\\&+\gamma^{\alpha}\{6(2-\alpha)M^2+3\gamma^{\alpha}(2-\alpha+\gamma^{\alpha})V\}\cdot V_i=\gamma^{3\alpha}K_i.\end{aligned}$$

Step 3: parameters (V_{ij}) as function of $(\alpha, \gamma, M, V, \beta_i, V_i, K_{ij})$

$$\begin{aligned}&\gamma^{\alpha}\{2(2-\alpha)M^2+\gamma^{\alpha}(2-\alpha+\gamma^{\alpha})V\}\cdot V_{ij}=\gamma^{3\alpha}K_{ij}-2\gamma^{2\alpha}(2-\alpha+\gamma^{\alpha})\cdot V_iV_j\\&-2(2-\alpha)\{\gamma^{\alpha}V-(1-\alpha)(6-\alpha)M^2\}\cdot\beta_i\beta_j-4(2-\alpha)\gamma^{\alpha}M(V_i\beta_j+\beta_iV_j)\end{aligned}$$

5. Statistical estimation of bivariate NVM logarithmic returns

We consider now two stock market indices for which all the mean, coskewness and cokurtosis quantities can be estimated. Return observations stem from the following seven different pairs of bivariate data from the Standard & Poors 500 (SP500) and the NASDAQ 100 (NDX) data sets:

SP500/NDX/3Y:

754 daily closing prices over 3 years from 04.01.2010 to 31.12.2012

SP500/NDX/5Y:

1259 daily closing prices over 5 years from 02.01.2008 to 31.12.2012

SP500/NDX/10Y:

2516 daily closing prices over 10 years from 02.01.2003 to 31.12.2012

SP500/NDX/15Y:

3773 daily closing prices over 15 years from 02.01.1998 to 31.12.2012

SP500/NDX/20Y:

5093 daily closing prices over 20 years from 04.01.1993 to 31.12.2012

SP500/NDX/25Y:

6302 daily closing prices over 25 years from 04.01.1988 to 31.12.2012

SP500/NDX/27Y:

6808 daily closing prices over 27 years from 02.01.1986 to 31.12.2012

These data sets are typical as they contain short to medium high volatile periods (recent 3 and 5 years), moderate long term periods (10 and 15 years), and long term periods (20,25 and 27 years). The last data set has been included because it contains the highest and lowest daily changes observed so far (drop in 22.9% and 16.3% for SP500 respectively NDX on 19.10.1987, increase of 17.2% for NDX on 03.01.2001).

The Table 5.1 below lists the required sample moment estimates for the bivariate logarithmic returns obtained from each of these combinations. Up to the 15Y and 20Y periods the coskewness vector has always negatively skewed components. The exception is the NDX. In the 15Y case one has also $S = S_1 + S_2 > 0$. Over the longest period of 27Y the coskewness components take the highest negative values. Up to the shortest 3Y period the overall cokurtosis coefficient $K = K_{11} + 2K_{12} + K_{22}$ exceeds 5 and is highest for the 5Y and 27Y periods. For specific fixed values of $\kappa = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ the bivariate NVM mixtures are fitted to the data following the moment method in Section 4.

Table 5.1: Sample moment estimates of bivariate log-returns

	moment estimates									
unit	10^-4		10^-6			10^-6				
SP500/NDX	µ1	µ2	S1	S2	S	K11	K12=K21	K22	K
3Y	3.05639	4.56635	-2.53599	-2.38737	-4.92336	0.49211	0.49395	0.50763	1.98765
5Y	-0.11603	2.07454	-3.14410	-2.33368	-5.47779	2.95173	2.94873	3.03195	11.8811
10Y	1.79008	3.78059	-1.88119	-1.36265	-3.24384	1.53194	1.54845	1.62493	6.25377
15Y	1.00817	2.57285	-0.55355	1.38095	0.82739	1.42779	1.85222	3.06706	8.19929
20Y	2.35612	4.00594	-0.87526	0.43794	-0.43732	1.11516	1.44658	2.39096	6.39928
25Y	2.72626	4.45475	-1.14828	-0.12651	-1.27479	0.93652	1.20443	1.97202	5.31740
27Y	2.81711	4.43105	-6.53485	-4.19231	-10.7272	2.18012	2.12053	2.63988	9.06107

One can argue that linear correlation cannot be fitted once the margins are fixed. However, the proposed moment method does not fit the margins separately, but provides an overall parsimonious fit of all its parameters regardless of the margins and the dependence structure. For this reason, it is important to discuss its goodness-of-fit capabilities. In particular, an analysis of the goodness-of-fit of the estimated margins is undertaken. To do so our goodness-of-fit (GoF) measure is based on statistics, which measure the difference between the empirical distribution functions $F_n(x)$ and the estimated marginal distribution functions $F(x)$. We use the Cramér-von Mises family of statistics defined by (e.g. D'Agostino and Stephens(1986), Cizek et al.(2005) and Burnecki et al.(2010))

$$T = n \cdot \int_{-\infty}^{\infty} [F_n(x) - F(x)]^2 w(x) dF(x), \tag{5.1}$$

where $w(x)$ is a suitable weighting function. If $w(x) = 1/\left[F(x)\bar{F}(x)\right]$ one gets the A^2 Anderson-Darling (1952) statistic. Consider the order statistics of the return data such that $r_1 \le r_2 \le ... \le r_n$ and let $\hat{F}(r_i), i = 1,...,n,$ be the estimated values of a marginal distribution function. Then one has

$$A^2 = -n - \sum_{i=1}^{n} \frac{2i-1}{n} \cdot \ln\left\{\hat{F}(r_i) \cdot \hat{\bar{F}}(r_{n-i+1})\right\}. \tag{5.2}$$

The values $\hat{F}(r_i)$ are obtained numerically by integration of the marginal densities

(Appendix 1) or their fast Fourier transform (FFT) approximations (Appendix 2). The Anderson-Darling statistic yields one of the most powerful test if the fitted distribution departs from the true distribution in the tails (e.g. D'Agostino and Stephens(1986)), and is recommended in this situation. Now, the observed sample return marginal data is skewed and has a much higher kurtosis than is allowed by a normal distribution, which indicates that the fit in the tails matters and justifies the use of (5.2). Needless to say, the proposed moment method is only a starting point for improved GoF estimation methods. However, a more complex data analysis is beyond the scope of the present study. To weight the influence of the margins, we use the Euclidean distance to define an overall GoF measure as $\|A\|^2 = (A_1^2)^2 + (A_2^2)^2$, with $A_i^2,\ i = 1,2,$ the Anderson-Darling statistics of the margins.

To calculate the FFT approximations of the marginal densities as specified in Appendix 2, one uses the following analytical expressions for their characteristic functions. Suppose that the marginal distributions are of the form $X_k = \xi_k + \beta_k \cdot W + \sqrt{W} \cdot Z_k,\ Z_k \sim N(0, \tau_k^2),\ k = 1,2$.

<u>NIG distribution</u>: $\phi_{X_k}(z) = \exp\{\xi_k \cdot iz + \alpha - \sqrt{\alpha^2 + \alpha \cdot (\tau_k^2 z^2 - 2\beta_k \cdot iz)}\}$

<u>VG distribution</u>: $\phi_{X_k}(z) = \exp\{\xi_k \cdot iz\} \cdot \{1 + \frac{1}{2}\nu \cdot (\tau_k^2 z^2 - 2\beta_k \cdot iz)\}^{-\nu^{-1}}$

<u>SHT distribution</u>:
$$\phi_{X_k}(z) = 2 \cdot e^{\xi_k \cdot iz} \cdot \{\tfrac{1}{2}(\alpha - 1) \cdot (\tau_k^2 z^2 - 2\beta_k \cdot iz)\}^{\frac{1}{2}\alpha} \cdot \Gamma(\alpha)^{-1} \cdot K_\alpha(\sqrt{2(\alpha - 1) \cdot (\tau_k^2 z^2 - 2\beta_k \cdot iz)})$$

where $\Gamma(\alpha)$ is the gamma function and $K_\lambda(x)$ is the Macdonald function (also called modified Bessel function of the 2nd kind, hyperbolic Bessel function of the 3rd kind, Basset function, modified Hankel function) (see (Oldham et al. (2009), Section 51).

<u>NTS distribution</u>: $\phi_{X_k}(z) = \exp\{\xi_k \cdot iz + \alpha^{-1} \cdot \{\gamma^\alpha - (\gamma^2 + \gamma^{2-\alpha} \cdot (\tau_k^2 z^2 - 2\beta_k \cdot iz))^{\frac{\alpha}{2}}\}\}$

The estimated parameters and GoF statistics of the different NVM mixtures are summarized and compared in the Tables 5.2 to 5.6. Table 5.7 summarizes the GoF $\|A\|$ ranking between these bivariate return distributions. Except for the 3Y period, best fitted by the bivariate VG, the best fit is always attained at the bivariate NTS and its NIG subfamily, followed by the bivariate VG (4 times) and the bivariate SHT (2 times). Let us attribute points to these rankings, say as much points as the rank is. Then, the NTS achieves 13 points, the NIG 25 points, the VG 39 points and the SHT 43 points. One can conclude that, in the present case study, the bivariate NTS and NIG perform best in terms of overall goodness-of-fit. A significant difference between the NTS and the NIG is only observed over the middle periods 5Y and 10Y, as seen in Table 5.5. One notes that the SHT moment method remains feasible over the range $\alpha \in (2,4), \alpha \neq 3$, though the third and fourth order cumulant do not exist. An explanation for this analytical continuation remains to be formulated in mathematical terms.

Table 5.2: Parameter estimates and GoF statistics for the bivariate NIG family

	parameter estimates								GoF statistics			GoF statistics		
unit		10^-3				10^-2			FFT method			numerical integral		
period	α	ξ1	ξ2	β1	β2	τ1	τ2	ρ	A1²	A2²	||A||	A1²	A2²	||A||
3Y	0.775	1.63850	1.52567	-1.33286	-1.06903	1.1397	1.1654	0.96500	0.87	1.00	1.32	0.85	1.01	1.323
	0.800	**1.66909**	**1.55028**	**-1.36345**	**-1.09365**	**1.1449**	**1.1707**	**0.96500**	**0.96**	**0.87**	**1.29**	**0.96**	**0.87**	**1.289**
	0.825	1.69959	1.57482	-1.39395	-1.11818	1.1498	1.1757	0.96500	1.09	0.73	1.31	1.07	0.74	1.305
5Y	0.375	0.46976	0.38601	-0.48136	-0.17855	1.6138	1.6394	0.95867	1.23	2.07	2.40	1.40	2.26	2.656
	0.400	**0.49001**	**0.39354**	**-0.50162**	**-0.18608**	**1.6327**	**1.6586**	**0.95866**	**1.44**	**1.56**	**2.12**	**1.59**	**1.74**	**2.360**
	0.425	0.51002	0.40098	-0.52162	-0.19352	1.6504	1.6765	0.95864	1.82	1.29	2.23	1.96	1.46	2.445
10Y	0.375	0.58586	0.50977	-0.40686	-0.13171	1.3680	1.4119	0.94649	3.52	8.67	9.36	3.77	9.02	9.781
	0.400	**0.60298**	**0.51533**	**-0.42398**	**-0.13727**	**1.3841**	**1.4284**	**0.94647**	**5.17**	**6.24**	**8.10**	**5.39**	**6.57**	**8.497**
	0.425	0.61989	0.52082	-0.44089	-0.14276	1.3991	1.4438	0.94645	7.06	4.43	8.33	7.26	4.73	8.669
15Y	0.525	0.51993	-0.31111	-0.41911	0.56839	1.3982	1.9718	0.69932	2.71	6.82	7.34	2.86	7.31	7.850
	0.550	**0.53329**	**-0.32923**	**-0.43247**	**0.58651**	**1.4089**	**1.9868**	**0.69927**	**3.87**	**6.23**	**7.33**	**4.00**	**6.70**	**7.806**
	0.575	0.54656	-0.34723	-0.44574	0.60451	1.4190	2.0010	0.69922	5.33	6.38	7.89	5.33	6.38	8.317
20Y	0.450	0.65518	0.06165	-0.41957	0.33894	1.2793	1.8061	0.70063	3.76	11.5	12.1	3.92	12.2	12.82
	0.475	**0.67037**	**0.04938**	**-0.43476**	**0.35122**	**1.2912**	**1.8229**	**0.70057**	**5.91**	**9.00**	**10.8**	**6.04**	**9.66**	**11.39**
	0.500	0.68543	0.03721	-0.44982	0.36338	1.3024	1.8387	0.70051	8.40	7.18	11.1	8.51	7.82	11.55
25Y	0.450	0.76206	0.21393	-0.48944	0.23155	1.2253	1.7192	0.70166	4.02	10.6	11.3	4.01	11.5	12.20
	0.475	**0.77979**	**0.20555**	**-0.50716**	**0.23993**	**1.2368**	**1.7352**	**0.70159**	**6.54**	**8.30**	**10.6**	**6.50**	**9.18**	**11.25**
	0.500	0.79735	0.19724	-0.52472	0.24823	1.2475	1.7502	0.70153	9.52	6.82	11.7	9.46	7.66	12.18
27Y	0.300	1.43306	0.59002	-1.15135	-0.14691	1.4675	1.6391	0.69981	26.8	37.9	46.4	25.4	39.0	46.50
	0.325	**1.49079**	**0.59780**	**-1.20908**	**-0.15469**	**1.4909**	**1.6643**	**0.69951**	**37.1**	**26.6**	**45.6**	**35.6**	**27.6**	**45.07**
	0.350	1.54753	0.60543	-1.26582	-0.16233	1.5125	1.6875	0.69926	48.2	18.3	51.6	58.3	13.3	59.82

Table 5.3: Parameter estimates and GoF statistics for the bivariate VG family

	parameter estimates								GoF statistics FFT method			GoF statistics numerical integral		
unit		10^-3				10^-2								
period	ν	ξ1	ξ2	β1	β2	τ1	τ2	ρ	$A1^2$	$A2^2$	||A||	$A1^2$	$A2^2$	||A||
3Y	1.02	1.88232	1.72341	-1.57668	-1.26678	1.1819	1.2062	0.96505	0.80	0.56	0.98	0.75	0.56	0.9311
	1.03	**1.87088**	**1.71422**	**-1.56524**	**-1.25758**	**1.1804**	**1.2048**	**0.96505**	**0.76**	**0.61**	**0.97**	**0.70**	**0.61**	**0.9308**
	1.04	1.85966	1.70520	-1.55402	-1.24856	1.1810	1.2033	0.96505	0.72	0.66	0.98	0.67	0.66	0.9386
5Y	1.60	0.66333	0.45815	-0.67494	-0.25070	1.7604	1.7867	0.95832	1.86	1.75	2.55	1.99	1.93	2.776
	1.65	**0.64915**	**0.45288**	**-0.66075**	**-0.24543**	**1.7520**	**1.7782**	**0.95832**	**1.65**	**1.94**	**2.55**	**1.79**	**2.13**	**2.784**
	1.70	0.63575	0.44790	-0.64735	-0.24044	1.7438	1.7700	0.95832	1.52	2.20	2.67	1.67	2.40	2.925
10Y	1.60	0.74951	0.56303	-0.57050	-0.18497	1.4923	1.5387	0.94608	9.44	6.75	11.6	9.61	7.11	11.954
	1.65	**0.73752**	**0.55914**	**-0.55851**	**-0.18108**	**1.4852**	**1.5314**	**0.94608**	**8.45**	**7.84**	**11.5**	**8.64**	**8.22**	**11.925**
	1.70	0.72618	0.55546	-0.54718	-0.17740	1.4783	1.5243	0.94608	7.64	9.07	11.9	7.84	9.46	12.291
15Y	1.30	0.64722	-0.48374	-0.54640	0.74103	1.4832	2.0914	0.69738	7.40	7.62	10.6	7.49	8.28	11.162
	1.35	**0.63267**	**-0.46401**	**-0.53185**	**0.72129**	**1.4752**	**2.0802**	**0.69738**	**6.40**	**8.15**	**10.4**	**6.51**	**8.82**	**10.963**
	1.40	0.61910	-0.44561	-0.51828	0.70290	1.4675	2.0693	0.69738	5.70	8.92	10.6	5.82	9.63	11.247
20Y	1.40	0.81036	-0.06371	-0.57475	0.46430	1.3783	1.9447	0.69874	14.4	10.4	17.7	14.3	11.2	18.167
	1.45	**0.79629**	**-0.05234**	**-0.56068**	**0.45294**	**1.3712**	**1.9347**	**0.69875**	**12.6**	**11.8**	**17.3**	**12.5**	**12.7**	**17.816**
	1.50	0.78311	-0.04169	-0.54749	0.44228	1.3643	1.9249	0.69875	11.2	13.5	17.6	11.1	14.4	18.233
25Y	1.45	0.92655	0.13616	-0.65392	0.30931	1.3140	1.8413	0.69970	16.4	14.2	21.7	**16.0**	**15.3**	**22.145**
	1.50	**0.91117**	**0.14344**	**-0.63854**	**0.30204**	**1.3074**	**1.8321**	**0.69970**	**14.7**	**15.9**	**21.7**	14.4	17.0	22.279
	1.55	0.89672	0.15027	-0.62409	0.29521	1.3009	1.8230	0.69971	13.5	18.0	22.5	13.2	19.1	23.240
27Y	2.00	1.86615	0.65195	-1.58443	-0.20884	1.6244	1.7971	0.69520	70.5	38.4	80.3	**67.6**	**39.6**	**78.313**
	2.05	**1.84044**	**0.64850**	**-1.55873**	**-0.20539**	**1.6176**	**1.7897**	**0.69522**	**67.7**	**42.9**	**80.2**	64.7	44.1	78.320
	2.10	1.81586	0.64520	-1.53414	-0.20210	1.6109	1.7824	0.69525	65.3	47.6	80.8	62.3	48.9	79.165

Table 5.4: Parameter estimates and GoF statistics for the bivariate SHT family

	parameter estimates								GoF statistics		
unit		10^-3				10^-2			FFT method		
period	α	ξ1	ξ2	β1	β2	τ1	τ2	ρ	$A1^2$	$A2^2$	$\lVert A \rVert$
3Y	2.52	1.27641	1.24598	-0.97077	-0.78934	1.1121	1.1211	0.96530	2.04	1.64	2.617
	2.53	**1.28831**	**1.25582**	**-0.98268**	**-0.79918**	**1.1163**	**1.1251**	**0.96531**	**2.10**	**1.56**	**2.615**
	2.54	1.30014	1.26560	-0.99450	-0.80897	1.1205	1.1290	0.96532	2.16	1.48	2.623
	4.0314	2.49386	1.61861	-2.18822	-1.16198	1.0955	1.0851	0.99582	3.65	1.99	4.161
	4.0315	**2.50859**	**1.64138**	**-2.20295**	**-1.18475**	**1.0972**	**1.0901**	**0.99472**	**3.69**	**1.91**	**4.156**
	4.0316	2.52257	1.66304	-2.21693	-1.20640	1.0988	1.0947	0.99371	3.72	1.85	4.156
5Y	2.21	0.32517	0.33298	-0.33677	-0.12553	1.4469	1.4649	0.95731	4.64	3.97	6.10
	2.22	**0.33450**	**0.33647**	**-0.34611**	**-0.12901**	**1.4608**	**1.4790**	**0.95729**	**4.79**	**3.76**	**6.09**
	2.23	0.34372	0.33991	-0.35532	-0.13245	1.4742	1.4925	0.95728	4.97	3.61	6.15
	4.0006599	**1.64801**	**0.74354**	**-1.65961**	**-0.53608**	**1.9226**	**2.0054**	**1.00000**	**35.24**	**28.29**	**45.19**
	4.001	1.64178	0.77301	-1.65338	-0.56556	1.9558	2.0230	0.98300	37.55	29.38	47.68
10Y	2.20	0.45570	0.46811	-0.27669	-0.09005	1.2141	1.2488	0.94493	9.65	8.86	13.10
	2.21	**0.46370**	**0.47072**	**-0.28469**	**-0.09266**	**1.2265**	**1.2615**	**0.94492**	**10.61**	**7.68**	**13.09**
	2.22	0.47159	0.47329	-0.29259	-0.09523	1.2384	1.2736	0.94490	11.61	6.67	13.39
	4.000546	**1.58664**	**0.76117**	**-1.40763**	**-0.38311**	**1.6205**	**1.7243**	**1.00000**	**91.38**	**41.43**	**100.34**
	4.001	1.57711	0.79583	-1.39811	-0.41777	1.6623	1.7457	0.97148	99.32	44.36	108.77
15Y	2.31	0.39923	-0.14743	-0.29841	0.40472	1.2770	1.8003	0.68962	6.18	11.22	12.81
	2.32	**0.40516**	**-0.15548**	**-0.30434**	**0.41276**	**1.2847**	**1.8113**	**0.68949**	**6.73**	**10.79**	**12.71**
	2.33	0.41104	-0.16346	-0.31023	0.42074	1.2923	1.8219	0.68937	7.31	10.44	12.75
	4.0000198	**1.25319**	**-1.30337**	**-1.15237**	**1.56066**	**1.4856**	**2.1840**	**1.00000**	**57.45**	**42.62**	**71.54**
	4.001	1.24867	-1.29938	-1.14785	1.55667	1.6431	2.3191	0.71095	95.22	59.71	112.39
20Y	2.26	0.53288	0.16046	-0.29726	0.24014	1.1598	1.6330	0.69238	11.60	14.45	18.53
	2.27	**0.53974**	**0.15491**	**-0.30413**	**0.24568**	**1.1685**	**1.6453**	**0.69228**	**12.78**	**13.23**	**18.40**
	2.28	0.54653	0.14942	-0.31092	0.25117	1.1770	1.6571	0.69218	14.03	12.19	18.59
	4.0000073	**1.50981**	**-0.62921**	**-1.27419**	**1.02981**	**1.3707**	**2.0807**	**1.00000**	**97.13**	**61.39**	**114.91**
	4.001	1.50841	-0.62762	-1.27280	1.02822	1.5433	2.1839	0.70820	161.9	81.43	181.18
25Y	2.25	0.61102	0.28549	-0.33839	0.15999	1.1043	1.5414	0.69317	11.09	15.01	18.67
	2.26	**0.61910**	**0.28167**	**-0.34647**	**0.16380**	**1.1130**	**1.5535**	**0.69308**	**12.47**	**13.75**	**18.56**
	2.27	0.62709	0.27789	-0.35447	0.16758	1.1214	1.5651	0.69298	13.95	12.72	18.88
	4.0000933	**1.77519**	**-0.27843**	**-1.50256**	**0.72390**	**1.2877**	**1.9917**	**1.00000**	**107.5**	**92.46**	**141.82**
	4.001	1.75876	-0.25866	-1.48613	0.70414	1.4614	2.0728	0.72943	191.8	114.8	223.60
27Y	2.12	0.94133	0.53564	-0.65962	-0.09253	1.2486	1.3533	0.68436	36.81	56.72	67.62
	2.13	**0.97124**	**0.53990**	**-0.68953**	**-0.09679**	**1.2714**	**1.3776**	**0.68433**	**44.13**	**47.61**	**64.92**
	2.14	1.00033	0.54405	-0.71862	-0.10095	1.2926	1.4003	0.68427	51.54	40.12	65.32
	4.012121	**5.05933**	**0.42347**	**-4.77761**	**0.01963**	**1.5635**	**1.9740**	**1.00000**	**272.3**	**107.0**	**292.60**
	4.02	4.94132	0.75073	-4.65961	-0.30762	1.7188	2.0721	0.82040	357.6	137.6	383.18

Table 5.5: Parameter estimates and GoF statistics for the NTS and NIG family

	parameter estimates									GoF statistics		
unit			10^-3				10^-2			FFT method		
period	α	γ	ξ1	ξ2	β1	β2	τ1	τ2	ρ	A1^2	A2^2	$\lVert A\rVert$
3Y	0.98	0.81	1.40834	1.34118	-1.10270	-0.88454	1.1518	1.1748	0.96508	0.93	0.86	1.27
	1	0.80	1.66909	1.55028	-1.36345	-1.09365	1.1449	1.1707	0.96500	0.96	0.87	1.29
5Y	0.73	0.48	0.32663	0.33300	-0.33823	-0.12555	1.6769	1.7005	0.95797	1.42	1.38	1.97
	1	0.40	0.49001	0.39354	-0.50162	-0.18608	1.6327	1.6586	0.95866	1.44	1.56	2.12
10Y	1	0.41	0.60977	0.51753	-0.43077	-0.13947	1.3902	1.4347	0.94646	5.90	5.44	8.03
	1.17	0.38	1.94972	0.95107	-1.77071	-0.57301	1.2985	1.4131	0.97209	5.42	3.93	6.69
15Y	0.89	0.59	0.56619	-0.37384	-0.46537	0.63113	1.4141	1.9942	0.69962	3.18	6.40	7.15
	1	0.54	0.52796	-0.32200	-0.42714	0.57928	1.4047	1.9809	0.69929	3.38	6.43	7.26
20Y	0.88	0.53	0.67288	0.04736	-0.43726	0.35324	1.3075	1.8454	0.69970	6.64	8.16	10.52
	1	0.48	0.67340	0.04694	-0.43778	0.35366	1.2935	1.8261	0.70055	6.38	8.58	10.69
25Y	0.98	0.48	0.76717	0.21151	-0.49455	0.23396	1.2382	1.7368	0.70118	6.34	8.42	10.54
	1	0.47	0.77625	0.20722	-0.50363	0.23826	1.2345	1.7320	0.70160	5.99	8.69	10.56
27Y	1	0.31	1.45628	0.59315	-1.17457	-0.15004	1.4771	1.6494	0.69968	30.79	32.95	45.10
	1.01	0.31	1.59380	0.61062	-1.31208	-0.16752	1.4669	1.6488	0.70285	31.46	32.15	44.98

Table 5.6: minimum $\lVert A\rVert$ GoF statistic on a grid for the NTS and NIG family

period	α	γ	$\lVert A\rVert$	period	α	γ	$\lVert A\rVert$	period	α	γ	$\lVert A\rVert$
3Y	0.97	0.81	1.27192	5Y	0.72	0.48	1.97700	10Y	0.99	0.4	8.17594
	0.97	0.82	1.26798		0.72	0.49	1.97522		0.99	0.41	8.04533
	0.97	0.83	1.27131		0.72	0.50	1.99798		0.99	0.42	8.13940
	0.98	0.80	1.27389		0.73	0.47	1.99341		1.00	0.40	8.10034
	0.98	**0.81**	**1.26767**		**0.73**	**0.48**	**1.97361**		**1.00**	**0.41**	**8.02945**
	0.98	0.82	1.26907		0.73	0.49	1.98078		1.00	0.42	8.18328
	0.99	0.80	1.27545		0.74	0.47	1.98457		1.16	0.36	7.09262
	0.99	0.81	1.27407		0.74	0.48	1.97372		1.16	0.37	6.73519
	0.99	0.82	1.28020		0.74	0.49	1.99005		1.16	0.38	6.80486
	1.00	0.79	1.29995		1.00	0.39	2.18379		1.17	0.37	6.77697
	1.00	**0.80**	**1.29445**		**1.00**	**0.40**	**2.12207**		**1.17**	**0.38**	**6.69209**
	1.00	0.81	1.29686		1.00	0.41	2.12355		1.17	0.39	7.06194
	1.01	0.79	1.34766		1.01	0.39	2.16884		1.18	0.38	7.34736
	1.01	0.80	1.34409		1.01	0.40	2.12238		1.18	0.39	7.15127
	1.01	0.81	1.34833		1.01	0.41	2.13935		1.18	0.40	7.59654
period	α	γ	$\lVert A\rVert$	period	α	γ	$\lVert A\rVert$	period	α	γ	$\lVert A\rVert$
15Y	0.88	0.59	7.16142	20Y	0.87	0.52	10.67624	25Y	0.97	0.47	10.75328
	0.88	0.60	7.15325		0.87	0.53	10.52789		0.97	0.48	10.53570
	0.88	0.61	7.20876		0.87	0.54	10.55249		0.97	0.49	10.61594
	0.89	0.58	7.18967		0.88	0.52	10.59404		0.98	0.47	10.65069
	0.89	**0.59**	**7.14939**		**0.88**	**0.53**	**10.51677**		**0.98**	**0.48**	**10.53517**
	0.89	0.60	7.17622		0.88	0.54	10.61423		0.98	0.49	10.72048
	0.90	0.58	7.16248		0.89	0.52	10.54217		0.99	0.47	10.58464
	0.90	0.59	7.15724		0.89	0.53	10.53861		0.99	0.48	10.57515
	0.90	0.60	7.22030		0.89	0.54	10.71029		0.99	0.49	10.86695
	1.00	0.53	7.28807		1.00	0.47	10.89829		1.00	0.46	10.78287
	1.00	**0.54**	**7.26230**		**1.00**	**0.48**	**10.69321**		**1.00**	**0.47**	**10.55718**
	1.00	0.55	7.33272		1.00	0.49	10.75092		1.00	0.48	10.65685
	1.01	0.52	7.36767		1.01	0.47	10.82585		1.01	0.46	10.68721
	1.01	0.53	7.28498		1.01	0.48	10.71449		1.01	0.47	10.57014
	1.01	0.54	7.30336		1.01	0.49	10.86734		1.01	0.48	10.78116

Table 5.7: Goodness-of-fit ||A|| ranking between NTS, NIG, VG and SHT

data set	FFT GoF statistics			
period	1	2	3	4
3Y	VG	NTS	NIG	SHT
5Y	NTS	NIG	VG	SHT
10Y	NTS	NIG	VG	SHT
15Y	NTS	NIG	VG	SHT
20Y	NTS	NIG	VG	SHT
25Y	NTS	NIG	SHT	VG
27Y	NTS	NIG	SHT	VG

Appendix 1: FFT approximation of the marginal densities and distribution functions

If no tractable expression for a probability density function is available, it is possible to approximate it using the fast Fourier transform (FFT) (e.g. Scherer et al. (2012)). We use the interpolation scheme by Jelonek (2012), Appendix B, which has been adapted here to the mid-point rule (MPR) for a higher accuracy.

Consider a finite interval $[a,b]$ that is divided into N disjoints subintervals of equal length $h=(b-a)N^{-1}$ and assume that the random variable X with pdf $f_X(x)$ has a known characteristic fnction $\phi_X(z), z\in C$. For $k=0,...,N-1$ set $x_k=a+hk$. For N sufficiently large the constant $c=\pi\cdot h^{-1}$ is also large and one has the pdf approximation

$$f_X(x_k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-izx_k}\cdot\phi_X(z)dz\approx\frac{1}{2\pi}\int_{-c}^{c}e^{-izx_k}\cdot\phi_X(z)dz=\int_{-N/2(b-a)}^{N/2(b-a)}e^{-2\pi i\cdot ux_k}\cdot\phi_X(2\pi\cdot u)du.$$

For $j=0,...,N$ set $u_j=(j-\frac{N}{2})(b-a)^{-1}$ and consider the mid-points

$$m_j=\tfrac{1}{2}(u_j+u_{j+1})=(j-\tfrac{N-1}{2})(b-a)^{-1},\quad j=0,...,N-1.$$

Applying the MPR to the right-hand side integral one obtains the finite sum approximation

$$\begin{aligned}f_X(x_k)&\approx(b-a)^{-1}\cdot\sum_{j=0}^{N-1}e^{-2\pi i\cdot m_jx_k}\cdot\phi_X(2\pi\cdot m_j)\\&=(b-a)^{-1}\cdot\sum_{j=0}^{N-1}e^{-2\pi i\cdot(\frac{a}{h}+k)(j-\frac{N-1}{2})\frac{1}{N}}\cdot\phi_X(\tfrac{2\pi}{b-a}(j-\tfrac{N-1}{2})).\end{aligned}$$

Since $e^{\pi i} = -1$ one has further $e^{-2\pi i\cdot(\frac{a}{h}+k)(j-\frac{N-1}{2})\frac{1}{N}} = (-1)^{(\frac{a}{b-a}+\frac{k}{N})(N-1)} \cdot (-1)^{(\frac{2a}{b-a})\cdot j} \cdot e^{-2\pi i\cdot k\frac{j}{N}}$. Inserted into the above sum, one gets the desired representation

$$f_X(x_k) \approx (b-a)^{-1} \cdot (-1)^{(\frac{a}{b-a}+\frac{k}{N})(N-1)} \cdot \sum_{j=0}^{N-1} (-1)^{(\frac{2a}{b-a})\cdot j} \cdot \phi_X(\tfrac{2\pi}{b-a}(j-\tfrac{N-1}{2})) \cdot e^{-2\pi i\cdot k\frac{j}{N}},$$

which one interprets as k -th component of a Discrete Fourier Transform (DFT)

$$\begin{aligned} &f_X(x_k) \approx C_k \cdot DFT(y)_k, \quad C_k = (b-a)^{-1} \cdot (-1)^{(\frac{a}{b-a}+\frac{k}{N})(N-1)}, \\ &y = (y_0, ..., y_{N-1}), \quad y_j = (-1)^{(\frac{2a}{b-a})\cdot j} \cdot \phi_X(\tfrac{2\pi}{b-a}(j-\tfrac{N-1}{2})), \quad j = 0, ..., N-1. \end{aligned}$$

An efficient software implementation of the DFT is based on the Fast Fourier Transform (FFT) algorithm by Cooley and Tukey (1965). For numerical approximation of the distribution function $F_X(x) = \int_{-\infty}^{x} f_x(t)dt$ one derives a similar DFT approximation in terms of the chf (e.g. Kim et al. (2010), Proposition 1) or one uses the recursive formula

$$F_X(x_k) = F_X(x_{k-1}) + hf_X(x_{k-1}), \quad k = 1, ..., N-1, \quad F_X(x_0) = 0,$$

and a simple piecewise linear interpolation for intermediate values:

$$F_X(x) = F_X(x_{k-1}) + h^{-1}(x - x_{k-1})\{F_X(x_k) - F_X(x_{k-1})\}, \quad x \in [x_{k-1}, x_k], \quad k = 1, ..., N-1.$$

Finally, we note that similar approximations can be obtained for the value-at-risk measure (VaR), the stop-loss transform and the related conditional value-at-risk measure (CVaR) (see Kim et al. (2010) for formulas in terms of the chf). They can be used for further important financial applications of the multivariate NVM mixtures in option pricing and risk management.

Appendix 2: Numerical integration of the NIG and VG marginal densities

Alternatively to the FFT approximation method in Appendix 1, the NIG and VG marginal distributions have been calculated more accurately using their analytical density expressions.

NIG marginal density

The probability density of the unit mean inverse Gaussian mixing random variable reads

$$f_W(w) = \sqrt{\tfrac{\alpha}{2\pi}} e^{\alpha} w^{-\frac{3}{2}} \exp\{-\tfrac{\alpha}{2}(w + w^{-1})\}.$$

The marginal random variables of the multivariate NIG distribution are of the form

$$X = \xi + \tfrac{\beta}{\tau^2} \cdot U + \sqrt{U} \cdot Z,\ Z \sim N(0,1),$$

where $U = \tau^2 \cdot W$ has the density

$$f_U(u) = \tau \sqrt{\tfrac{\alpha}{2\pi}} e^{\alpha} u^{-\frac{3}{2}} \exp\{-\tfrac{1}{2}(\tfrac{\alpha}{\tau^2} u + \alpha\tau^2 \cdot u^{-1})\}.$$

It follows that

$$\begin{aligned} f_X(u) &= \int_0^{\infty} f_{X|U=u}(x) f_U(u) du \\ &= \int_0^{\infty} \tfrac{1}{\sqrt{2\pi u}} \exp\{-\tfrac{1}{2u}(x - \xi - \tfrac{\beta}{\tau^2} u)^2\} \cdot \tau \sqrt{\tfrac{\alpha}{2\pi}} e^{\alpha} u^{-\frac{3}{2}} \exp\{-\tfrac{1}{2}(\tfrac{\alpha}{\tau^2} u + \alpha\tau^2 \cdot u^{-1})\} du \\ &= \tfrac{\sqrt{\alpha}}{\pi\tau} \frac{\sqrt{\alpha + (\frac{\beta}{\tau})^2} \cdot K_1(\sqrt{\alpha + (\frac{\beta}{\tau})^2} \cdot \sqrt{\alpha + (\frac{x-\xi}{\tau})^2})}{\sqrt{\alpha + (\frac{x-\xi}{\tau})^2}} \exp\{\alpha + \tfrac{\beta}{\tau}(\tfrac{x-\xi}{\tau})\}, \end{aligned}$$

where $K_1(x)$ is the Macdonald function of order 1.

VG marginal density

A bilateral gamma (BG) random variable is defined by (e.g. Küchler and Tappe (2008))

$$X = \xi + \alpha^{-1} \cdot G_1 - \beta^{-1} \cdot G_2 \sim BG(\xi, \gamma, \alpha, \delta, \beta),\ \gamma, \alpha, \delta, \beta > 0, -\infty < \xi < \infty,$$

with independent $G_1 \sim \Gamma(\gamma,1), G_2 \sim \Gamma(\delta,1)$ (standardized gamma's with scale parameter 1). It suffices to restrict the attention to the BG with vanishing location $\xi = 0$. The BG pdf, denoted by $f(x) = f(x;\gamma,\alpha,\delta,\beta)$, is the convolution $f(x) = (f_1 * f_2)(x)$ of the two gamma pdf's:

$$f_1(x) = \Gamma(\gamma)^{-1}\alpha^{\gamma} x^{\gamma-1} e^{-\alpha x} \cdot 1\{x \ge 0\}, \quad f_2(x) = \Gamma(\delta)^{-1}\beta^{\delta}|x|^{\delta-1} e^{-\beta|x|} \cdot 1\{x \le 0\}. \qquad (A.1)$$

The following "generalized gamma function" representation seems new. It is equivalent to the representation (A.6) below in terms of the confluent hyper-geometric function of the 2nd kind.

Theorem A.1 (*Generalized gamma function representation*). The probability density function of the bilateral gamma $BG(\xi = 0,\gamma,\alpha,\delta,\beta)$ is given by

$$\begin{aligned} f(x) &= \Gamma(\gamma)^{-1}\Gamma(\delta)^{-1}\left(\tfrac{\beta}{\alpha+\beta}\right)^{\delta}\alpha^{\gamma} x^{\gamma-1} e^{-\alpha x} \cdot \Gamma(\delta,\gamma,(\alpha+\beta)x), \quad x > 0, \\ f(x) &= \Gamma(\gamma)^{-1}\Gamma(\delta)^{-1}\left(\tfrac{\alpha}{\alpha+\beta}\right)^{\gamma}\beta^{\delta}|x|^{\delta-1} e^{-\beta|x|} \cdot \Gamma(\gamma,\delta,(\alpha+\beta)|x|), \quad x < 0, \end{aligned} \qquad (A.2)$$

with the *generalized gamma function*

$$\Gamma(a,b,x) = \int_0^{\infty} t^{a-1}(1+x^{-1}t)^{b-1} e^{-t}\,dt. \qquad (A.3)$$

Proof. Using the symmetry relation $f(x;\gamma,\alpha,\delta,\beta) = f(-x;\delta,\beta,\gamma,\alpha)$ it suffices to consider the case $x \in (0,\infty)$. Through elementary integration (change of variables $y = -tx$) one obtains

$$\begin{aligned} f(x) &= (f_1 * f_2)(x) = \int_{-\infty}^{0} f_1(x-y) f_2(y)\,dy = \Gamma(\gamma)^{-1}\Gamma(\delta)^{-1}\alpha^{\gamma}\beta^{\delta} e^{-\alpha x} \cdot I(x), \\ I(x) &= \int_{-\infty}^{0} (x-y)^{\gamma-1}(-y)^{\delta-1} e^{(\alpha+\beta)y}\,dy = \int_0^{\infty} x^{\gamma+\delta-1}(1+t)^{\gamma-1} t^{\delta-1} e^{-(\alpha+\beta)xt}\,dt, \end{aligned}$$

The transformation $t = c(x)^{-1}u$ with $c(x) = (\alpha + \beta)x$ yields further

$$I(x) = x^{\gamma+\delta-1}c(x)^{-\delta} \cdot \int_0^\infty (1 + c(x)^{-1}u)^{\gamma-1}u^{\delta-1}e^{-u}du = x^{\gamma-1}(\alpha+\beta)^{-\delta} \cdot \Gamma(\delta, \gamma, c(x)).$$

Insert into the first integral expression for $f(x)$ to get (A.2). ◊

In virtue of the limiting property $\lim_{x\to\infty}\Gamma(a,b,x) = \int_0^\infty t^{a-1}e^{-t}dt = \Gamma(a)$ the naming of the integral (A.3) is justified. Furthermore, one has also trivially $\Gamma(a,1,x) = \Gamma(a)$. Another justification arises from the fact that when $\alpha \to \infty$ or $\beta \to \infty$ the pdf converges to a left- and right-tail gamma pdf respectively, as should be. Moreover, a close look at the *confluent hyper-geometric function of the 2nd kind*, introduced by Tricomi (1947) and also called Tricomi function, shows the relationship

$$\Gamma(a,b,x) = \Gamma(a)x^a U(a, a+b, x), \tag{A.4}$$

where the *Tricomi function* is defined by (e.g. Oldham et al. (2009), 48:3:6 and 48:3.7)

$$U(a,b,x) = \Gamma(a)^{-1} \cdot \int_0^\infty t^{a-1}(1+t)^{b-a-1}e^{-xt}dt = \Gamma(a)^{-1} \cdot \int_0^1 t^{a-1}(1-t)^{-b}e^{-xt(1-t)^{-1}}dt. \tag{A.5}$$

The generalized gamma function is a transformed Tricomi function and (A.2) rewrites

$$\begin{aligned} f(x) &= \Gamma(\gamma)^{-1}\alpha^\gamma x^{\gamma-1}e^{-\alpha x}(\beta x)^\delta \cdot U(\delta, \gamma+\delta, (\alpha+\beta)x), \quad x > 0, \\ f(x) &= \Gamma(\delta)^{-1}\beta^\delta |x|^{\delta-1}e^{-\beta|x|}(\alpha|x|)^\gamma \cdot U(\gamma, \gamma+\delta, (\alpha+\beta)|x|), \quad x < 0. \end{aligned} \tag{A.6}$$

In the variance-gamma special case $VG(\rho,\alpha,\beta) = BG(\xi = 0, \gamma = \rho, \alpha, \delta = \rho, \beta)$ the relevant Tricomi function reduces to a Macdonald function of the type (Oldham et al. (2009), 48:4:3 and 48:13:6)

$$U(a,2a,x)=\frac{x^{\frac{1}{2}-a}}{\sqrt{\pi}}e^{\frac{1}{2}x}K_{a-\frac{1}{2}}(\tfrac{1}{2}x). \qquad \text{(A.7)}$$

Inserting these expressions into the Tricomi representation (A.6) one obtains the VG pdf

$$f(x)=\frac{(\alpha\beta)^{\rho}}{\sqrt{\pi}\Gamma(\rho)}\left(\frac{|x|}{\alpha+\beta}\right)^{\rho-\frac{1}{2}}\cdot\exp\left(-\tfrac{1}{2}(\alpha-\beta)x\right)\cdot K_{\rho-\frac{1}{2}}(\tfrac{1}{2}(\alpha+\beta)|x|), \quad x\neq 0. \qquad \text{(A.8)}$$

This closed-form expression has been first derived in Madan et al. (1998) for the parameterization

$$\left(\theta,\sigma^{2},\nu\right)=\left((\alpha^{-1}-\beta^{-1})\rho, 2(\alpha\beta)^{-1}\rho, \rho^{-1}\right). \qquad \text{(A.9)}$$

However, in its original form the VG pdf takes the less symmetrical form

$$\begin{aligned} f(x)&=\frac{2\exp(\sigma^{-2}\theta x)}{\nu^{\nu^{-1}}\sqrt{2\pi}\sigma\cdot\Gamma(\nu^{-1})}\cdot\left(\frac{x^{2}}{\theta^{2}+2\nu^{-1}\sigma^{2}}\right)^{\frac{1}{2}\nu^{-1}-\frac{1}{4}} \\ &\cdot K_{\nu^{-1}-\frac{1}{2}}\left(\sigma^{-2}\sqrt{(\theta^{2}+2\nu^{-1}\sigma^{2})x^{2}}\right), \quad x\neq 0 \end{aligned} \qquad \text{(A.10)}$$

Conflict of Interests

The author declares that there is no conflict of interests.

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