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A LOWER BOUND FOR THE NUMBER OF CONJUGACY CLASSES IN POSITIVE PERMUTATION BRAIDS

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Abstract. We study the problem of finding lower and upper bounds for the number of conjugacy clases in positive permutation braids. For such braids with associated type cycle (n), all possible values of their crossing numbers, the minimum and maximum crossing numbers and a more sharbend lower bound of the number of their conjugacy classes in S_n^+ are given. Also for positive permutation braids with associate type cycle $(n_1, n_2, \dots, n_k)$ the minimum crossing number is given..

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1. Introduction

The Artin's braid group B_n and the symmetric group S_n have, respectively, the presentations:

$$(1) \qquad B_n = \left\{ \begin{array}{c} \sigma_i,\ i = 1, 2, ..., n-1 :\ \sigma_i\sigma_j \ = \ \sigma_j\sigma_i \ \text{ if } |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i \ = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ if } i = 1, 2, ..., n-2 \end{array} \right\}$$

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$$(2) \qquad S_n = \left\{ \begin{array}{c} \tau_i = (i\ i+1),\ i = 1, 2, ..., n-1 :\ \tau_i \tau_j \ = \ \tau_j\ \tau_i \text{ if } |i-j| > 1, \\ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \text{ if } i = 1, 2, ..., n-2,\ \tau_i^2 = 1\ \forall\ i \end{array} \right\}$$

So we have the natural homomorphism $\theta : B_n \rightarrow S_n$, such that $\theta(\sigma_i) = \tau_i$ for all i. There are several known algorithms for solving the word and the conjugacy problems in braid groups [1]. The first algorithm for these problems was given by Garside [2]. Then it was improved by Elrifai [3], Elrifai and Morton [4]. A positive braid in B_n is the braid which can be written as a word in positive powers of generators σ_i, and without use of the inverse elements σ_i^{-1}. The set of all positive braids form a monoid, denoted B_n^+. The positive permutation braids, PPBs S_n^+, were first defined by Elrifai [3], where a braid is a positive permutation braid if it is positive and each pair of its strings cross at most once. PPBs represent a geometric analogue of permutations, and $S_n^+ \subseteq B_n^+ \subseteq B_n$. The algebraic crossing number of a braid is the algebraic sum of the powers of the letters in it. In [5] Elrifai and Benkhalifa introduced a conjugacy invariant matrix, called classification crossing matrix, for positive braids. They also proved that this matrix is a complete conjugacy invariant for PPBs when $n \leq 5$. In [6] Morton and Hadji studied the problem in point of view of knot theory. They proved that PPBs which close to the trivial knot or to the trefoil knot are all conjugate.

The knot is an embedded circle in $S^3(R^3)$, and a link is a disjoint collection of knots. The closure (closed braid) of a braid in B_n is formed by joining the top points to the bottom. Each knot or link can be represented as a closed braid. Two closed braids are equivalent (as links) if and only if their braid representatives are related by a finite sequence of Markov moves, $b \longleftrightarrow aba^{-1}$ for any a, b in B_n (conjugation) and $b \longleftrightarrow b\sigma_n^{\pm 1}$ for any $b \, \epsilon B_n$ (stabilizer). The half twist braid $\Delta_n = (\sigma_1\sigma_2...\sigma_{n-1})(\sigma_1\sigma_2...\sigma_{n-2})...(\sigma_1\sigma_2)(\sigma_1)$ in B_n plays an important role in braid theory. Through this article we use symbols $\alpha, \beta, ...$ for elements in S_n, and $\alpha^+, \beta^+, ...$ for elements in S_n^+. Let $c(\alpha) = c(\alpha^+)$, be the crossing number of the strings of α as a permutation, or α^+ as a PPB. For more details about braid and link theories, we refer to [7].

In this article, the following results are given,

1: For each positive permutation braid α^+ in S_n^+ with associated permutation of type cycle (n), it is proved that:

- $n-1 \leq c(\alpha^+) \leq \frac{(n-1)^2}{2}$ if n is odd, or $\frac{(n-1)^2+1}{2}$ if n is even.
- For each integer k in $\{(n-1), (n-1)+2, (n-1)+4, ..., (n-1)^2/2 \text{ if } n \text{ odd}\}$, there exists at least one α^+ such that $c(\alpha^+) = k$.
- For each integer k in $\{(n-1), (n-1)+2, (n-1)+4, ..., [(n-1)^2+1]/2 \text{ if } n \text{ even}\}$, there exists at least one α^+ such that $c(\alpha^+) = k$.
- A more sharpened lower bound of the number of the conjugacy classes in S_n^+ of type cycle (n) is $\frac{(n-1)(n-3)}{2}$ if n is odd, or $\frac{(n-1)(n-3)+1}{2}$, if n is even, $n \geq 4$.

2: For PPBs with associate type cycle $(n_1, n_2, \ldots, n_k)$ the minimum crossing number is $n-r$.

2. Preliminaries

In S_n^+, the identity braid e has $c(e) = 0$ and the half twist braid Δ_n, where each pair of strings cross exactly once, has crossing number $c(\Delta_n) = n(n-1)/2$. Therefore $0 \leq c(\alpha) \leq n(n-1)/2 \ \forall \ \alpha \in S_n^+$. Also crossing number is a conjugacy invariant but it is not a complete invariant for conjugation. There are several examples of PPB words with the same crossing and the same type cycle but they are not conjugate [6].

3. Main results

Proposition 3.1. *For elements in S_n^+ with associate type cycle (n), the minimum crossing number is $n-1$.*

Proof. Suppose that in the word α^+ one of the generators σ_k does not appear at all, then the diagram of α^+ will be splitted as in figure 1b, which means that the presentation of α^+ does not a cycle. Hence to have a cycle, each generator σ_i must appear at least once. Then minimal crossing will be reached if each generator appears exactly once. So that the minimal crossing is the number of generators, i.e. it equals $n-1$. This completes the proof.

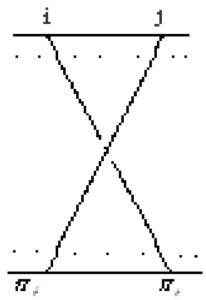

Figure 1a:

An inversion

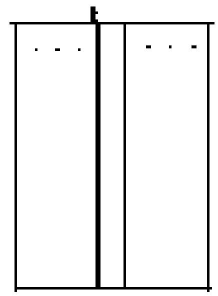

Figure 1b:

Splitting braid

Proposition 3.2. *For elements in S_n^+ with associate type cycle (n), the maximum crossing number is equal to $\frac{(n-1)^2}{2}$ if n is odd, or $\frac{(n-1)^2+1}{2}$ if n is even.*

Proof. Let α^+ of type cycle (n), but Δ_n^+ is the only PPB which has the maximum crossing number $\frac{n(n-1)}{2}$. The closure of Δ_n^+ is a link with $\frac{n}{2}$ components if n is even, and $\frac{n+1}{2}$ if n is odd. Then to have a knot you must switch some of the crossing to make bridges between these components, so we need at least a number of bridges which equals the number of components minus one. Therefore the maximum crossing number equal

$$\frac{n(n-1)}{2} - (\frac{(n+1)}{2} - 1) = \frac{(n-1)^2}{2}, \quad if \quad n - odd$$

or

$$\frac{n(n-1)}{2} - (\frac{n}{2} - 1) = \frac{(n-1)^2+1}{2}, \quad if \quad n - even$$

This completes the proof.

Theorem 3.3. *In S_n^+, there is at least one α^+ with type cycle (n), such that $c(\alpha^+)$ covers the set*

$$(n-1), (n-1)+2, (n-1)+4, ..., (n-1)^2/2 \; if \; n \; odd$$

or

$$(n-1), (n-1)+2, (n-1)+4, ..., [(n-1)^2+1]/2 \; if \; n \; even\}$$

Proof. For PPBs of type cycle (n), proposition 1 implies the existence of at least one α with $c(\alpha^{+}) = n-1$. Also proposition 2 implies the existence of at least one α with crossing number $\frac{(n-1)^2}{2}$ if n is odd, or $\frac{(n-1)^2+1}{2}$ if n is even. But, to preserve the type cycle (n), we must increase the crossing by even natural numbers. So that $c(\alpha^{+})$ covers the set $\{(n-1),(n-1)+2,(n-1)+4,...,(n-1)^2/2\ \}$ if n is odd, and $\{\ (n-1),(n-1)+2,(n-1)+4,...,[(n-1)^2+1]/2\}$ if n is even. This completes the proof.

Theorem 3.4. A lower bound of the number of the conjugacy classes in S_n^{+} of type cycle (n) is $\frac{(n-1)(n-3)}{2}$ if n is odd, or $\frac{(n-1)(n-3)+1}{2}$, if n is even, $n \geq 4$.

Proof. In order to preserve the associated type cycle of a PPB, we must increase the crossing by even natural numbers. Then each crossing will be $(n-1)+2k, k = 0,1,2,....$ So the upper bound of k is the maximum crossing number minus $(n-1)$, which implies that the number of conjugacy classes with different crossings of a PPB in S_n^{+} which associates a type cycle (n) is $\frac{(n-1)^2}{2}-(n-1) = \frac{(n-1)(n-3)}{4}$, if n is odd, and $\frac{(n-1)^2+1}{2}-(n-1) = \frac{(n-1)(n-3)+1}{4}$ if n is even. But the algebraic crossing number is invariant under conjugation, i.e. the words with different crossing numbers are not conjugate. Hence we have at least a number of conjugacy classes which is equal to the number of classes with different crossings. This completes the proof.

Corollary 3.5. PPBs with associate type cycle $(n_1, n_2, \ldots, n_k)$ have minimum crossing number $n-r$.

Proof. Let α be a PPB with permutation $\alpha = \alpha_1 \circ \alpha_2 \circ ... \circ \alpha_r$ of type cycle $(n_1, n_2, \ldots, n_k)$. Let x_i be the least integer in the support of the cycle α_i for each $i = 1,2,...,r$, then strings from x_i to $\alpha(x_i)$ and from x_{i+1} to $\alpha(x_{i+1})$ never cross each other, otherwise we have the inversion $(\alpha(x_i), \alpha(x_{i+1}))$. Therefore in the portion $\alpha_i \circ \alpha_{i+1}$ we will lost one crossing. Hence the minimum number of crossings in α is $(n-1)-(r-1) = n-r$. This completes the proof.

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