Available online at http://scik.org
J. Math. Comput. Sci. 3 (2013), No. 6, 1601-1616
ISSN: 1927-5307

STABILITY OF SVIS MODEL WHERE THE VACCINE IS UTTERLY INFECTIVE

KAWA AHMED HASAN

Department of Mathematics, University of Sulaimani, School of Science,

Faculty of Science and Science Educations, Sulaimani, Iraq

Abstract: Pulse vaccination is an important strategy for eliminating infectious diseases. A mathematical SIS epidemic model with pulse vaccination is formulated in this paper. The effect of immigrants on the dynamical behavior of SVIS model is considered analytically as well as numerically. The stability analysis (locally as well as globally) of all possible equilibrium points is also established.

Keywords: Continuous dynamical systems, SIS model, Vaccination, Equilibrium and stability analysis Stability, Numerical simulation.

2000 AMS Subject Classification: 47H17; 47H05; 47H09

1. Introduction :

Studies of epidemic models that incorporate diseases causing death and varying total population have become one of the important areas in the mathematical theory of epidemiology. Largely inspired by the works of Anderson and May [1], vaccination is an effective way to control the transmission of a disease. The study of vaccination, treatment, and associated behavioral changes related to disease transmission has been the subject of intense theoretical analysis. A population with a constant flow of infective immigrants within the simple SIS framework proposed by Kermack and Mckendrick in [10] is considered in a

---

*Corresponding author

Received October 13, 2013

model that incorporates the impact of a vaccination program. Gumel and Moghads [5] proposed a model for the dynamics of an infectious disease in the presence of a preventive vaccine considering non-linear incidence rate ,Li and Ma[11] studied the SIS model with vaccination and temporary immunity. Moghadas S.M [13] proposed SVIS model and he gave analysis of an epidemic model with stability of equilibrium points by using the Poincare index. Hethcote and Tudor [10] discussed endemic infectious disease model for which infection confers permanent immunity with no disease–related mortality but with vaccination.

In this paper, we consider an SIS model with vaccination, the vaccination is given to the newly born and the susceptible. The sufficient conditions for the existence and locally stability of the equilibrium points of this model are obtained. Also the global stability of each possible equilibrium points are proved by using Lyapunov theorem. Moreover, The local and global dynamics of our model are studied numerically.

2. Model formulation :

Consider an SIS disease when a vaccination program is in effect and there is a constant flow of incoming immigrants. A population of size $N(t)$ is partitioned into three classes of individuals; susceptible, infections and vaccinated, with sizes denoted by $S(t), I(t)$ and $V(t)$, respectively. A constant flow of A_2 new members arrives into the population in a unit time with the fraction p of A_2 arriving infected $(0 \leq p \leq 1)$. The susceptible population is vaccinated at a constant rate A_7, and the rate at which the vaccine wears off is A_6. The population is replenished in two ways, birth and immigration, assume that newborns enter the susceptible class at the constant rate is A_1. The per capita natural death rate is $A_4 > 0$ in each class. A constant $A_5 > 0$ of infective recovers in unite time. Accordingly the dynamics of SVIS epidemic model with constant number of immigrant the model can be represented by the following non linear system

$$\frac{dS}{dt} = A_1 + (1-p)A_2 - \frac{A_3 S I^2}{1+I^2} - A_4 S + A_5 I + A_6 V - A_7 S$$

$$\frac{dV}{dt} = A_7 S - \frac{A_8 V I^2}{1+I^2} - A_4 V - A_6 V \tag{1}$$

$$\frac{dI}{dt} = pA_2 + \frac{A_3 S I^2}{1+I^2} + \frac{A_8 V I^2}{1+I^2} - A_4 I - A_5 I$$

Where A_3, A_8 are infection constant rate coefficients for the susceptible individuals and vaccinated individuals respectively , the non linear term $\frac{I^2}{1+I^2}$ represents the incidence rate which is known as Holling-type III functional response .The initial condition for system(1) may be taken as any point in the region

Theorem 3.1: All solutions of system (1) with non-negative initial condition are uniformly bounded

Proof. consider the following function $W(t) = S(t) + V(t) + I(t)$,time derivative of $W(t)$ along the trajectory of system(1) gives the following differential equation

$$\bar{W}(t) + A_4 W(t) = A_1 + A_2 \tag{2}$$

Which has an integrating factor $e^{A_4 t}$ and hence a solution is $W(t) = \frac{A_1 + A_2}{A_4} + ce^{-A_4 t}$ where $c = W(0) - \frac{A_1 + A_2}{A_4}$,that means $W(t) = \frac{A_1 + A_2}{A_4}(1 - e^{-A_4 t}) + W(0)e^{-A_4 t}$ hence all solutions of system(1) that initiate in the region R_+^3 are eventually confined in the region

$$M = \{(S, V, I) : W = S + V + I = \frac{A_1 + A_2}{A_4}\ \}$$

3. Existence of Equilibrium points

An investigation of system (1) shows that there are at most two possible non-negative equilibrium points, the existence conditions of them are gives as the following

1) The disease free equilibrium point $\bar{E} = (\bar{S}, \bar{V}, 0)$ always exists

where $\bar{S} = \frac{(A_1 + A_2)(A_4 + A_6)}{A_4(A_4 + A_6 + A_7)}$ and $\bar{V} = \frac{A_7(A_1 + A_2)}{A_4(A_4 + A_6 + A_7)}$

2) The endemic equilibrium point $\tilde{E} = (\tilde{S}, \tilde{V}, \tilde{I})$ exists in the region R_+^3 if and only if there is a positive solution to the following non-linear equations

$$A_1 + (1-p)A_2 - \frac{A_3 S I^2}{1+I^2} - A_4 S + A_5 I + A_6 V - A_7 S = 0 \quad (3.a)$$

$$A_7 S - \frac{A_8 V I^2}{1+I^2} - A_4 V - A_6 V = 0 \quad (3.b)$$

$$pA_2 + \frac{A_3 S I^2}{1+I^2} + \frac{A_8 V I^2}{1+I^2} - A_4 I - A_5 I = 0 \quad (3.c)$$

By adding (3.a) , (3.b) and (3.c) we get $A_1 + A_2 - A_4 S - A_4 V - A_4 I = 0$ that is

$$\tilde{S} = \frac{A_1 + A_2 - A_4(\tilde{V} + \tilde{I})}{A_4} \quad (4)$$

Clearly $\tilde{S} > 0$ if $0 < \tilde{V} + \tilde{I} < \frac{A_1 + A_2}{A_4}$,from Eq.(3.b) we obtain that

$$\tilde{V} = \frac{A_7 \tilde{S}(1+\tilde{I}^2)}{\tilde{A} + \tilde{B}\tilde{I}^2} \quad (5)$$

Where $\tilde{A} = A_4 + A_6$ and $\tilde{B} = A_8 + \tilde{A}$ .Substituting the value of $\tilde{V}$ in Eq.(3.c) we get

$$\tilde{S} = \frac{(1+\tilde{I}^2)[c\tilde{I} - pA_2](\tilde{A} + \tilde{B}\tilde{I}^2)}{\tilde{I}^2[A_7 A_8 (1+\tilde{I}^2) + A_3(\tilde{A} + \tilde{B}\tilde{I}^2)]} \quad (6)$$

Clearly from Eq.(5) we note $\tilde{S} > 0$ if $c\tilde{I} > pA_2$

Now by substitution Eq(5) and Eq.(6) in Eq.(3.c) and then simplifying the resulting term gives the following polynomial equation

$$a_5 \tilde{I}^5 + a_4 \tilde{I}^4 + a_3 \tilde{I}^3 + a_2 \tilde{I}^2 + a_1 \tilde{I} + a_0 = 0 \quad (7)$$

Where

$$a_0 = pA_2 A_4 (A_7 + \tilde{A}) > 0 \quad (8.a)$$

$$a_1 = -cA_4 A_7 + A_4 (pA_2 \tilde{B} - \tilde{A} c) \quad (8.b)$$

$$a_2 = (A_1 + A_2)(A_7 A_8 + A_3 \tilde{A}) + 2pA_2 A_4 A_7 + A_4 (pA_2 \tilde{A} - \tilde{B}) \quad (8.c)$$

$$a_3 = -2cA_4A_7 + A_4(pA_2\tilde{B} - \tilde{A}c) - A_4(A_7A_8 + A_3\tilde{A}) \tag{8.d}$$

$$a_4 = (A_7A_8 + A_3\,\tilde{\mathrm{B}})(\mathrm{A}_1 + A_2) + pA_2A_4A_7 - A_4c\tilde{B} \tag{8.e}$$

$$a_5 = -cA_4A_7 - A_4(A_7A_8 + A_3\,\tilde{\mathrm{B}}) < 0 \tag{8.f}$$

Straightforward computation shows that Eq.(7) has a positive root namely $\tilde{I}$ provided that one set of the following sets of conditions holds

$$a_1 > 0\,,\ a_2 > 0 \ \text{ with } a_3 > 0 \tag{9.a}$$

$$a_1 < 0\,,\ a_2 < 0\,,\ a_3 < 0 \ \text{ with } a_4 < 0 \tag{9.b}$$

$$a_1 > 0\,,\ a_3 < 0 \ \text{ with } a_5 < 0 \tag{9.c}$$

Substitution the value of $\tilde{I}$ in Eq.(6) gives the value of $\tilde{S}$ and then Substituting the value of $\tilde{S}$ and $\tilde{I}$ in Eq.(5) gives the value of $\tilde{V}$.

4. Stability analysis

In the following, the local stability analysis for the above equilibrium points is studied. The general Variational matrix of the system (1) at $(\mathrm{S},\mathrm{V},\mathrm{I})$ is given by

$$\mathrm{J}(\mathrm{S},\mathrm{V},\mathrm{I}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where $a_{11} = -\dfrac{A_3\,\mathrm{I}^2}{1+I^2} - A_4 - A_7 < 0\,, a_{12} = A_6 > 0\,,$ $a_{13} = A_5 - \dfrac{2A_3SI}{(1+I^2)^2}\,,$ $a_{21} = A_7 > 0\,,$

$a_{22} = -A_4 - \mathrm{A}_6 - \dfrac{A_8I^2}{(1+I^2)^2} < 0\,,$ $a_{23} = \dfrac{-2A_8VI}{(1+I^2)^2} < 0\,,$ $a_{31} = \dfrac{A_3I^2}{1+I^2} > 0\,,$ $a_{32} = \dfrac{A_8I^2}{1+I^2} > 0\,,$

$$a_{33} = \frac{2A_3\,\mathrm{SI}}{(1+I^2)^2} + \frac{2A_8\,\mathrm{VI}}{(1+I^2)^2} - A_4 - A_5$$

Therefore, the Variational matrix about the equilibrium points $\bar{E}$ given below:

$$\mathrm{J}(\bar{E}) = \begin{pmatrix} -A_4 - A_7 & A_6 & A_5 \\ A_7 & -A_4 - A_6 & 0 \\ 0 & 0 & -A_4 - A_5 \end{pmatrix}$$

Therefore the eigenvalues of $\mathrm{J}(\bar{E})$ satisfy the following relations:

$$\lambda_1 + \lambda_2 = -(2A_4 + A_7 + A_6) \tag{10a}$$

$$\lambda_1.\lambda_2 = A_4(A_4 + A_6 + A_7) \tag{10b}$$

$$\lambda_3 = -(A_4 + A_5) \tag{10c}$$

Theorem 4.1: The disease free equilibrium point $\bar{E}$ of system (1) is always locally asymptotically stable in the $Int.R_+^3$.

Proof. From equations (10a),(10b) and (10c) we have $\lambda_1 + \lambda_2 < 0$, $\lambda_1.\lambda_2 > 0$ and $\lambda_3 < 0$ hence all the eigenvalues have negative real parts and by Routh-Hurwitz criterion the disease free equilibrium point $\bar{E}$ is locally asymptotically stable in the $Int.R_+^3$.

Theorem 4.2: Assume that the endemic equilibrium point $\tilde{E} = (\tilde{S}, \tilde{V}, \tilde{I})$ of system(1) exists then it is locally asymptotically stable if the following conditions are satisfied:

$$\frac{2A_3\,\tilde{S}\tilde{I}}{(1+\tilde{I}^2)^2} + \frac{2A_8\,\tilde{V}\tilde{I}}{(1+\tilde{I}^2)^2} - A_4 < A_5 < \frac{2A_3\,\tilde{S}\tilde{I}}{(1+\tilde{I}^2)^2} \tag{11a}$$

$$R_1 < (A_8{}^2 - R_1)\tilde{I}^2 \tag{11b}$$

$$R_2 < (A_3{}^2 - R_2)\tilde{I}^2 \tag{11c}$$

where $R_1 = A_3A_6 - A_4A_8 - A_6A_8$, $R_2 = A_7A_8 - A_3A_4 - A_3A_7$

proof. The characteristic equation of the $\mathrm{J}(\tilde{E})$ can be written as:

$$\lambda^3 + d_1\lambda^2 + d_2\lambda + d_3 = 0$$

Where : $d_1 = -(a_{11} + a_{22} + a_{33})$, $d_2 = (a_{11}a_{22} - a_{12}a_{21}) + a_{33}(a_{11} + a_{22}) - (a_{13}a_{31} + a_{23}a_{32})$

$d_3 = a_{33}(a_{12}a_{21} - a_{11}a_{22}) + a_{31}(a_{13}a_{22} - a_{12}a_{23}) + a_{32}(a_{11}a_{23} - a_{13}a_{21})$.

Eq. (11a) gives that $a_{33} < 0$ and hence $d_1 > 0$,for d_3 we have

$$a_{12}a_{21} - a_{11}a_{22} = -[(A_4 + \frac{A_8\,\tilde{I}^2}{1+\tilde{I}^2})(A_4 + \frac{A_3\,\tilde{I}^2}{1+\tilde{I}^2}) + A_6(A_4 + \frac{A_3\,\tilde{I}^2}{1+\tilde{I}^2}) + A_7(A_4 + \frac{A_8\,\tilde{I}^2}{1+\tilde{I}^2})] < 0$$

and Eq.(11a) gives that $a_{13}<0$ so $d_3>0$,on the other hand we have $\Delta=d_1d_2-d_3$

$$=(\mathrm{a}_{11}+a_{22})[(a_{12}a_{21}-\mathrm{a}_{11}\mathrm{a}_{22})+d_1a_{33}]+(a_{13}a_{31}+a_{23}a_{32})a_{33}$$
$$+a_{13}(a_{11}a_{31}+a_{21}a_{32})+a_{23}(a_{22}a_{32}+a_{12}a_{31})$$

And equations (11b),(11c) give that $a_{11}a_{31}+a_{21}a_{32}<0$, $a_{22}a_{32}+a_{12}a_{31}<0$ and hence $\Delta>0$

Therefore, all the requirements of Routh-Hurwitz criterion[12]are satisfied. Hence $\tilde{E}$ is locally asymptotically stable.

Theorem 4.3: The disease free equilibrium point $\bar{E}$ of system (1) is globally asymptotically stable in the sub region

$$\Omega=\{(S,V,I):0<I<V_1 \ \ or \ \ I>V_2, 0<I<S_1 or\, I>S_2, \frac{A_6}{S}+\frac{A_7}{V}\le 2\sqrt{\frac{(A_4+A_7)(A_4+A_6)}{SV}}\}$$

Where $V_1=\dfrac{A_8\bar{V}-\sqrt{A_8^2\bar{V}^2-4A_4^2}}{2A_4}$, $V_2=\dfrac{A_8\bar{V}+\sqrt{A_8^2\bar{V}^2-4A_4^2}}{2A_4}$,

$$S_1=\frac{A_3\bar{S}-\sqrt{A_3^2\bar{S}^2-4A_5^2}}{2A_5}, S_2=\frac{A_3\bar{S}+\sqrt{A_3^2\bar{S}^2-4A_5^2}}{2A_5}$$

Proof. Consider the function $L(\mathrm{S,V,I})=\int_{\bar{S}}^{S}\frac{u_1-\bar{S}}{u_1}du_1+\int_{\bar{V}}^{V}\frac{u_2-\bar{V}}{u_2}du_2+I$

By differentiating L with respect to t along the solution of the system (1), we get

$$\frac{dL}{dt}=\frac{S-\bar{S}}{S}.\frac{dS}{dt}+\frac{V-\bar{V}}{V}.\frac{dV}{dt}+\frac{dI}{dt}$$

$$=\frac{S-\bar{S}}{S}\left[A_4\bar{S}-A_6\bar{V}+A_7\bar{S}-\frac{A_3SI^2}{1+I^2}-A_4S+A_6V-A_7S+A_5I\right]$$

$$+\frac{V-\bar{V}}{V}\left[A_7S+A_4\bar{V}+A_6\bar{V}-\frac{A_8VI^2}{1+I^2}-A_4V-A_6V-A_7\bar{S}\right]$$

$$+\frac{A_8VI^2}{1+I^2}+\frac{A_3SI^2}{1+I^2}-A_5I-A_4I \tag{12}$$

$$=-\frac{(A_4+A_7)}{S}(S-\bar{S})^2-\frac{(A_4+A_6)}{V}(\mathrm{V}-\bar{V})^2+(\frac{A_6}{S}+\frac{A_7}{V})(S-\bar{S})(\mathrm{V}-\bar{V})$$

$$+(\frac{A_3I}{1+I^2}-\frac{A_5}{S})\mathrm{I\bar{S}}+(\frac{A_8\bar{V}I}{1+I^2}-A_4)\mathrm{I} \tag{13}$$

Now for any (S, V, I) in Ω and by Eq.(13) we get

$$\frac{dL}{dt} < -\frac{(A_4+A_7)}{S}(S-\bar{S})^2 - \frac{(A_4+A_6)}{V}(\mathrm{V}-\bar{V})^2 + (\frac{A_6}{S}+\frac{A_7}{V})(S-\bar{S})(\mathrm{V}-\bar{V})$$

$$< -\frac{(A_4+A_7)}{S}(S-\bar{S})^2 - \frac{(A_4+A_6)}{V}(\mathrm{V}-\bar{V})^2 + 2\sqrt{\frac{(A_4+A_7)(A_4+A_6)}{SV}}(S-\bar{S})(\mathrm{V}-\bar{V})$$

$$< -[\sqrt{\frac{(A_4+A_7)}{S}}(S-\bar{S}) - \sqrt{\frac{(A_4+A_6)}{V}}(\mathrm{V}-\bar{V})]^2 < 0$$

$\frac{dL}{dt}$ is negative definite and hence L is a Lyapunov function with respect to $\bar{E}$ hence, $\bar{E}$ is globally asymptotically stable in the sub region Ω.

Theorem 4.4: Assume that the endemic equilibrium point $\tilde{E} = (\tilde{S}, \tilde{V}, \tilde{\mathrm{I}})$ of system(1) is locally asymptotically stable then it is globally asymptotically stable in the sub region Υ that satisfies the following conditions

$$\frac{(A_3\tilde{\mathrm{S}}+A_8\tilde{\mathrm{V}})\tilde{\mathrm{K}}}{\tilde{H}} < (A_4+A_5)(1+\mathrm{I}^2) \tag{14a}$$

$$(A_5+(A_3+A_5)\mathrm{I}^2 - \frac{A_3\tilde{\mathrm{S}}\tilde{\mathrm{K}}}{\tilde{H}})^2 < ((A_4+A_5)(1+\mathrm{I}^2) - \frac{(A_3\tilde{\mathrm{S}}+A_8\tilde{\mathrm{V}})\tilde{\mathrm{K}}}{\tilde{H}})((\mathrm{A}_3+A_4+A_7)\mathrm{I}^2+A_4+A_7) \tag{14b}$$

$$((A_6+A_7)(1+\mathrm{I}^2))^2 < ((\mathrm{A}_3+A_4+A_7)\mathrm{I}^2+A_4+A_7)((\mathrm{A}_4+A_6+A_8)\mathrm{I}^2+A_4+A_6) \tag{14c}$$

$$(A_8\mathrm{I}^2 - \frac{A_8\tilde{\mathrm{V}}\tilde{\mathrm{K}}}{\tilde{H}})^2 < ((\mathrm{A}_4+A_6+A_8)\mathrm{I}^2+A_4+A_6)((A_4+A_5)(1+\mathrm{I}^2) - \frac{(A_3\tilde{\mathrm{S}}+A_8\tilde{\mathrm{V}})\tilde{\mathrm{K}}}{\tilde{H}}) \tag{14d}$$

Where $\tilde{H} = 1+\tilde{I}^2$, $\tilde{K} = I+\tilde{I}$

Proof. Consider the function $\mathrm{P(S,V,I)} = \int_{\tilde{S}}^{S}\frac{u_1-\tilde{S}}{u_1}du_1 + \int_{\tilde{V}}^{V}\frac{u_2-\tilde{V}}{u_2}du_2 + \int_{\tilde{I}}^{I}\frac{u_3-\tilde{I}}{u_3}du_3$

By differentiating P with respect to t along the solution of the system (1), we get

$$\frac{dP}{dt} = (S-\tilde{S})\left[\frac{A_3\tilde{S}\tilde{I}^2}{1+\tilde{I}^2} - \frac{A_3SI^2}{1+I^2} + (A_4+A_7)(\tilde{S}-S) + A_5(I-\tilde{I}) + A_6(V-\tilde{V})\right]$$

$$+(V-\tilde{V})\left[\frac{A_8\tilde{V}\tilde{I}^2}{1+\tilde{I}^2}-\frac{A_8VI^2}{1+I^2}+A_7(S-\tilde{S})+(A_4+A_6)(\tilde{V}-V)\right]$$

$$+(I-\tilde{I})\left[\frac{A_8VI^2}{1+I^2}-\frac{A_8\tilde{V}\tilde{I}^2}{1+\tilde{I}^2}+\frac{A_3SI^2}{1+I^2}-\frac{A_3\tilde{S}\tilde{I}^2}{1+\tilde{I}^2}+(A_4+A_5)(\tilde{I}-I)\right] \quad (15)$$

$$\frac{dP}{dt}=-\left[\frac{A_3I^2}{1+I^2}+A_4+A_7\right](S-\tilde{S})^2-\left[\frac{A_8I^2}{1+I^2}+A_4+A_6\right](\mathrm{V}-\tilde{V})^2$$

$$-\left[A_4+A_5-\frac{A_3\tilde{S}\tilde{K}}{\tilde{\mathrm{H}}(1+I^2)}-\frac{A_8\tilde{V}\tilde{K}}{\tilde{\mathrm{H}}(1+I^2)}\right](\mathrm{I}-\tilde{\mathrm{I}})^2+\left[A_5-\frac{A_3\tilde{S}\tilde{K}}{\tilde{\mathrm{H}}(1+I^2)}+\frac{A_3I^2}{1+I^2}\right](S-\tilde{S})(\mathrm{I}-\tilde{\mathrm{I}})$$

$$+\left[A_6+A_7\right](S-\tilde{S})(V-\tilde{V})+\left[\frac{A_8I^2}{1+I^2}-\frac{A_8\tilde{V}\tilde{K}}{\tilde{\mathrm{H}}(1+I^2)}\right](V-\tilde{V})(I-\tilde{I}) \quad (16)$$

Now from Eq.(16) and Eq.(14)(a-d) we have

$$\frac{dP}{dt}<-\frac{1}{1+I^2}\left[\sqrt{\frac{(A_3+A_4+A_7)I^2+A_4+A_7}{2}}(S-\tilde{S})-\sqrt{\frac{(A_4+A_6+A_8)I^2+A_4+A_6}{2}}(\mathrm{V}-\tilde{\mathrm{V}})\right]^2$$

$$-\frac{1}{1+I^2}\left[\sqrt{\frac{(A_3+A_4+A_7)I^2+A_4+A_7}{2}}(S-\tilde{S})-\sqrt{\frac{(A_4+A_5)(1+\mathrm{I}^2)-\frac{(A_3\tilde{\mathrm{S}}+A_8\tilde{\mathrm{V}})\tilde{\mathrm{K}}}{\tilde{H}}}{2}}(\mathrm{I}-\tilde{\mathrm{I}})\right]^2$$

$$-\frac{1}{1+I^2}\left[\sqrt{\frac{(A_4+A_6+A_8)I^2+A_4+A_6}{2}}(\mathrm{V}-\tilde{V})-\sqrt{\frac{(A_4+A_5)(1+\mathrm{I}^2)-\frac{(A_3\tilde{\mathrm{S}}+A_8\tilde{\mathrm{V}})\tilde{\mathrm{K}}}{\tilde{H}}}{2}}(\mathrm{I}-\tilde{\mathrm{I}})\right]^2$$

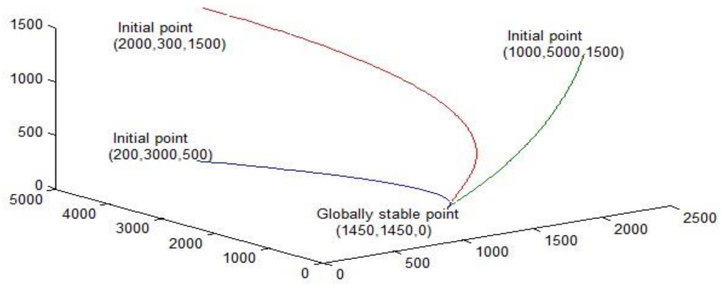
<0

So $\frac{dP}{dt}$ is negative definite and P is a Lyapunov function with respect to $\tilde{E}$ hence, $\tilde{E}$ is globally asymptotically stable in the sub region Υ.

4.Numerical analysis

In this section the global dynamics of system (1) is studied numerically. The objectives of this study are confirming our analytical results and understand the effects of immigration and the existence of vaccine on the dynamics of *SVIS* epidemic system. Consequently, system (1) is solved numerically, for different sets of parameters and different sets of initial conditions. It is observed that, for the following set of parameters, system (1) is solved for different sets of initial values and then the trajectories of system (1) as a function of time are drawn in Fig (1).

$$A_1 = 250, A_2 = 60, A_3 = 0.0002, A_4 = 0.15, A_5 = 0.18,$$
$$A_6 = 0.12, A_7 = 0.2, A_8 = 0.00003, p = 0 \tag{17}$$

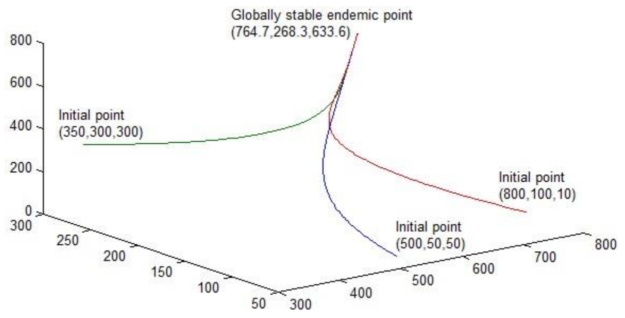


Fig(1):Phase plot of system (1) starting from different initial points for data given in Eq.(17) ;blue color for S, green color for V, and red color for I .

Clearly Fig.(1) shows the convergent of system (1) to the globally asymptotically stable $(1450,1450,0)$ which confirm our analytic results .However ,for the following set of parameters

$$A_1 = 200, A_2 = 50, A_3 = 0.2, A_4 = 0.15, A_5 = 0.25,$$
$$A_6 = 0.12, A_7 = 0.2, A_8 = 0.3, p = 0.4 \tag{18}$$

The trajectories of system(1) starting from different sets of initial data are drawn in Fig(2)



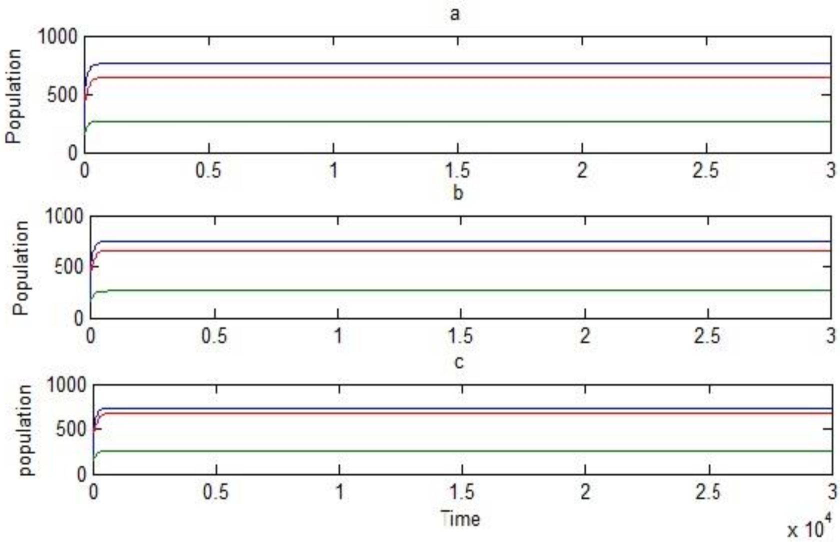
Fig(2):Phase plot of system (1) starting from different initial points for data given in Eq.(18)

Similarly, Fig(2) shows the approaching of system(1) to the endemic equilibrium point $\tilde{E} = (764.7, 268.3, 633.6)$,and study the effect of varying the rate of infected immigrant individuals on the dynamics of system (1) shows in the following table

Table (1): The effect of varying the rate of infection immigrant individuals

Parameters kept fixed	Parameter	Dynamical behavior of the system (1)
As given in Eq. (18)	$p = 0.5$	The system (1) approaches asymptotically to (758.8,266.3,641.6)
	$p = 0.7$	The system (1) approaches asymptotically to (747,262.1,657.6)
	$p = 0.9$	The system (1) approaches asymptotically to (735.2,258,673.5)

And the trajectories of system(1) as given in table (1) are drawn in Fig(3)(a-c)



Fig(3) Time series of the solutions of system(1) : (a) for p=0.5 (b) for p=0.7 (c) for p=0.9

According to Fig(3) ,as the fraction of infected immigrant individuals increases (through increasing p) the trajectory of system(1) approaches asymptotically to the endemic equilibrium point .In fact as p increases it is observed that the number of susceptible and vaccinated individuals decreases but the number of infective individuals increases .

Now the effect of varying the vaccination converge rate, the number of individuals who lose vaccine immunity and return to susceptible (failure in vaccine) are discuss in Tables (2),(3).

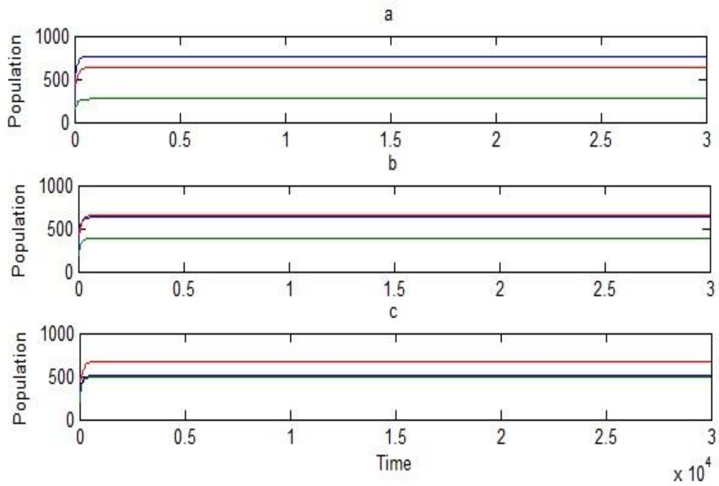
Table (2): The effect of varying the vaccination converge rate

Parameters kept fixed	Parameter	Dynamical behavior of the system (1)
As given in Eq. (18)	$A_7 = 0.2$	The system (1) approaches asymptotically to (764.7,268.3,633.6)
	$A_7 = 0.35$	The system (1) approaches asymptotically to (627.9,385.6,653.2)
	$A_7 = 0.55$	The system (1) approaches asymptotically to (507,489.2,670.4)

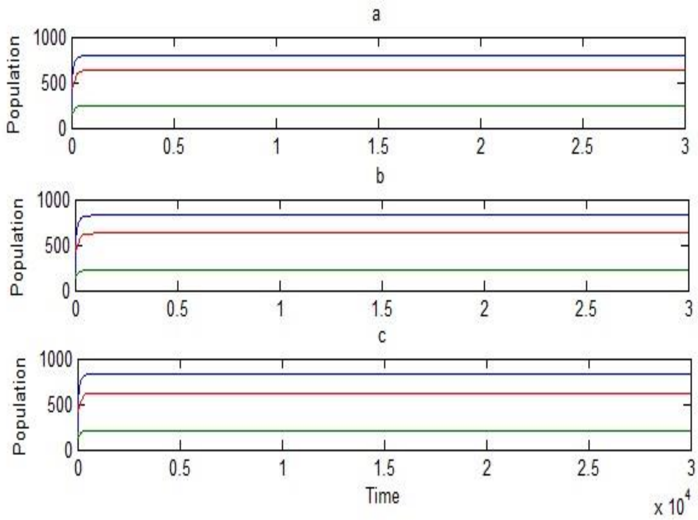
Table (3): The effect of varying the number of individuals who lose vaccine immunity and return to susceptible.

Parameters kept fixed	Parameter	Dynamical behavior of the system (1)
As given in Eq. (18)	$A_6 = 0.2$	The system (1) approaches asymptotically to (793.1,244,629.6)
	$A_6 = 0.3$	The system (1) approaches asymptotically to (822.9,219.2,625.4)
	$A_6 = 0.35$	The system (1) approaches asymptotically to (834.4,208.6,623.7)

And the trajectories of system(1) as given in Table (2),(3) are drawn in Fig(4),(5)(a-c)



Fig(4) Time series of the solutions of system(1):a) for $A_7 = 0.2$ *(b) for* $A_7 = 0.35$ *(c) for* $A_7 = 0.55$



Fig(5)Time series of the solutions of system(1):(a) for $A_6 = 0.2$ *(b) for* $A_6 = 0.3$ *(c) for* $A_6 = 0.35$

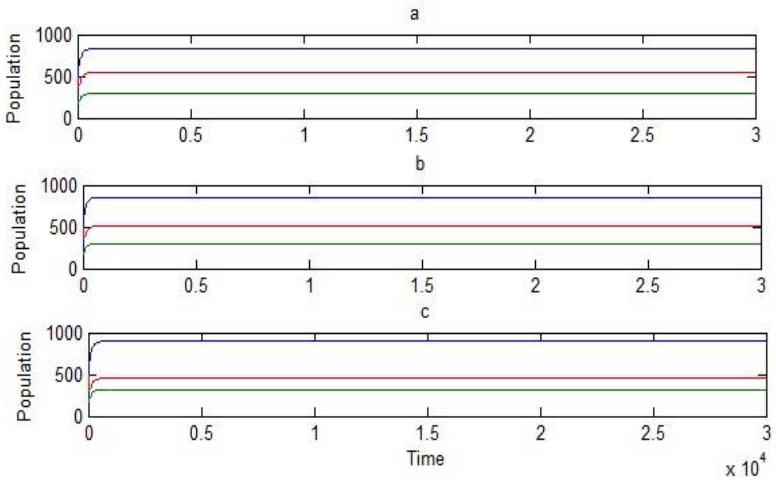
From Fig(4) ,as the rate of vaccination coverage increases the endemic equilibrium point of system(1) still coexists and table but the number of susceptible decrease whereas the number of infective individuals and vaccinated individuals increases .

Finally in the Table (4) shows that increases the value of recover rate causes increasing in S,V and decreasing in I but the system(1) in this cases still approaches to endemic equilibrium point

Table (4): The effect of varying the value of recover rate

Parameters kept fixed	Parameter	Dynamical behavior of the system (1)
As given in Eq. (18)	$A_5 = 0.35$	The system (1) approaches asymptotically to (829.3,291,546.3)
	$A_5 = 0.4$	The system (1) approaches asymptotically to (855.4,300.1,511.1)
	$A_5 = 0.5$	The system (1) approaches asymptotically to (898.6,315.3,452.8)

And the trajectories of system(1) as given in table (4) is drawn in fig(6)(a-c)



Fig(6)Time series of the solutions of system(1) : (a) for $A_5 = 0.35$ (b) for $A_5 = 0.4$ (c) for $A_5 = 0.5$

5.Discussion and Conclusions

In this paper, a mathematical model of SVIS epidemic model with immigrants has been studied and analyzed. The existence, uniqueness and boundedness of the solutions of system (1) have been investigated. The local and global dynamical behaviors of the model, in addition, have been studied analytically .The basin of attraction Ω, Υ of each equilibrium point has been found. Finally, according to the numerical simulation for the set of data that given in Eq.(17),(18) and some different initial conditions in Ω, Υ showed that $\bar{E}, \tilde{E}$ are globally asymptotically stable for different value of p, A_5, A_6 and A_7

Conflict of Interests

The author declares that there is no conflict of interests.

REFERENCES

[1] Anderson R.M., May R.M., Population biology of infectious disease: Part I, Nature 280(1979), 361-376.

[2] Christopher M., Jorge X., A simple vaccination model with multiple endemic states, Mathematical Biosciences., 164 (2000) 183-201.

[3] Farrington C.P ., On vaccine efficacy and reproduction numbers, Mathematical Biosciences., 185 (2003)

89–109.

[4] Guckenkerimer j., Holmes P., Nonlinear Oscillation dynamical systems and Bifurcations of vector fields., Springer-Verlag, New York, Inc., 1983.

[5] Gumel A.B., Moghadas S.M., A qualitative study of a vaccination model with nonlinear incidence, Appl. Math. Comput., 143(2003), 409-419.

[6] Halloran M.E., Haber M., Longini I.M., Struchiner C., Direct and indirect effects in vaccine efficacy and effectiveness, Am. J. Epidemiol., 133 (1991) 323.

[7] Hethcote H.W., Optimal ages of vaccination for measles, Math. Biosci., 89 (1988) 29.

[8] Hethcote H.W ., D.W. Tudor., Integral equations models for endemic infectious disease, J. Math. Biol., 9(1980), 37-47.

[9] Jianquani LI .,Zhien MA., Qualitative Analyses of SIS Epidemic Model with Vaccination Varying Total Population, Mathematical and Computer Modelling., 35 (2002) 1235-1243.

[10] Kermack W. O ., McKendrick A. G., Contributions to the mathematical theory of epidemics, part II. Proc. Roy. Soc. London 138(1932) 55-83.

[11] Li J., Ma Z., Qualitative analysis of SIS epidemic model with vaccination and varying total population size, Math. Comput. Model., 20(2002), 43-1235.

[12] May R.M. (1973) “Stability and complexity in model ecosystems”, Princeton, New Jersey: Princeton University Press

[13] May R.M., Anderson R.M., Population biology of infectious disease: Part II, Nature 280(1979), 455-461.

[14] Moghadas S.M., Analysis of an epidemic model with bistable equilibria using the Poincare index, Appl. Math. Comput., 149(2004), 689-702.

[15] Wiggins S., Introduction to applied nonlinear dynamical system and chaos, Spring-Verlag, New York, Inc., 1990.

[16] YicangI Zhou., Hanwu Liu., Stability of Periodic Solutions for an SIS Model with Pulse Vaccination,Mathematical and Computer Modelling, 38 (2003) 299-308