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A SELF-STARTING FOUR-STEP FIFTH-ORDER BLOCK INTEGRATOR FOR STIFF AND OSCILLATORY DIFFERENTIAL EQUATIONS

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Abstract. This paper examines the derivation and implementation of a self-starting four-step fifth order block integrator for direct integration of stiff and oscillatory first-order ordinary differential equations using interpolation and collocation procedures. The method was developed by collocation and interpolation of the combination of power series and exponential function to generate a continuous implicit linear multistep method. The paper further investigates the properties of the block integrator and found it to be zero-stable, consistent and convergent. The efficiency of the integrator was also tested on some sampled stiff and oscillatory problems and found to perform better than some existing ones.

Keywords: block integrator, exponential function, oscillatory, stiff.

2000 AMS Subject Classification: 65L05, 65L06, 65D30

1. Introduction

In this paper, we present a self-starting four-step fifth-order block integrator for direct integration of stiff and oscillatory problems of the form,

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(1) $$y' = f(x,y), \;\; y(a) = y_0, \;\; x \in [a,b],$$

where $f : \Re \times \Re^m \to \Re^m$, $y, y_0 \in \Re^m$, f satisfies Lipchitz condition which guarantees the existence and uniqueness of solution of (1). The development of numerical integration formulas for stiff as well as oscillatory differential equations has attracted considerable attention in the past; see Fatunla [7]. A special problem arising in the solution of ODEs is stiffness. This problem occurs in single linear and nonlinear ODEs, higher-order linear and nonlinear ODEs and systems of linear and nonlinear ODEs; see Hoffman [9]. It is also important to note that mathematical models of physical situations in kinetic chemical reactions, process control and electrical circuit theory often results to stiff ODEs; see Fatunla [7]. According to Sanugi and Evans [12], an interesting and important class of IVPs which can also arise in practice consists of differential equations whose solutions are known to be periodic or to oscillate with a known frequency. Examples of such problems can be found in the field of ecology, medical sciences and oscillatory motion in a nonlinear force field.

Almost invariably, most conventional numerical integration solvers cannot efficiently cope with stiff and oscillatory problems of the form (1) as they lack adequate stability characteristics, see Fatunla [7]. The degree of stiffness of a problem depends on the definition of stiffness that is applied; see Okunuga *et al.,* [11]. There are various definitions of stiffness in the literature as regards to ODEs. Lambert [10] gave a simple definition of stiffness of an ODE in such a manner that problem (1) possesses some stiffness if $R_e(\lambda_i) < 0, i = 1(1)m$, where λ is the eigen value of the problem.

Definition 1.1. [10] A stiff equation is a differential equation for which certain numerical methods for solving the equation are numerically unstable, unless the step size is taken to be extremely small. The main idea is that the equation includes some terms that can lead to rapid variation in the solution.

Definition 1.2. [4] A nontrivial solution (function) of an ODE is called oscillating if it does not tend either to a finite limit or to infinity (i.e. if it has an infinite number of roots). The differential equation is called oscillating, if it has at least one oscillating solution.

There are different concepts of the oscillation of a solution. The most widespread are oscillation at a point (usually taken to $+\infty$) and oscillation on an interval.

More recently, authors like Awoyemi *et al.* [3], Chollom *et al.* [5], Okunuga *et al.* [11], Yakubu *et al.* [13], Ajie *et al.* [2], Adebayo and umar [1], among others, have all proposed block methods to generate numerical solution to (1). These authors proposed methods in which the approximate solution ranges from power series, Chebychev's, Lagrange's and Laguerre's polynomials.

In this paper, the derivation of the continuous self-starting block integrator is carried out using an approximate solution which is a combination of power series and exponential function. This would help in coming up with a more computationally reliable integrator that could solve stiff and oscillatory problems of the form (1).

2. Preliminaries

2.1. Derivation Technique of the Self-Starting Block Integrator

We consider an approximate solution that combines power series and exponential function of the form:

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j + a_{r+s} \sum_{j=0}^{r+s} \frac{\alpha^j x^j}{j!}. \tag{2}$$

Interpolation and collocation procedures are used by choosing interpolation point s at a grid point and collocation points r at all points giving rise to $\xi = s + r$ system of equations whose coefficients are determined by using appropriate procedures. The first derivative of (2) is given by:

$$y'(x) = \sum_{j=0}^{r+s-1} j a_j x^{j-1} + a_{r+s} \sum_{j=0}^{r+s} \frac{\alpha^j x^{j-1}}{(j-1)!}, \tag{3}$$

where $a_j, \alpha^j \in \Re$ for $j = 0(1)5$ and $y(x)$ is continuously differentiable. Let the solution of (1) be sought on the partition $\pi_N : a = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} < \ldots x_N = b$, of the integration interval $[a,b]$ with a constant step-size h, given by: $h = x_{n+1} - x_n, n = 0, 1, 2, \ldots, N$

Then, substituting (3) in (1), we obtain that

$$f(x,y)=\sum_{j=0}^{r+s-1} ja_j x^{j-1}+a_{r+s}\sum_{j=0}^{r+s}\frac{\alpha^j x^{j-1}}{(j-1)!}. \tag{4}$$

Now, interpolating (2) at point $x_{n+s}, s=0$ and collocating (4) at points $x_{n+r}, r=0(1)4$, leads to the following system of equations:

$$AX=U, \tag{5}$$

where

$$A=[a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5]^T,$$

$$U=[y_n \quad f_n \quad f_{n+1} \quad f_{n+2} \quad f_{n+3} \quad f_{n+4}]^T$$

and

$$X=\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & \left(1+\alpha x_n+\frac{\alpha^2x_n^2}{2!}+\frac{\alpha^3x_n^3}{3!}+\frac{\alpha^4x_n^4}{4!}+\frac{\alpha^5x_n^5}{5!}\right) \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & \left(\alpha+\alpha^2x_n+\frac{\alpha^3x_n^2}{2!}+\frac{\alpha^4x_n^3}{3!}+\frac{\alpha^5x_n^4}{4!}\right) \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & \left(\alpha+\alpha^2x_{n+1}+\frac{\alpha^3x_{n+1}^2}{2!}+\frac{\alpha^4x_{n+1}^3}{3!}+\frac{\alpha^5x_{n+1}^4}{4!}\right) \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & \left(\alpha+\alpha^2x_{n+2}+\frac{\alpha^3x_{n+2}^2}{2!}+\frac{\alpha^4x_{n+2}^3}{3!}+\frac{\alpha^5x_{n+2}^4}{4!}\right) \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & \left(\alpha+\alpha^2x_{n+3}+\frac{\alpha^3x_{n+3}^2}{2!}+\frac{\alpha^4x_{n+3}^3}{3!}+\frac{\alpha^5x_{n+3}^4}{4!}\right) \\ 0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & \left(\alpha+\alpha^2x_{n+4}+\frac{\alpha^3x_{n+4}^2}{2!}+\frac{\alpha^4x_{n+4}^3}{3!}+\frac{\alpha^5x_{n+4}^4}{4!}\right) \end{bmatrix}.$$

Solving (5), for $a_j's, j=0(1)5$ and substituting back into (2) gives a continuous linear multistep method of the form:

$$y(t)=\alpha_0(x)y_n+h\sum_{j=0}^{4}\beta_j(x)f_{n+j}, \tag{6}$$

where

$$(7) \qquad \begin{bmatrix} \alpha_0 = 1 \\ \beta_0 = \frac{1}{720}(6t^5 - 75t^4 + 350t^3 - 750t^2 + 720t) \\ \beta_1 = -\frac{1}{360}(12t^5 - 135t^4 + 520t^3 - 720t^2) \\ \beta_2 = \frac{1}{60}(3t^5 - 30t^4 + 95t^3 - 90t^2) \\ \beta_3 = -\frac{1}{360}(12t^5 - 105t^4 + 280t^3 - 240t^2) \\ \beta_4 = \frac{1}{720}(6t^5 - 45t^4 + 110t^3 - 90t^2) \end{bmatrix},$$

where $t = \frac{x - x_n}{h}$. Evaluating (6) at $t = 1(1)4$ gives a continuous discrete block scheme of the form:

$$(8) \qquad A^{(0)} Y_m = E y_n + h d f(y_n) + h b F(Y_m),$$

where

$$Y_m = [y_{n+1} \quad y_{n+2} \quad y_{n+3} \quad y_{n+4}]^T, \quad y_n = [y_{n-3} \quad y_{n-2} \quad y_{n-1} \quad y_n]^T,$$

$$F(Y_m) = [f_{n+1} \quad f_{n+2} \quad f_{n+3} \quad f_{n+4}]^T, \quad f(y_n) = [f_{n-3} \quad f_{n-2} \quad f_{n-1} \quad f_n],^T$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$d = \begin{bmatrix} 0 & 0 & 0 & \frac{251}{720} \\ 0 & 0 & 0 & \frac{29}{90} \\ 0 & 0 & 0 & \frac{27}{80} \\ 0 & 0 & 0 & \frac{14}{45} \end{bmatrix}, \; b = \begin{bmatrix} \frac{323}{360} & \frac{-11}{30} & \frac{53}{360} & \frac{-19}{720} \\ \frac{62}{45} & \frac{4}{15} & \frac{2}{45} & \frac{-1}{90} \\ \frac{51}{40} & \frac{9}{10} & \frac{21}{40} & \frac{-3}{80} \\ \frac{64}{45} & \frac{8}{15} & \frac{64}{45} & \frac{14}{45} \end{bmatrix}.$$

2.2 Analysis of Basic Properties of the Self-Starting Block Integrator

2.2.1. Order of the Self-Starting Block Integrator

Let the linear operator $L\{y(x); h\}$ associated with the block integrator (8) be defined as

$$(9) \qquad L\{y(x); h\} = A^{(0)} Y_m - E y_n - h(d f(y_n) + b F(Y_m))$$

expanding using Taylor series and comparing the coefficients of h gives

$$L\{y(x);h\} = c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + ... + c_p h^p y^p(x) + c_{p+1} h^{p+1} y^{p+1}(x) + ... \tag{10}$$

Definition 2.1. [8] The linear operator L and the associated continuous linear multistep method (6) are said to be of order p if $c_0 = c_1 = c_2 = ... = c_p = 0$ and $c_{p+1} \neq 0$. c_{p+1} is called the error constant and the local truncation error is given by

$$t_{n+k} = c_{p+1} h^{(p+1)} y^{(p+1)}(x_n) + \bigcirc(h^{p+2}). \tag{11}$$

For our method

$$L\{y(x);h\} = \left[\begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} \\ \\ -h \begin{bmatrix} \frac{251}{720} & \frac{323}{360} & \frac{-11}{30} & \frac{53}{360} & \frac{-19}{720} \\ \frac{29}{90} & \frac{62}{45} & \frac{4}{15} & \frac{2}{45} & \frac{-1}{90} \\ \frac{27}{80} & \frac{51}{40} & \frac{9}{10} & \frac{21}{40} & \frac{-3}{80} \\ \frac{14}{45} & \frac{64}{45} & \frac{8}{15} & \frac{64}{45} & \frac{14}{45} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \end{array}\right]. \tag{12}$$

Expanding (12) in Taylor series, we find that

$$\begin{bmatrix} \sum_{j-0}^{\infty} \frac{(h)^j}{j!} y_n^j - y_n - \frac{251h}{720} y'_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \begin{array}{c} \frac{323}{360}(1)^j - \frac{11}{30}(2)^j \\ + \frac{53}{360}(3)^j - \frac{19}{720}(4)^j \end{array} \right\} \\ \sum_{j-0}^{\infty} \frac{(2h)^j}{j!} y_n^j - y_n - \frac{29h}{90} y'_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \begin{array}{c} \frac{62}{45}(1)^j + \frac{4}{15}(2)^j \\ + \frac{2}{45}(3)^j - \frac{1}{90}(4)^j \end{array} \right\} \\ \sum_{j-0}^{\infty} \frac{(3h)^j}{j!} y_n^j - y_n - \frac{27h}{80} y'_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \begin{array}{c} \frac{51}{40}(1)^j + \frac{9}{10}(2)^j \\ + \frac{21}{40}(3)^j - \frac{3}{80}(4)^j \end{array} \right\} \\ \sum_{j-0}^{\infty} \frac{(4h)^j}{j!} y_n^j - y_n - \frac{14h}{45} y'_n - \sum_{j=0}^{\infty} \frac{h^{j+1}}{j!} y_n^{j+1} \left\{ \begin{array}{c} \frac{64}{45}(1)^j + \frac{8}{15}(2)^j \\ + \frac{64}{45}(3)^j + \frac{14}{45}(4)^j \end{array} \right\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{13}$$

Hence, $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \bar{c}_4 = \bar{c}_5 = 0$, $\bar{c}_6 = [1.88(-02) \quad 1.11(-02) \quad 1.88(-02) \quad -8.47(-03)]^T$. Therefore, the self-starting block integrator is of order five.

2.2.2. Zero Stability

Definition 2.2. [8] The block integrator (8) is said to be zero-stable, if the roots $z_s, s = 1, 2, ..., k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z) = \det(zA^{(0)} - E)$ satisfies $|z_s| \leq 1$ and every root satisfying $|z_s| \leq 1$ have multiplicity not exceeding the order of the differential equation. Moreover, as $h \to 0, \rho(z) = z^{r-\mu}(z-1)^{\mu}$ where μ is the order of the differential equation, r is the order of the matrices $A^{(0)}$ and E (see Awoyemi *et al.* [3] for details).

For our integrator,

$$\rho(z) = \left| z \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0 \tag{14}$$

$\rho(z) = z^3(z-1) = 0 \Longrightarrow z_1 = z_2 = z_3 = 0,\ z_4 = 1.$ Hence, the self-starting block integrator is zero-stable.

2.2.3. Consistency

The block integrator (8) is consistent since it has order $p = 5 \geq 1$.

2.2.4. Convergence

The self-starting block integrator is convergent by consequence of Dahlquist theorem below.

Theorem 2.1. [6] *The necessary and sufficient conditions that a continuous LMM be convergent are that it be consistent and zero-stable.*

2.2.5. Region of Absolute Stability

Definition 2.3. [14] Region of absolute stability is a region in the complex z plane, where $z = \lambda h$. It is defined as those values of z such that the numerical solutions of $y' = -\lambda y$ satisfy $y_j \to 0$ as $j \to \infty$ for any initial condition.

To determine the absolute stability region of the self-starting block integrator, we adopt the boundary locus method. This is achieved by substituting the test equation

(15) $$y' = -\lambda y$$

into the block formula (8). This gives

(16) $$A^{(0)}Y_m(w) = Ey_n(w) - h\lambda Dy_n(w) - h\lambda BY_m(w).$$

Thus,

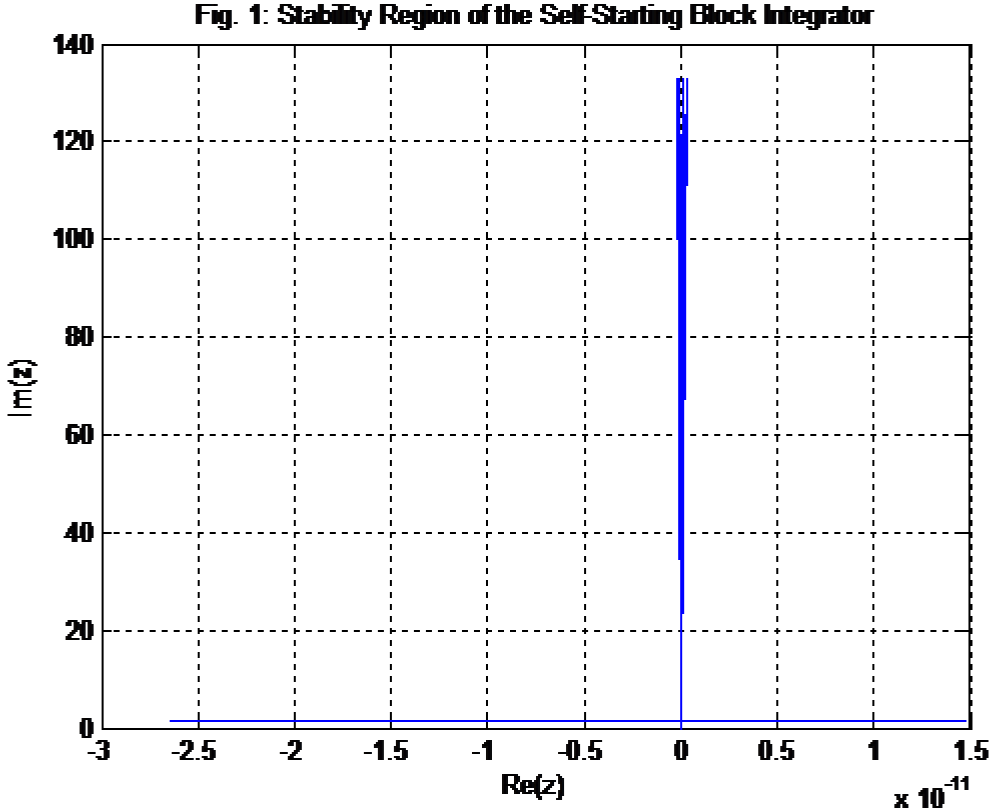
(17) $$\bar{h}(w) = -\left(\frac{A^{(0)}Y_m(w) - Ey_n(w)}{Dy_n(w) + BY_m(w)}\right)$$

since $\bar{h}$ is given by $\bar{h} = \lambda h$ and $w = e^{i\theta}$. Equation (17) is our characteristic or stability polynomial. For our integrator, equation (17) is given by

(18)
$$\bar{h}(w) = -h^4\left(\frac{1}{5}w^3 - \frac{1}{5}w^4\right) - h^3\left(\frac{5}{6}w^4 + \frac{5}{6}w^3\right) - h^2\left(\frac{7}{4}w^3 - \frac{7}{4}w^4\right) - h\left(2w^4 + 2w^3\right) + w^4 - w^3.$$

This gives the stability region shown in fig. 1 below.

Fig. 1: Stability Region of the Self-Starting Block Integrator



According to Fatunla [8], stiff algorithms have unbounded RAS. Also, Lambert [10] showed that the stability region for L-stable schemes must encroach into the positive half of the complex plane.

3. Main results

We shall evaluate the performance of the self-starting block integrator on some challenging stiff and oscillatory problems which have appeared in the literature and compare the results with solutions from some methods of similar derivation. The following notations shall be used in the tables below;

ERR- |Exact Solution-Computed Result|

ERO- Error in Okunuga *et al.* [11]

ERA- Error in Adebayo and Umar [1]

3.1. Numerical Examples

Problem 3.1. Consider the highly stiff ODE

$$y' = -10(y-1)^2, y(0) = 2, \tag{19}$$

which has the exact solution

$$y(x) = 1 + \frac{1}{(1+10x)}. \tag{20}$$

This problem was earlier discussed by Lambert [10], he showed that many predictor-corrector and block methods become unstable with this problem, including the hybrid methods. However, the newly derived block intgerator is used for the integration of this problem within the interval $0 \leq x \leq 0.1$.Okunuga *et al.* [11] solved this stiff problem by adopting a new 2-point block method with step-size ratio at $r = 1$.

Problem 3.2. Consider the Prothero-Robinson oscillatory ODE

$$y' = L(y - \sin x) + \cos x, \ \ L = -1, \ \ y(0) = 0 \tag{21}$$

with the exact solution

$$y(x) = \sin x. \tag{22}$$

Adeboye and Umar [1] solved this problem using generalized rational approximation method via Pade approximants with step number $k = 6$.

Table 3.1. Showing the result for stiff problem 3.1

x	Exact Solution	Computed Solution	ERR	ERO
0.0100	1.9090909090909092	1.9090874944198133	$3.414671e-006$	$1.07e-03$
0.0200	1.8333333333333335	1.8333305836980267	$2.749635e-006$	$2.38e-03$
0.0300	1.7692307692307692	1.7692173398056399	$1.342943e-005$	$2.21e-03$
0.0400	1.7142857142857144	1.7141948078088534	$9.090648e-005$	$5.36e-03$
0.0500	1.6666666666666665	1.6665869698207376	$7.969685e-005$	$7.53e-03$
0.0600	1.6250000000000000	1.6249300511359952	$6.994886e-005$	$9.00e-03$
0.0700	1.5882352941176470	1.5881725936414972	$6.270048e-005$	$9.98e-03$
0.0800	1.5555555555555556	1.5554953845413360	$6.017101e-005$	$1.06e-02$
0.0900	1.5263157894736841	1.5262616763962231	$5.411308e-005$	$1.10e-02$
0.1000	1.5000000000000000	1.4999511902154874	$4.880978e-005$	$1.12e-02$

Table 3.2. Showing the result for Prothero-Robinson oscillatory problem 3.2

x	Exact Solution	Computed Solution	ERR	ERA
0.1000	0.0998334166468282	0.0998334166192123	$1.452952e-011$	$2.0e-11$
0.2000	0.1986693307950612	0.1986693307257173	$1.621117e-011$	$3.0e-11$
0.3000	0.2955202066613396	0.2955202066226441	$2.131013e-011$	$1.0e-10$
0.4000	0.3894183423086505	0.3894183423146642	$1.379910e-011$	$2.0e-10$
0.5000	0.4794255386042030	0.4794255386278611	$2.744084e-011$	$1.0e-10$
0.6000	0.5646424733950355	0.5646424733206178	$1.111424e-011$	$2.0e-10$
0.7000	0.6442176872376911	0.6442176872324257	$2.865663e-011$	$1.0e-10$
0.8000	0.7173560908995228	0.7173560901091701	$1.921784e-010$	$2.0e-10$
0.9000	0.7833269096274835	0.7833269097514037	$1.239202e-010$	$3.0e-10$
1.0000	0.8414709848078966	0.8414709849550068	$1.471102e-010$	$3.0e-10$

0.1. **Discussion of Results.** We have considered two numerical examples in this paper. The first problem (which is stiff) was solved by Okunuga *et al.* [11] where they applied 2-point block method with step-size ratio at $r = 1$ while the second problem (which is oscillatory) was solved by Adebayo *et al.* [1] where they adopted generalized rational approximation method via Pade approximants with step number $k = 6$. We solved the two problems using the self-starting block integrator developed. Tables 3.1 and 3.2 above showed that the block integrator gives better results than the existing ones.

4. Conclusion

In this paper, we have presented a self-starting four-step fifth-order block numerical integrator for the solution of stiff and oscillatory first-order ordinary differential equations. The approximate solution (basis function) adopted in this research produced a block integrator with L-stable stability region. This made it possible for the block integrator to perform well on stiff and oscillatory problems. The block integrator proposed was also found to be zero-stable, consistent and convergent. The new integrator was also found to perform better than some existing methods.

Conflict of Interests

The authors declare that there is no conflict of interests.

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