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# AN OVERVIEW OF WADA'S REPRESENTATIONS

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**Abstract.** M. Wada has constructed several linear representations of the braid group. An overview of three interesting types of these representations is given in our work. As an extension to the results obtained about Wada's representations of types 1 and 2, we prove that type 3 is of Burau type and that all the three types of Wada's representations are equivalent. We determine the hecke algebras that these representations arise from.

**Keywords**: Artin representation, braid group, Burau representation, Magnus representation.

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## 1. Introduction

Let $B_n$ be the braid group on $n$ strands. This group has a standard presentation

$$< \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i, \text{if} \mid i-j \mid > 1; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ for } 1 \leq i \leq n-2 > .$$

There is a well known representation, due to Artin, in the group $Aut(F_n)$ of automorphisms of a free group $F_n$ generated by $x_1, \ldots, x_n$. The automorphism corresponding to the braid generator $\sigma_i$, with $i \in \{1, \ldots, n-1\}$, takes $x_i$ to $x_ix_{i+1}x_i^{-1}$, $x_{i+1}$ to $x_i$ and fixes all other generators. In [4],

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M. Wada had discovered several linear representations of the braid group by automorphisms of $F_n$. The following are the most interesting representations:

**(1)** For an arbitrary non-zero integer $k$, the automorphism corresponding to the braid generator $\sigma_i$, with $i \in \{1,\ldots,n-1\}$, takes $x_i$ to $x_i^k x_{i+1} x_i^{-k}$, $x_{i+1}$ to $x_i$ and fixes all other generators.

**(2)** The automorphism corresponding to the braid generator $\sigma_i$, with $i \in \{1,\ldots,n-1\}$, takes $x_i$ to $x_i x_{i+1}^{-1} x_i$, $x_{i+1}$ to $x_i$ and fixes all other generators.

**(3)** The automorphism corresponding to the braid generator $\sigma_i$, with $i \in \{1,\ldots,n-1\}$, takes $x_i$ to $x_i^2 x_{i+1}$, $x_{i+1}$ to $x_{i+1}^{-1} x_i^{-1} x_{i+1}$ and fixes all other generators.

Abdulrahim made a complete study of the first two types (see [1] and [2]). In this paper, we discuss Wada's representation of type 3 and compare our results to those obtained in [1] and [2]. This completes the study of Wada's representations.

In section 2, we let $z \in \mathbb{C}^*$ and we define the reduced Burau representation $\beta_n(z) : B_n \to GL_{n-1}(\mathbb{C})$, when it is irreducible. We also state some theorems that give a characterization for irreducible representations, where the matrix of one of the generators of the braid group is a pseudoreflection.

In section 3, we define Wada's representations of types 1 and 2 and present some previous theorems concerning the irreducibility and unitarity of these representations.

In section 4, we consider Wada's representation of type 3 and show that this representation is of Burau type.

In section 5, we make a comparison between Wada's representations of types 1, 2 and 3 and we present our main theorem, Theorem 23.

In section 6, we introduce the Hecke algebra and conclude with Theorem 30, which determines the Hecke algebras that Wada's representations arise from.

## 2. Burau Representation

**Definition 1.** *An $n \times n$ matrix $H$ is a pseudoreflection if $H$ can be written as $H = I_n - AB$ for some column and row vectors $A$ and $B$ respectively. Here, $I_n$ is the $n \times n$ identity matrix.*

According to the standard Burau representation, the automorphism corresponding to $\sigma_i$, sends $x_i$ to $x_i x_{i+1} x_i^{-1}$, $x_{i+1}$ to $x_i$ and fixes all other generators. Applying Magnus representation to the image of the braid group, we obtain the Burau representation $B_n \to GL_n(\mathbb{C})$. The automorphism $\sigma_i$ is mapped to the following $n \times n$ matrix

$$\sigma_i(z) = I_{i-1} \oplus \begin{pmatrix} 1-z & z \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1} \text{ for } i = 1, 2, \ldots, n-1.$$

It is clear that this representation is reducible.

**Definition 2.** *The complex reduced Burau representation $\beta_n(z) : B_n \to GL_{n-1}(\mathbb{C})$ is given by $\beta_n(z)(\sigma_i) = I_{n-1} - C_i D_i$, where*

$$C_1 = (z+1, 1, \underbrace{0, \ldots, 0}_{n-3})^T,$$

$$C_i = (\underbrace{0, \ldots, 0}_{i-2}, z, z+1, 1, \underbrace{0, \ldots, 0}_{n-i-2})^T, \textit{ for } i = 2, \ldots, n-2,$$

*and*

$$C_{n-1} = (\underbrace{0, \ldots, 0}_{n-3}, z, z+1)^T.$$

*Here, $\{D_1, \ldots, D_{n-1}\}$ is the standard basis of $\mathbb{C}^{n-1}$ and $T$ is the transpose.*

The associated matrix given by the inner product is

$$(D_i C_j) = \begin{pmatrix} z+1 & z & 0 & \ldots & \ldots & 0 \\ 1 & z+1 & z & 0 & \ldots & \vdots \\ 0 & 1 & z+1 & z & \ldots & \vdots \\ 0 & 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ldots & 1 & z+1 & z \\ 0 & 0 & \ldots & 0 & 1 & z+1 \end{pmatrix}.$$

Regarding the question of whether or not the above representation is irreducible for certain values of $z$, we state some lemmas and theorems which are proved by E. Formanek.

**Lemma 1.** [3] *For $z \in \mathbb{C}^*$, $\beta_n(z) : B_n \to GL_{n-1}(\mathbb{C})$ is irreducible if and only if $z$ is not a root of $f_n(t) = t^{n-1} + t^{n-2} + \cdots + t + 1$.*

We now state some theorems proved by E. Formanek [3], which will be needed in our work.

**Lemma 2.** *[3, p. 286] If $n > 3$ and $z$ is a root of $f_n(t) = t^{n-1} + t^{n-2} + \cdots + t + 1$ then $\hat{\beta}_n(z) : B_n \to GL_{n-2}(\mathbb{C})$ is a composition factor of $\beta_n(z) : B_n \to GL_{n-1}(\mathbb{C})$, where $\hat{\beta}_n(z)$ is defined by:*

$$\hat{\beta}_n(z)(\sigma_i) = \beta_{n-1}(\sigma_i)(i = 1, \ldots, n-2) \text{ and } \hat{\beta}_n(z)(\sigma_{n-1}) = I_{n-1} - PQ,$$

*where $P$ is the column vector given by $P = (0, \ldots, 0, z)^T$ and $Q$ is the row vector given by $Q = (-1)^{n-2} z(1, -(1+z), (1+z+z^2), \ldots, (-1)^{n-3}(1+z+\cdots+z^{n-3}))$.*

**Theorem 1.** *[3, p. 287] Let $\rho : B_n \to GL_r(\mathbb{C})$ be an irreducible representation, where $n > 4$ and $r > 1$. Suppose that $\rho(\sigma_1)$ is a pseudoreflection. Then either*

*(a) the representation $\rho$ is equivalent to $\beta_n(z) : B_n \to GL_{n-1}(\mathbb{C})$, where $z \in \mathbb{C}^*$ is not a root of $f_n(t) = t^{n-1} + \cdots + t + 1$; or*
*(b) the representation $\rho$ is equivalent to $\hat{\beta}_n(z) : B_n \to GL_{n-2}(\mathbb{C})$, where $z \in \mathbb{C}^*$ is a root of $f_n(t)$.*

**Definition 3.** *A representation of $B_n$ is of* Burau type *if it is of degree $> 1$ and it is equivalent to the tensor product of a one-dimensional representation and the irreducible representation $\beta_n(z)$ or its composition factor, namely $\hat{\beta}_n(z)$.*

**Theorem 2.** *[3, p. 282] Let $X_1 = I - A_1B_1$, $\ldots$, $X_r = I - A_rB_r$ be $r$ invertible pseudoreflections in $M_r(\mathbb{C})$, where $r > 1$. Let $\tau$ be the directed graph whose vertices are $1, \ldots, r$, and which has a directed edge from $i$ to $j$ ($i \neq j$) precisely when $B_iA_j \neq 0$. Let $G$ be the subgroup of $GL_r(\mathbb{C})$ generated by $X_1, \ldots, X_r$. Then the following holds.*

*(a) $G$ is an irreducible subgroup of $GL_r(\mathbb{C})$ if and only if for each $i \neq j$ ($1 \leq i, j \leq r$), the graph $\tau$ contains a directed path from $i$ to $j$ and $(B_iA_j) \in M_r(\mathbb{C})$ is invertible.*

*(b) Suppose that* $G = \langle X_1,\dots,X_r\rangle$ *and* $H = \langle Y_1,\dots,Y_r\rangle$ *are irreducible subgroups of* $GL_r(\mathbb{C})$*, generated by pseudoreflections* $X_i = I - A_iB_i$*,* $Y_i = I - C_iD_i$*. Then there is a matrix* $T \in GL_r(\mathbb{C})$ *such that* $TX_iT^{-1} = Y_i$ *for* $i = 1,\dots,r$ *if and only if there exist* $a_1,\dots,a_r \in \mathbb{C}$ *such that* $D_iC_j = a_i^{-1}a_jB_iA_j$ *(that is,* $(B_iA_j)$ *and* $(D_iCj)$ *are conjugates by a diagonal matrix).*

## 3. Proved Results about Wada's Representations of Type 1 and Type 2

We recall some known results about Wada's representations of type 1 and type 2, which are proved in [1] and [2] respectively. More precisely, theorems about irreducibility and unitarity of such representations are presented.

**Wada's Representation of Type 1:**

**Definition 4.** [1] *Let k be a nonzero integer and t an independent indeterminate. Wada's representation of type 1 asserts that the automorphism corresponding to* $\sigma_i$*, with* $i \in \{1,\dots,n-1\}$*, takes* $x_i$ *to* $x_i^k x_{i+1} x_i^{-k}$*,* $x_{i+1}$ *to* $x_i$ *and fixes all other generators. By applying Magnus representation to the image of the braid group under this representation, we obtain the representation* $B_n \to GL_n(\mathbb{C})$*, where the automorphism* $\sigma_i$ *is mapped to the following* $n\times n$ *matrix*

$$\sigma_i = I_{i-1} \oplus \begin{pmatrix} 1-t^k & t^k \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}, \text{ for } i = 1,2,\dots,n-1.$$

This is a generalization of Burau representation by letting $z = t^k$. It is clear that this representation is reducible. To determine the composition factor, we present the following definition.

**Definition 5.** [1] *Let k be a non zero integer. Wada's representation of type 1, namely* $\phi_k^{(1)} : B_n \to GL_{n-1}(\mathbb{C})$*, is defined as* $\phi_k^{(1)}(\sigma_i) = I_{n-1} - A_{i,k}^{(1)} B_{i,k}^{(1)}$*, where*

$$A_{1,k}^{(1)} = (t^k+1, -1, \underbrace{0,\dots,0}_{n-3})^T,$$

$$A_{i,k}^{(1)} = (\underbrace{0,\dots,0}_{i-2}, -t^k, t^k+1, -1, \underbrace{0,\dots,0}_{n-i-2})^T, \text{ for } i = 2,\dots,n-2,$$

*and*

$$A^{(1)}_{n-1,k} = (\underbrace{0,\dots,0}_{n-3}, -t^k, t^k+1)^T.$$

*Here, $\{B^{(1)}_1, \dots, B^{(1)}_{n-1}\}$ is the standard basis of $\mathbb{C}^{n-1}$.*

These representations are irreducible by [3]. Notice that the representation $\phi^{(1)}_k : B_n \to GL_{n-1}(\mathbb{C})$ is (conjugate to) the reduced Burau representation, $\beta_n(t^k)$.

**Theorem 3.** *[1, p. 1323] The images of the generators under $\phi^{(1)}_k$ are unitary relative to a hermitian positive definite matrix.*

**Wada's Representation of Type 2:**

**Definition 6.** [2] *Wada's representation of type 2 asserts that the automorphism corresponding to $\sigma_i$ takes $x_i$ to $x_i x_{i+1}^{-1} x_i$, $x_{i+1}$ to $x_i$ and fixes all other generators. By applying Magnus representation to the image of the braid group under Wada's representation, we obtain the representation $B_n \to GL_n(\mathbb{C})$. The automorphism $\sigma_i$ is mapped to the following $n \times n$ matrix*

$$\sigma_i(z) = I_{i-1} \oplus \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}, \textit{for } i = 1, \dots, n-1.$$

It is clear that this representation is reduced to a subrepresentation of degree $n-1$.

**Definition 7.** [2] *Wada's representation of type 2, $\phi^{(2)}_n : B_n \to GL_{n-1}(\mathbb{C})$ is defined as $\phi^{(2)}_n(\sigma_i) = I_{n-1} - A^{(2)}_i B^{(2)}_i$, where*

$$A^{(2)}_1 = (0, -1, \underbrace{0,\dots,0}_{n-3})^T,$$

$$A^{(2)}_i = (\underbrace{0,\dots,0}_{i-2}, 1, 0, -1, \underbrace{0,\dots,0}_{n-i-2})^T, \textit{ for } i = 2, \dots, n-2,$$

*and*

$$A^{(2)}_{n-1} = (0, \dots, 0, \underbrace{1}_{n-2}, 0).$$

*Here, $\{B^{(2)}_1, \dots, B^{(2)}_{n-1}\}$ is the standard basis of $\mathbb{C}^{n-1}$.*

The associated matrix given by the inner product $(B_i^{(2)}A_j^{(2)})$ is

$$(B_i^{(2)}A_j^{(2)}) = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ -1 & 0 & 1 & 0 & \dots & \vdots \\ 0 & -1 & 0 & 1 & \dots & \vdots \\ 0 & 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \dots & -1 & 0 & 1 \\ 0 & 0 & \dots & 0 & -1 & 0 \end{pmatrix}.$$

**Lemma 3.** *[2, p. 561] Wada's representation, $\phi_n^{(2)} : B_n \to GL_{n-1}(\mathbb{C})$ is irreducible if and only if $n$ is an odd integer.*

**Theorem 4.** *[2, p. 562] Let $\phi_n^{(2)}$ be Wada's representation, $\phi_n^{(2)} : B_n \to\ GL_{n-1}(\mathbb{C})$ then one of the following is true:*

*(a) if $n$ is odd, then $\phi_n^{(2)}$ is equivalent to $\beta_n(-1) : B_n \to GL_{n-1}(\mathbb{C})$, where $\beta_n(z)$ is the complex specialization of the reduced Burau representation of $B_n$ and $z \in \mathbb{C}^*$;*

*(b) if $n$ is even, then the composition factors of $\phi_n^{(2)}$ are the irreducible representation $\hat{\beta}_n(-1)$ : $B_n \to GL_{n-2}(\mathbb{C})$ and the trivial one.*

**Theorem 5.** *[2, p. 564] The images of the generators of $B_n$ under Wada's representations of type 2 are unitary relative to a hermitian matrix.*

## 4. Wada's Representation of Type 3

In this section, we study Wada's representation of type 3 and determine its properties in order to make a comparison between the three types in section 5.

**Definition 8.** *The representation of type 3, discovered by M. Wada, asserts that the automorphism corresponding to $\sigma_i$ takes $x_i$ to $x_i^2 x_{i+1}$, $x_{i+1}$ to $x_{i+1}^{-1} x_i^{-1} x_{i+1}$ and fixes all other generators.*

Let $F_n$ be the free group of rank $n$ with free basis $x_1, \ldots, x_n$. It is easy to see that $F_n = < g_1, \ldots, g_n >$, where $g_1 = x_1$, $g_2 = g_1 x_2$, $\ldots$, $g_n = g_{n-1} x_n$. The action of the braid generator $\sigma_i$ on the basis $\{g_1, \ldots, g_n\}$ is given by

$$\sigma_1 : \begin{cases} g_1 \to g_1 g_2, & \\ g_j \to g_j, & \text{if } j \neq 1. \end{cases}$$

For $1 < i < n$, we have that

$$\sigma_i : \begin{cases} g_i \to g_i g_{i-1}^{-1} g_{i+1}, & \\ g_j \to g_j, & \text{if } j \neq i. \end{cases}$$

Let $\rho : \mathbb{Z}[F_n] \to \mathbb{Z}[t^{\pm 1}]$, where $\mathbb{Z}[t^{\pm 1}]$ is the ring of Laurent polynomials with independent indeterminate $t$. The map $\rho$ is defined as $\rho(g_i) = \begin{cases} 1, & \text{if } i \text{ is even,} \\ t, & \text{if } i \text{ is odd.} \end{cases}$

Using Magnus representation of subgroups of the automorphism group of the free group $F_n = \{g_1, \ldots, g_n\}$, we determine Wada's representation $\alpha$: $B_n \to GL_n(\mathbb{Z}[t^{\pm 1}])$. The images of the generators under Wada's representation of type 3 are given by

$$\alpha(\sigma_1) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \oplus I_{n-2},$$

$$\alpha(\sigma_k) = I_{k-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ -t & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-k-1}, \text{ if } k \text{ is odd } (1 < k < n),$$

and

$$\alpha(\sigma_k) = I_{k-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ -t^{-1} & 1 & t^{-1} \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-k-1}, \text{ if } k \text{ is even } (1 < k < n).$$

**Lemma 4.** *Wada's representation of type 3, namely $\alpha$: $B_n \rightarrow GL_n(\mathbb{C})$, is a reducible representation.*

*Proof.* Let $u$ be the column vector in $\mathbb{C}^n$ defined as

$$u = \begin{cases} (1,0,1,0,\ldots,1,0)^T & \text{if } n \text{ is even,} \\ (1,0,1,0,\ldots,0,1)^T & \text{if } n \text{ is odd.} \end{cases}$$

It is easy to see that the subspace generated by $u$ is invariant under the representation $\alpha$ because $\alpha(\sigma_k)(u) = u$ for every $k \in \{1,2,\ldots,n-1\}$. □

Having that Wada's representation of type 3 is reducible, we then reduce this representation to a subrepresentation of a lower degree. More precisely, we have the following definition.

**Definition 9.** *Wada's representation of type* 3*, $\phi_n^{(3)} : B_n \rightarrow GL_{n-1}(\mathbb{C})$ is a family of linear representations defined as $\phi_n^{(3)}(\sigma_k) = I_{n-1} - A_k^{(3)} B_k^{(3)}$, where*

$$B_1^{(3)} = (0, -t, \underbrace{0, \ldots, 0}_{n-3}),$$

$$B_k^{(3)} = \begin{cases} (0,\ldots,0,t,\underbrace{0}_{k},-t,0,\ldots,0), & \textit{if } k \textit{ is odd } (1 < k < n-1), \\ (0,\ldots,0,t^{-1},\underbrace{0}_{k},-t^{-1},0,\ldots,0), & \textit{if } k \textit{ is even } (1 < k < n-1). \end{cases}$$

*and*

$$B_{n-1}^{(3)} = \begin{cases} (\underbrace{0,\ldots,0}_{n-3},t^{-1},0), & \textit{if } n \textit{ is odd,} \\ (\underbrace{0,\ldots,0}_{n-3},t,0), & \textit{if } n \textit{ is even.} \end{cases}$$

*Here,* $\{A_1^{(3)},\ldots,A_{n-1}^{(3)}\}$ *are the standard basis of* $\mathbb{C}^{n-1}$.

Direct computations show that the associated matrix given by the inner product $(B_i^{(3)}A_j^{(3)})$ is

$$(B_i^{(3)}A_j^{(3)}) = \begin{cases} \begin{pmatrix} 0 & -t & 0 & 0 & \ldots & 0 & 0 & 0 \\ t^{-1} & 0 & -t^{-1} & 0 & \ldots & 0 & 0 & 0 \\ 0 & t & 0 & -t & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & t & 0 & -t \\ 0 & 0 & 0 & 0 & \ldots & 0 & t^{-1} & 0 \end{pmatrix}, & \text{if } n \text{ is odd,} \\ \\ \begin{pmatrix} 0 & -t & 0 & 0 & \ldots & 0 & 0 & 0 \\ t^{-1} & 0 & -t^{-1} & 0 & \ldots & 0 & 0 & 0 \\ 0 & t & 0 & -t & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & t^{-1} & 0 & -t^{-1} \\ 0 & 0 & 0 & 0 & \ldots & 0 & t & 0 \end{pmatrix}, & \text{if } n \text{ is even.} \end{cases}$$

Next, we prove our next lemma.

**Lemma 5.** *Wada's representation of type* 3, $\phi_n^{(3)} : B_n \to GL_{n-1}(\mathbb{C})$ *is irreducible if and only if n is an odd integer.*

*Proof.* If $n$ is an odd integer, direct computations show that the determinant of the $(n-1)\times(n-1)$ matrix $(B_i^{(3)}A_j^{(3)})$ equals one. By Theorem 2 (a), we get that $\phi_n^{(3)}$ is irreducible.

On the other hand if $n$ is an even integer then the determinant of $(B_i^{(3)}A_j^{(3)})$ equals zero, and so the representation is reducible.

□

**Lemma 6.** *If $n$ is even then $\phi_n^{(3)} : B_n \to GL_{n-1}(\mathbb{C})$ has an invariant subspace of dimension one.*

*Proof.* It is easy to see that the subspace generated by the vector $(1,0,1,0,\ldots,0,1)^T$ is invariant under $\phi_n^{(3)}$. Here, $T$ is the transpose. □

**Lemma 7.** *If $n$ is odd then $\phi_n^{(3)} : B_n \to GL_{n-1}(\mathbb{C})$ is of Burau type. In particular, $\phi_n^{(3)}$ is equivalent to $\beta_n(-1) : B_n \to GL_{n-1}(\mathbb{C})$, where $\beta_n(z)$ is the complex specialization of the reduced Burau representation and $z \in \mathbb{C}^*$.*

*Proof.* If $n$ is odd then by Lemma 5, we have that $\phi_n^{(3)}$ is irreducible. It follows that by Theorem 1, $\phi_n^{(3)}$ is equivalent to $\beta_n(z)$ for some non-zero complex number $z$ such that $f_n(z) \neq 0$. To find such a $z$, direct computations show that, by letting $z=-1$, the matrices $(B_i^{(3)}A_j^{(3)})$ and $(D_iC_j)$ are conjugates by an $(n-1)\times(n-1)$ diagonal matrix, where the diagonal entries are $\{t,1,\ldots,t,1\}$ and $\beta_n(z)(\sigma_i) = I_{n-1} - C_iD_i$ (See Definition 2). □

**Lemma 8.** *If $n$ is even then the composition factors of $\phi_n^{(3)} : B_n \to GL_{n-1}(\mathbb{C})$ are the irreducible representation $\hat{\beta}_n(-1) : B_n \to GL_{n-2}(\mathbb{C})$ and the trivial one.*

*Proof.* The composition factor of $\phi_n^{(3)} : B_n \to GL_{n-1}(\mathbb{C})$ is $\hat{\phi}_n^{(3)} : B_n \to GL_{n-2}(\mathbb{C})$, which is defined by:

$\hat{\phi}_n^{(3)}(\sigma_i) = \phi_{n-1}^{(3)}(\sigma_i)$ for $i = 1,\ldots,n-2$ and $\hat{\phi}_n^{(3)}(\sigma_{n-1}) = I_{n-2} - XY$, where

$$X = (-1,0,-1,0\ldots,-1,0)^T, Y = (0,\ldots,0,t).$$

Since $\hat{\phi}_n^{(3)} : B_n \to GL_{n-2}(\mathbb{C})$ is the extension of the irreducible representation $\phi_{n-1}^{(3)} : B_{n-1} \to GL_{n-2}(\mathbb{C})$ to $B_n$, it follows that $\hat{\phi}_n^{(3)}$ is irreducible. By Theorem 1, the non-trivial composition factor of $\phi_n$, namely $\hat{\phi}_n^{(3)} : B_n \to GL_{n-2}(\mathbb{C})$, is equivalent to $\hat{\beta}_n(z) : B_n \to GL_{n-2}(\mathbb{C})$ for some

$z \in \mathbb{C}$ which is a root of $f_n(t) = t^{n-1} + \cdots + t + 1$. Along the same computations as in Lemma 7, one can show, by letting $z = -1$, that the irreducible representation $\hat{\phi}_n^{(3)}$ and the representation $\hat{\beta}_n(-1) : B_n \to GL_{n-2}(\mathbb{C})$ are equivalent. □

## 5. Comparison between Wada's Representations

In this section, we make a comparison between Wada's representations of the three different types and present our main theorem, Theorem 6.

**Theorem 6.** *Wada's representations of types 2 and 3 are equivalent.*

*Proof.* This is clear by Theorem 4, Lemma 7 and Lemma 8. □

**Lemma 9.** *If $t^k = -1$ then Wada's representations of types 1, 2 and 3 are equivalent.*

*Proof.* We have that Wada's representation of type 1 is the complex specialization of Burau representation with $z = t^k$. Using the fact that Wada's representations of types 2 and 3 of degree $n-1$ ($n-2$) are equivalent to $\beta_n(-1)$ ($\hat{\beta}_n(-1)$), we get that Wada's representations of types 1, 2 and 3 are equivalent when $t^k = -1$. □

**Lemma 10.** *let $\alpha$ and $\beta$ be two equivalent representations of a group $G$ of degree $n$. Then $\alpha$ is unitary relative to a hermitian invertible matrix $M$ if and only if $\beta$ is unitary relative to a hermitian invertible matrix $N$.*

*Proof.* Let $\alpha$ be a unitary representation relative to a hermitian matrix $M$. Then $\alpha(g)M\alpha(g)^* = M$ for all $g \in G$. Since $\alpha$ and $\beta$ are equivalent representations, it follows that there exists an $n \times n$ invertible matrix $T$ such that $T^{-1}\alpha(g)T = \beta(g)$.

We now verify that $\beta$ is unitary relative to $T^{-1}M(T^{-1})^*$.

$$\begin{aligned}
\beta(g)T^{-1}M(T^{-1})^*\beta(g)^* &= T^{-1}T\beta(g)T^{-1}M(T^{-1})^*\beta(g)^*T^*(T^{-1})^* \\
&= T^{-1}\alpha(g)M(T\beta(g)T^{-1})^*(T^{-1})^* \\
&= T^{-1}(\alpha(g)M\alpha(g)^*)(T^{-1})^* \\
&= T^{-1}M(T^{-1})^*.
\end{aligned}$$

Since $(T^{-1}M(T^{-1})^*)^* = T^{-1}M(T^{-1})^*$, it follows that $T^{-1}M(T^{-1})^*$ is hermitian. Using Theorem 5, Theorem 6 and Lemma 10, we get the following Lemma. $\square$

**Lemma 11.** *Wada's representations of types 2 and 3 are unitary.*

## 6. Wada's Representations And Hecke Algebra

In this section, we prove our main result, which states that Wada's representations of types 1, 2 and 3 arise from Hecke algebras.

**Definition 10.** *The Hecke algebra $H_n(q)$ is the complex algebra defined by the presentation*

$$< s_1, \ldots, s_n \mid s_i s_j = s_j s_i, \mid i-j \mid > 1, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, (s_i)^2 = (1-q)s_i + q > .$$

*Here, q is any nonzero complex number.*

Under direct computations, we easily verify the next lemma.

**Lemma 12.** *The minimal polynomial of Wada's representations of type* 3 *is*
$(x-1)^2$ *for all* $n > 2$.

**Lemma 13.** *Let $\alpha$ and $\beta$ be two equivalent representations of a group $G$ of degree $n$ and $q$ be a nonzero complex number. Then $\alpha$ arises from a Hecke algebra $H_n(q)$ if and only if $\beta$ arises from $H_n(q)$.*

*Proof.* Suppose that $\alpha$ arises from a Hecke algebra $H_n(q)$. Then, for every $g \in G$, we have

$$(\alpha(g))^2 = (1-q)\alpha(g) + qI_n.$$

Since $\alpha$ and $\beta$ are equivalent representations, it follows that there exists an invertible matrix $T$ in $M_n(\mathbb{C})$ such that $T^{-1}\alpha(g)T = \beta(g)$.
Multiplying (1) by $T^{-1}$ from the left and by $T$ from the right, we get that

$$T^{-1}(\alpha(g))^2 T = T^{-1}((1-q)\alpha(g) + qI_n)T.$$

This implies that $(\beta(g))^2 = (1-q)\beta(g) + qI_n$. Therefore, $\beta$ arises from $H_n(q)$. $\square$

Having shown that Wada's representations of types 1, 2 and 3 are of Burau type, we easily get our main theorem, Theorem 7.

**Theorem 7.** *Wada's representations of types 1, 2 and 3 arise from hecke algebras.*

*Proof.* It is easy to see that *type 1* arises from $H_n(t^k)$, whereas *type 2* and *type 3* arise from $H_n(-1)$. $\square$

## Conflict of Interests

The authors declare that there is no conflict of interests.

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