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A GENERALIZED CORRELATION ON MULTIVECTOR FIELDS

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Abstract. In this paper, we study the Clifford correlation and investigate its important properties. We establish the correlation theorem for the Clifford Fourier transform (CFT). Based on the concept of the Clifford algebra we derive some properties of relationship between the CFT and the Clifford correlation.

Keywords: clifford (geometric) algebra; clifford correlation; clifford Fourier transform.

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1. Introduction

The classical Fourier transform (FT) is one of the most frequently used tools in signal processing and many other scientific fields [9]. The generalization of the FT to the Clifford algebra is so-called the Clifford Fourier transform (CFT). It was first introduced from the mathematical aspect by Brackx et al. [4]. Unfortunately, some fundamental and important properties of the FT such as convolution, correlation and uncertainty principle can not be established in this generalization [18]. An alternative definition of the CFT also has been proposed by Brackx et al. [4] which its properties are further investigated by De Bie et al. (see [5, 6]). Later, based on the concept of the Clifford algebra some definitions of the CFT have been introduced recently

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(see, for example, [1, 2, 6, 11, 16, 17]). Moreover, several important properties of the CFT are already known such as shift, modulation, convolution, vector differential, vector derivative and directional uncertainty principle, which are generalization of the corresponding properties of the FT with some modifications.

In [12, 13, 14, 15], the authors proposed the convolution and correlation theorems for the quaternion Fourier transform (QFT). Some of their properties are also investigated. This paper concentrates on the correlation theorems for the CFT proposed by the authors in [8, 11, 16]. Because of the noncommutative property of the Clifford (geometric) multiplication, we find some properties of the relationship between the CFT and the correlation of two Clifford-valued functions.

Let us briefly review some basic facts from the real Clifford algebra that we will use throughout this paper. We write $Cl_{n,0}$ for the real Clifford algebra over the real n -dimensional Euclidean vector space \mathbb{R}^n . The Clifford algebra is generated over \mathbb{R} by the orthonormal vector basis of reduced products

$$(0.1) \quad \{1, e_1, e_2, \dots, e_n, e_{12}, e_{31}, e_{23}, \dots, i_n = e_1 e_2 \cdots e_n\}.$$

Here i_n is called the unit oriented pseudoscalar. We remember that $i_n^2 = -1$ for $n = 2, 3 \pmod{4}$. The noncommutative multiplication of the basis vectors satisfies the rules $e_i e_j + e_j e_i = 2\delta_{ij}$, where δ_{ij} denotes the Dirac distribution whose support is $\{i, j\}$. The general elements of the Clifford algebra are called multivectors. Every multivector $f \in Cl_{n,0}$ can be represented in the form

$$(0.2) \quad f = \sum_A f_A e_A,$$

where $f_A \in \mathbb{R}$, $e_A = e_{\alpha_1 \alpha_2 \cdots \alpha_k} = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k}$, and $1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k \leq n$ with $\alpha_j \in \{1, 2, \dots, n\}$. For convenience, we introduce $\langle f \rangle_k = \sum_{|A|=k} f_A e_A$ to denote k -vector part of f ($k = 0, 1, 2, \dots, n$), then

$$(0.3) \quad f = \sum_{k=0}^{k=n} \langle f \rangle_k = \langle f \rangle_0 + \langle f \rangle_1 + \langle f \rangle_2 + \cdots + \langle f \rangle_n,$$

where $\langle \dots \rangle_0 = \langle \dots \rangle$.

Every multivector $f \in Cl_{n,0}, n = 2 \pmod{4}$ may be decomposed as a sum of even part grade $f_{\text{even}} = \langle f_{\text{even}} \rangle$ and odd grade part $f_{\text{odd}} = \langle f_{\text{odd}} \rangle$. It means that we have

$$(0.4) \quad f = f_{\text{even}} \oplus f_{\text{odd}},$$

where

$$(0.5) \quad \begin{aligned} f_{\text{even}} &= \langle f \rangle + \langle f \rangle_2 + \dots + \langle f \rangle_r, & r = 2s, s \in \mathbb{N}, s \leq \frac{n}{2}, \\ f_{\text{odd}} &= \langle f \rangle_1 + \langle f \rangle_3 + \dots + \langle f \rangle_r, & r = 2s + 1, s \in \mathbb{N}, s < \frac{n}{2}. \end{aligned}$$

Based on equation (0.3) we obtain the reverse \tilde{f} of a multivector f as

$$(0.6) \quad \tilde{f} = \sum_{k=0}^{k=n} (-1)^{k(k-1)/2} \langle f \rangle_k,$$

It is the anti-involution for which

$$(0.7) \quad \widetilde{fg} = \tilde{g}\tilde{f}.$$

Decomposition the multivector f in (0.4) gives the following important result and easy to prove (see [8]).

Proposition 1.1. *Given a multivector $f \in Cl_{n,0}$ with $n = 3 \pmod{4}$. For $\lambda \in \mathbb{R}$ we have*

$$(0.8) \quad fe^{i_n\lambda} = e^{-i_n\lambda} f_{\text{odd}} + e^{i_n\lambda} f_{\text{even}},$$

and

$$(0.9) \quad \bar{f}e^{i_n\lambda} = e^{-i_n\lambda} \widetilde{f_{\text{odd}}} + e^{i_n\lambda} \widetilde{f_{\text{even}}}.$$

The multivector $f \in Cl_{n,0}$ is called a paravector if (0.3) takes the form

$$(0.10) \quad f = \langle f \rangle + \langle f \rangle_1 = f_0 + \sum_{i=1}^n f_i e_i.$$

From equation (0.10) it is not difficult to check that the geometric product $f\tilde{f}$ is a scalar valued.

The scalar product of multivectors f, \tilde{g} is defined as the scalar part of the geometric product $f\tilde{g}$ of multivectors

$$(0.11) \quad \langle f\tilde{g} \rangle = f * \tilde{g} = \sum_A f_A g_A,$$

which leads to a cyclic product symmetry

$$(0.12) \quad \langle pqr \rangle = \langle qrp \rangle, \quad \forall p, q, r \in Cl_{n,0}.$$

In particular, if $f = g$ in (0.11), then we obtain the modulus (or magnitude) $|f|$ of a multivector $f \in Cl_{n,0}$ defined as

$$(0.13) \quad |f|^2 = f * \tilde{f} = \sum_A f_A^2.$$

It is convenient to introduce an inner product for two multivector functions $f, g : \mathbb{R}^n \rightarrow Cl_{n,0}$ as follows:

$$(0.14) \quad (f, g)_{L^2(\mathbb{R}^n; Cl_{n,0})} = \int_{\mathbb{R}^n} f(x) \widetilde{g(x)} d^n x = \sum_{A,B} e_A \tilde{e}_B \int_{\mathbb{R}^n} f_A(x) g_B(x) d^n x.$$

Thus for $f = g$ we take the scalar part of (0.14) and obtain the associated norm as

$$(0.15) \quad \|f\|_{L^2(\mathbb{R}^n; Cl_{n,0})}^2 = \int_{\mathbb{R}^n} \sum_A f_A^2(x) d^n x.$$

2. Clifford Fourier Transform (CFT)

Firstly, let us introduce some notation, which will be used in the next section. The *translation* and *modulation* of $f \in L^2(\mathbb{R}^n; Cl_{n,0})$ are defined by

$$(0.16) \quad \tau_a f(x) = f(x - a), \quad M_{\omega_0} f(x) = e^{i_n \omega_0 \cdot x} f(x),$$

respectively, and the *time-frequency shift* is defined by their composition:

$$(0.17) \quad M_{\omega_0} \tau_a f(x) = e^{i_n \omega_0 \cdot x} f(x - a), \quad a, \omega_0 \in \mathbb{R}^n.$$

Just as in the classical case, we obtain the canonical commutation relation:

$$(0.18) \quad \tau_a M_{\omega_0} f = e^{-i_n \omega_0 \cdot a} M_{\omega_0} \tau_a f.$$

The $Cl_{n,0}$ Clifford Fourier transform (CFT) is a generalization of the FT to the Clifford algebra obtained by replacing the FT kernel with the Clifford Fourier kernel. For detailed discussions of the properties of the CFT and their proofs, see, e.g., [8, 16].

Definition 2.1. Let $f \in L^2(\mathbb{R}^n; Cl_{n,0})$. The CFT $\mathcal{F}\{f\}$ of f is defined by

$$(0.19) \quad \mathcal{F}\{f\}(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^n} f(x) e^{-i_n \omega \cdot x} d^n x,$$

with $x, \omega \in \mathbb{R}^n$.

Note that we can prove that $\mathcal{F}\{f\} \in L^2(\mathbb{R}^n; Cl_{n,0})$ for $f \in L^2(\mathbb{R}^n; Cl_{n,0})$. Decomposing the multivector f into f_{even} and f_{odd} , (0.19) can be rewritten as

$$(0.20) \quad \mathcal{F}\{f\}(\omega) = \int_{\mathbb{R}^n} e^{i_n \omega \cdot x} f_{\text{odd}}(x) d^n x + \int_{\mathbb{R}^n} e^{-i_n \omega \cdot x} f_{\text{even}}(x) d^n x.$$

The Clifford exponential $e^{-i_n \omega \cdot x}$ is often called the *Clifford Fourier kernel*. For dimension $n = 3 \pmod{4}$, this kernel commutes with all elements of the Clifford algebra $Cl_{n,0}$, but for $n = 2 \pmod{4}$ it does not. Notice that the different commutation rules of the pseudoscalar i_n play a crucial rule in establishing the correlation theorems of the CFT and their important properties.

Theorem 2.1. Suppose that $\mathcal{F}\{f\} \in L^2(\mathbb{R}^n; Cl_{n,0})$. Then the CFT of $f \in L^2(\mathbb{R}^n; Cl_{n,0})$ is invertible and its inverse is calculated by the formula

$$(0.21) \quad \mathcal{F}^{-1}[\mathcal{F}\{f\}(\omega)](x) = f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(\omega) e^{i_n \omega \cdot x} d^n \omega.$$

For the sake of simplicity, if not otherwise stated, n is assumed to be $n = 2 \pmod{4}$ in the rest of this section.

3. Correlation on Clifford Algebra

In this section, we introduce the Clifford correlation and relationship between the Clifford correlation and Clifford convolution. We describe some useful properties of the Clifford correlation. We find that they are quite similar to those of the classical correlation.

Definition 3.1. Let $f, g : \mathbb{R}^n \rightarrow Cl_{n,0}$ be two multivector fields. The Clifford cross correlation of two multivectors $f, g \in L^2(\mathbb{R}^n; Cl_{n,0})$ is denoted $f \circ g$, and is defined by

$$\begin{aligned}
 (f \circ g)(x) &= f(x) \circ g(x) \\
 &= \int_{\mathbb{R}^n} \widetilde{f(y)} g(y+x) d^n y. \\
 (0.22) \quad &= \int_{\mathbb{R}^n} \sum_{A,B} \widetilde{e_A} e_B f_A(y) g_B(y+x) d^n y.
 \end{aligned}$$

Like the Clifford convolution (see [16]), it is clear that the Clifford correlation of f and g is a binary operation, which combines shifting, geometric product and integration.

Furthermore, when $f = g$ in equation (0.22), we obtain the Clifford auto correlation

$$(0.23) \quad (f \circ f)(x) = f(x) \circ f(x) = \int_{\mathbb{R}^n} \widetilde{f(y)} f(y+x) d^n y.$$

This means that the Clifford auto correlation is the Clifford cross correlation of a multivector with itself. The normalized Clifford autocorrelation function is defined by

$$(0.24) \quad \gamma(x) = \frac{\int_{\mathbb{R}^n} \widetilde{f(y)} * f(y+x) d^n y}{\int_{\mathbb{R}^n} \widetilde{f(y)} * f(y) d^n y}.$$

It is clear that $\gamma(0) = 0$.

It is not difficult to see the relationship between the Clifford convolution and Clifford correlation is given by

$$(0.25) \quad (f \circ g)(x) = \widetilde{f(-x)} \star g(x),$$

where \star denotes the Clifford convolution defined by

$$(0.26) \quad (f \star g)(x) = \int_{\mathbb{R}^n} f(x-y) g(y) d^n y.$$

In the following, we collect useful properties of the Clifford correlation, which will be used in the next section.

Lemma 3.1. For Clifford functions $f, g, h \in L^2(\mathbb{R}^n; Cl_{n,0})$ and for Clifford constants α and β we get

$$(0.27) \quad (f\alpha + g\beta) \circ h = \widetilde{\alpha}(f \circ h) + \widetilde{\beta}(g \circ h).$$

We also get for Clifford constants α and β

$$(0.28) \quad h \circ (f\alpha + g\beta) = (h \circ f)\alpha + (h \circ g)\beta.$$

Lemma 3.2. *Given a Clifford function $f \in L^2(\mathbb{R}^n; Cl_{n,0})$, let $\tau_a f$ denotes the shifted (translated) function defined by $\tau_a f = f(x - a)$. Then we get*

$$(0.29) \quad \tau_a(f \circ g)(x) = (\tau_{-a}f \circ g)(x) = (f \circ \tau_a g)(x),$$

and

$$(0.30) \quad \tau_a(g \circ f)(x) = (\tau_{-a}g \circ f)(x) = (g \circ \tau_a f)(x), \quad a \in \mathbb{R}^n.$$

Proof. We only prove (0.29), the proof of (0.30) can be proved similarly. A direct calculation gives

$$(0.31) \quad \begin{aligned} \tau_a(f \circ g)(x) &= \int_{\mathbb{R}^n} \tilde{f}(y)g(y + x - a) d^n y \\ &= \int_{\mathbb{R}^n} \widetilde{f(z + a)}g(z + x) d^n z \\ &= \int_{\mathbb{R}^n} \widetilde{\tau_{-a}f(z)}g(z + x) d^n z \\ &= (\tau_{-a}f \circ g)(x). \end{aligned}$$

For the last identity we have used the change of variables $z = y - a$. On the other hand, we easily get

$$\begin{aligned} \tau_a(f \circ g)(x) &= \int_{\mathbb{R}^n} \tilde{f}(y)g(y + x - a) d^n y \\ &= \int_{\mathbb{R}^n} \tilde{f}(y)\tau_a g(y + x) d^n y \\ &= (f \circ \tau_a g)(x). \end{aligned}$$

It can be easily seen that Lemma 3.2 shows that the Clifford correlations do not commute with translations (shifting). This fact is quite different from the Clifford convolution which commutes with translations.

Lemma 3.3. For all Clifford functions $f, g \in L^2(\mathbb{R}^n; Cl_{n,0})$ we have

$$(0.32) \quad \widetilde{(f \circ g)}(x) = (g \star \tilde{f})(-x).$$

Proof. A straightforward computation gives

$$\begin{aligned} \widetilde{(f \circ g)}(x) &= \int_{\mathbb{R}^n} \{\tilde{f}(y)g(y+x)\}^{\sim} d^n y \\ &\stackrel{(0.7)}{=} \int_{\mathbb{R}^n} \widetilde{g(y+x)} f(y) d^n y \\ &= \int_{\mathbb{R}^n} \tilde{g}(z) f(z-x) d^n z \\ &= \int_{\mathbb{R}^n} \tilde{g}(z) f(-(x-z)) d^n z \\ &= (\tilde{g} \star f)(-x), \end{aligned}$$

where in the last equality we have used the definition of the Clifford convolution of (0.26). This completes the proof.

Lemma 3.4. For all Clifford functions $f, g, h \in L^2(\mathbb{R}^n; Cl_{n,0})$ we have

$$(0.33) \quad ((f \star h) \circ g)(x) = (h(-) \circ (f \star g))(x).$$

Proof. We have, by the definition of the Clifford correlation (0.22) and Clifford convolution (0.26),

$$\begin{aligned} ((f \star h) \circ g)(x) &= \int_{\mathbb{R}^n} \widetilde{(f \star h)}(y) g(y+x) d^n y \\ &\stackrel{(0.7)}{=} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \{f(x)h(y-x)\}^{\sim} d^n x g(y+x) d^n y \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{h}(y-x) \tilde{f}(x) g(x+y) d^n x d^n y \\ &= \int_{\mathbb{R}^n} \tilde{h}(y-x) (f \circ g)(y) d^n y \\ &= (h(-) \circ (f \star g))(x), \end{aligned}$$

which was to be proved.

TABLE 1. Basic properties of the Clifford correlation.

Basic property	Clifford correlation
Linearity	$(\alpha f + \beta g) \circ h = \tilde{\alpha}(f \circ h) + \tilde{\beta}(f \circ h)$
	$h \circ (f\alpha + g\beta) = (h \circ f)\alpha + (h \circ g)\beta$
Shifting	$\tau_a(f \circ g) = (f \circ \tau_a g) = (\tau_{-a} f \circ g)$
	$\tau_a(g \circ f) = (g \circ \tau_a f) = (\tau_{-a} g \circ f), \quad a \in \mathbb{R}^n$
Reversion	$\widetilde{(f \circ g)} = (\tilde{g} \star f(-))$
	$(f \star h) \circ g = h(-) \circ (f \star g)$

We summarize in Table 1 the above basic properties of the Clifford correlation.

4. Correlation of CFT

In this section we shall present the main results of this paper. Let first us to describe the very important generalization of the CFT of the geometric product of two Clifford-valued functions.

Theorem 4.1. *Let $f, g \in L^2(\mathbb{R}^n; Cl_{n,0})$ be Clifford-valued functions. Let g_{odd} (g_{even}) denotes the odd (even) grade part of g . Then*

$$(0.34) \quad \mathcal{F}\{fg\}(\omega) = \frac{1}{(2\pi)^n} (\mathcal{F}\{f\} \star \mathcal{F}\{g_{\text{odd}}\})(-\omega) + \frac{1}{(2\pi)^n} (\mathcal{F}\{f\} \star \mathcal{F}\{g_{\text{even}}\})(\omega).$$

Proof. With the definition of the CFT (0.19) and inversion of the CFT (0.21), we easily obtain

$$\begin{aligned} \mathcal{F}\{fg\}(\omega) &= \int_{\mathbb{R}^n} f(x)g(x) e^{-i_n \omega \cdot x} d^n x \\ &= \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(v) e^{i_n v \cdot x} d^n v \right) g(x) e^{-i_n \omega \cdot x} d^n x. \end{aligned}$$

Splitting the multivector g into its even grade and odd grade parts and then taking inverse CFT we immediately get

$$\begin{aligned}
& \mathcal{F}\{fg\}(\omega) \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(v) e^{i_n v \cdot x} d^n v \right) (g_{\text{odd}}(x) + g_{\text{even}}(x)) e^{-i_n \omega \cdot x} d^n x \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(v) g_{\text{odd}}(x) e^{-i_n \omega \cdot x} e^{-i_n v \cdot x} d^n v d^n x \\
&\quad + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(v) g_{\text{even}}(x) e^{-i_n \omega \cdot x} e^{i_n v \cdot x} d^n v d^n x \\
&= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(v) \int_{\mathbb{R}^n} \mathcal{F}\{g_{\text{odd}}\}(\xi) e^{i_n \xi \cdot x} d^n \xi e^{-i_n \omega \cdot x} e^{-i_n v \cdot x} d^n v d^n x \\
&\quad + \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(v) \int_{\mathbb{R}^n} \mathcal{F}\{g_{\text{even}}\}(\xi) e^{i_n \xi \cdot x} d^n \xi e^{-i_n \omega \cdot x} e^{i_n v \cdot x} d^n v d^n x \\
&= \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(v) \mathcal{F}\{g_{\text{odd}}\}(\xi) \int_{\mathbb{R}^n} e^{i_n(\xi - \omega - v) \cdot x} d^n x d^n v d^n \xi \\
&\quad + \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(v) \mathcal{F}\{g_{\text{even}}\}(\xi) \int_{\mathbb{R}^n} e^{i_n(\xi - \omega + v) \cdot x} d^n x d^n v d^n \xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(v) \mathcal{F}\{g_{\text{odd}}\}(\xi) \delta(\xi - \omega - v) d^n v d^n \xi \\
&\quad + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(v) \mathcal{F}\{g_{\text{even}}\}(\xi) \delta(\xi - \omega + v) d^n v d^n \xi \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(v) \int_{\mathbb{R}^n} \mathcal{F}\{g_{\text{odd}}\}(\xi) \delta(\xi - \omega - v) d^n \xi d^n v \\
&\quad + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(v) \int_{\mathbb{R}^n} \mathcal{F}\{g_{\text{even}}\}(\xi) \delta(\xi - \omega + v) d^n \xi d^n v \\
&= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(v) \mathcal{F}\{g_{\text{odd}}\}(v - \omega) d^n v + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathcal{F}\{f\}(v) \mathcal{F}\{g_{\text{even}}\}(\omega - v) d^n v \\
&= \frac{1}{(2\pi)^n} (\mathcal{F}\{f\} \star \mathcal{F}\{g_{\text{odd}}\})(-\omega) + \frac{1}{(2\pi)^n} (\mathcal{F}\{f\} \star \mathcal{F}\{g_{\text{even}}\})(\omega),
\end{aligned}$$

and proof is complete.

As an immediate consequence of the above theorem we have the following which resembles the analogous result for the classical FT (see [9]).

Corollary 4.2. *Let $g \in L^2(\mathbb{R}^n; Cl_{n,0})$ with $n = 3 \pmod{4}$. Then one has for every multivector $f \in L^2(\mathbb{R}^n; Cl_{n,0})$*

$$\mathcal{F}\{fg\}(\omega) = \frac{1}{(2\pi)^n}(\mathcal{F}\{f\} \star \mathcal{F}\{g\})(\omega).$$

We next prove the following theorem which describes the relationship between the Clifford cross correlation and its CFT.

Theorem 4.3. *Let $f, g \in L^2(\mathbb{R}^n; Cl_{n,0})$ be two Clifford-valued functions. Let $f_{\text{odd}}(f_{\text{even}})$ and $g_{\text{odd}}(g_{\text{even}})$ denote the odd (even) grade components of f and g , respectively. Then*

$$\begin{aligned} \mathcal{F}\{f \circ g\}(\omega) &= (\{\mathcal{F}\{f_{\text{odd}}\}(\omega)\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(-\omega)\}^{\sim})\mathcal{F}\{g_{\text{odd}}\}(\omega) \\ (0.35) \quad &+ (\{\mathcal{F}\{f_{\text{odd}}\}(-\omega)\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(\omega)\}^{\sim})\mathcal{F}\{g_{\text{even}}\}(\omega). \end{aligned}$$

Proof. Using the definition of the CFT, it is obtained that

$$\begin{aligned} \mathcal{F}\{f \circ g\}(\omega) &= \int_{\mathbb{R}^n} (f \circ g) e^{-in\omega \cdot x} d^n x \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{f}(y) g(y+x) d^n y e^{-in\omega \cdot x} d^n x. \end{aligned}$$

Decomposing the multivector g into g_{even} and g_{odd} and making the change of variables $y+x = z$ we immediately obtain

$$\begin{aligned} &\mathcal{F}\{f \circ g\}(\omega) \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{f}(y) (g_{\text{odd}}(y+x) + g_{\text{even}}(y+x)) e^{-in\omega \cdot x} d^n x d^n y \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{f}(y) g_{\text{odd}}(y+x) e^{-in\omega \cdot x} d^n x d^n y + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{f}(y) g_{\text{even}}(y+x) e^{-in\omega \cdot x} d^n x d^n y \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{f}(y) g_{\text{odd}}(z) e^{-in\omega \cdot (z-y)} d^n z d^n y + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{f}(y) g_{\text{even}}(z) e^{-in\omega \cdot (z-y)} d^n z d^n y \\ &= \int_{\mathbb{R}^n} \tilde{f}(y) e^{-in\omega \cdot y} d^n y \mathcal{F}\{g_{\text{odd}}\}(\omega) + \int_{\mathbb{R}^n} \tilde{f}(y) e^{in\omega \cdot y} d^n y \mathcal{F}\{g_{\text{even}}\}(\omega). \end{aligned}$$

We split multivector f into f_{even} and f_{odd} and apply equation (0.7) to get

$$\begin{aligned} \mathcal{F}\{f \circ g\}(\omega) &= \int_{\mathbb{R}^n} (\widetilde{f_{\text{odd}}}(y) + \widetilde{f_{\text{even}}}(y)) e^{-in\omega \cdot y} d^n y \mathcal{F}\{g_{\text{odd}}\}(\omega) \\ &+ \int_{\mathbb{R}^n} (\widetilde{f_{\text{odd}}}(y) + \widetilde{f_{\text{even}}}(y)) e^{in\omega \cdot y} d^n y \mathcal{F}\{g_{\text{even}}\}(\omega). \end{aligned}$$

We know that the Clifford Fourier kernel commutes with the even multivector, but it anti-commutes with the odd multivector, then we obtain

$$\begin{aligned}
\mathcal{F}\{f \circ g\}(\boldsymbol{\omega}) &= \left(\int_{\mathbb{R}^n} \{e^{i_n \boldsymbol{\omega} \cdot y} f_{\text{odd}}(y)\}^{\sim} d^n y + \int_{\mathbb{R}^n} \{e^{i_n \boldsymbol{\omega} \cdot y} f_{\text{even}}(y)\}^{\sim} d^n y \right) \mathcal{F}\{g_{\text{odd}}\}(\boldsymbol{\omega}) \\
&\quad + \left(\int_{\mathbb{R}^n} \{e^{-i_n \boldsymbol{\omega} \cdot y} f_{\text{odd}}(y)\}^{\sim} d^n y + \int_{\mathbb{R}^n} \{e^{-i_n \boldsymbol{\omega} \cdot y} f_{\text{even}}(y)\}^{\sim} d^n y \right) \mathcal{F}\{g_{\text{even}}\}(\boldsymbol{\omega}) \\
&= \left(\int_{\mathbb{R}^n} \{f_{\text{odd}}(y) e^{-i_n \boldsymbol{\omega} \cdot y}\}^{\sim} d^n y + \int_{\mathbb{R}^n} \{f_{\text{even}}(y) e^{i_n \boldsymbol{\omega} \cdot y}\}^{\sim} d^n y \right) \mathcal{F}\{g_{\text{odd}}\}(\boldsymbol{\omega}) \\
&\quad + \left(\int_{\mathbb{R}^n} \{f_{\text{odd}}(y) e^{i_n \boldsymbol{\omega} \cdot y}\}^{\sim} d^n z + \int_{\mathbb{R}^n} \{f_{\text{even}}(y) e^{-i_n \boldsymbol{\omega} \cdot y}\}^{\sim} d^n z \right) \mathcal{F}\{g_{\text{even}}\}(\boldsymbol{\omega}) \\
&= (\{\mathcal{F}\{f_{\text{odd}}\}(\boldsymbol{\omega})\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(-\boldsymbol{\omega})\}^{\sim}) \mathcal{F}\{g_{\text{odd}}\}(\boldsymbol{\omega}) \\
&\quad + (\{\mathcal{F}\{f_{\text{odd}}\}(-\boldsymbol{\omega})\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(\boldsymbol{\omega})\}^{\sim}) \mathcal{F}\{g_{\text{even}}\}(\boldsymbol{\omega}),
\end{aligned}$$

which was to be proved.

From Theorem Theorem 4.3 we deduce the following lemma, which is very useful in proving the next results.

Lemma 4.4. *If $f \in L^2(\mathbb{R}^n; Cl_{n,0})$, then the CFT of the reverse of a multivector f equals to*

$$\begin{aligned}
\mathcal{F}\{\tilde{f}\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^n} \{f_{\text{odd}}(y) e^{-i_n \boldsymbol{\omega} \cdot y}\}^{\sim} d^n y + \int_{\mathbb{R}^n} \{f_{\text{even}}(y) e^{i_n \boldsymbol{\omega} \cdot y}\}^{\sim} d^n y \\
(0.36) \quad &= \{\mathcal{F}\{f_{\text{odd}}\}(\boldsymbol{\omega})\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(-\boldsymbol{\omega})\}^{\sim},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}\{\tilde{f}\}(-\boldsymbol{\omega}) &= \int_{\mathbb{R}^n} \{f_{\text{odd}}(y) e^{i_n \boldsymbol{\omega} \cdot y}\}^{\sim} d^n y + \int_{\mathbb{R}^n} \{f_{\text{even}}(y) e^{-i_n \boldsymbol{\omega} \cdot y}\}^{\sim} d^n y \\
(0.37) \quad &= \{\mathcal{F}\{f_{\text{odd}}\}(-\boldsymbol{\omega})\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(\boldsymbol{\omega})\}^{\sim}.
\end{aligned}$$

For the special case of the above theorem, an easy computation gives the following remark.

Remark 4.1. *Suppose that the multivector $g \in L^2(\mathbb{R}^n; Cl_{n,0})$. If f is a paravector, then equation (0.35) reduces to*

$$\begin{aligned}
\mathcal{F}\{f \circ g\}(\boldsymbol{\omega}) &= (\mathcal{F}\{f_0\}(-\boldsymbol{\omega}) + \sum_{i=1}^n e_i \mathcal{F}\{f_i\}(\boldsymbol{\omega})) \mathcal{F}\{g_{\text{odd}}\}(\boldsymbol{\omega}) \\
&\quad + (\mathcal{F}\{f_0\}(\boldsymbol{\omega}) + \sum_{i=1}^n e_i \mathcal{F}\{f_i\}(-\boldsymbol{\omega})) \mathcal{F}\{g_{\text{even}}\}(\boldsymbol{\omega}).
\end{aligned}$$

We also get the following corollary.

Corollary 4.5. *Suppose that the multivector $f \in L^2(\mathbb{R}^n; Cl_{n,0})$. For $g \in L^2(\mathbb{R}^n; Cl_{n,0})$ with $n = 3 \pmod{4}$, we have*

$$\mathcal{F}\{f \circ g\}(\omega) = (\{\mathcal{F}\{f_{\text{odd}}\}(-\omega)\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(\omega)\}^{\sim})\mathcal{F}\{g\}(\omega).$$

In the case $f, g \in L^2(\mathbb{R}^n; Cl_{n,0})$ with $n = 3 \pmod{4}$, we get

$$\mathcal{F}\{f \circ g\}(\omega) = \widetilde{\mathcal{F}\{f\}(\omega)}\mathcal{F}\{g\}(\omega).$$

Now we give an explicit proof of the reversion property of relationship between the CFT of correlation of two multivectors.

Theorem 4.6. *If $f, g \in L^2(\mathbb{R}^n; Cl_{n,0})$, then the following holds*

$$\begin{aligned} \mathcal{F}\{\widetilde{f \circ g}\}(\omega) &= (\{\mathcal{F}\{f_{\text{odd}}\}(\omega)\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(-\omega)\}^{\sim})\mathcal{F}\{g_{\text{odd}}\}(-\omega) \\ (0.38) \quad &+ (\{\mathcal{F}\{f_{\text{odd}}\}(-\omega)\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(\omega)\}^{\sim})\mathcal{F}\{g_{\text{even}}\}(-\omega). \end{aligned}$$

Proof. Decomposing the multivector f as $f_{\text{odd}} + f_{\text{even}}$ and applying the definition of the CFT, we immediately get

$$\begin{aligned} \mathcal{F}\{\widetilde{f \circ g}\}(\omega) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widetilde{g(y+x)} f(y) e^{-i_n \omega \cdot x} d^n y d^n x \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widetilde{g(y+x)} f_{\text{odd}}(y) e^{-i_n \omega \cdot x} d^n x d^n y \\ &\quad + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widetilde{g(y+x)} f_{\text{even}}(y) e^{-i_n \omega \cdot x} d^n x d^n y. \end{aligned}$$

Changing variables $z = y + x$ and applying Lemma 4.4 gives

$$\begin{aligned}
 & \mathcal{F}\{\widetilde{f \circ g}\}(\omega) \\
 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{g}(z) f_{\text{odd}}(y) e^{-i_n \omega \cdot (z-y)} d^n z d^n y + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{g}(z) f_{\text{even}}(y) e^{-i_n \omega \cdot (z-y)} d^n z d^n y \\
 &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{g}(z) e^{i_n \omega \cdot z} f_{\text{odd}}(y) e^{i_n \omega \cdot y} d^n y d^n z + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{g}(z) e^{-i_n \omega \cdot z} f_{\text{even}}(y) e^{i_n \omega \cdot y} d^n y d^n z \\
 &= \int_{\mathbb{R}^n} \tilde{g}(z) e^{i_n \omega \cdot z} d^n z \mathcal{F}\{f_{\text{odd}}\}(-\omega) + \int_{\mathbb{R}^n} \tilde{g}(z) e^{-i_n \omega \cdot z} d^n z \mathcal{F}\{f_{\text{even}}\}(-\omega) \\
 &= \mathcal{F}\{\tilde{g}\}(-\omega) \mathcal{F}\{f_{\text{odd}}\}(-\omega) + \mathcal{F}\{\tilde{g}\}(\omega) \mathcal{F}\{f_{\text{even}}\}(-\omega),
 \end{aligned}$$

which proves the theorem according to Lemma 4.4.

We next provide the relation between the shift property of the Clifford cross correlation of two Clifford functions and its CFT.

Theorem 4.7. *Let $f, g \in L^2(\mathbb{R}^n; Cl_{n,0})$ be Clifford-valued functions. Then*

$$\begin{aligned}
 \mathcal{F}\{f \circ \tau_a g\}(\omega) &= (e^{-i_n \omega \cdot a} \{\mathcal{F}\{f_{\text{odd}}\}(\omega)\}^{\sim} + e^{i_n \omega \cdot a} \{\mathcal{F}\{f_{\text{even}}\}(-\omega)\}^{\sim}) \mathcal{F}\{g_{\text{odd}}\}(\omega) \\
 &\quad + (e^{-i_n \omega \cdot a} \{\mathcal{F}\{f_{\text{odd}}\}(-\omega)\}^{\sim} + e^{i_n \omega \cdot a} \{\mathcal{F}\{f_{\text{even}}\}(\omega)\}^{\sim}) \mathcal{F}\{g_{\text{even}}\}(\omega).
 \end{aligned}$$

Consequently, for $g \in L^2(\mathbb{R}^n; Cl_{n,0})$, $n = 3 \pmod{4}$, we have

$$\mathcal{F}\{f \circ \tau_a g\}(\omega) = (e^{-i_n \omega \cdot a} \{\mathcal{F}\{f_{\text{odd}}\}(-\omega)\}^{\sim} + e^{i_n \omega \cdot a} \{\mathcal{F}\{f_{\text{even}}\}(\omega)\}^{\sim}) \mathcal{F}\{g_{\text{even}}\}(\omega).$$

Proof. An application of the CFT definition yields

$$\begin{aligned}
 \mathcal{F}\{f \circ \tau_a g\}(\omega) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{f}(y) g(y+x-a) d^n y e^{-i_n \omega \cdot x} d^n x \\
 &= \int_{\mathbb{R}^n} \tilde{f}(y) \left[\int_{\mathbb{R}^n} g(y+x-a) e^{-i_n \omega \cdot x} d^n x \right] d^n y.
 \end{aligned}$$

Making the change of variables $z = x - y - a$ and decomposing multivector g into its even grade and odd grade parts give

$$\begin{aligned} &\mathcal{F}\{f \circ \tau_a g\}(\omega) \\ &= \int_{\mathbb{R}^n} \tilde{f}(y) \left[\int_{\mathbb{R}^n} g(z) e^{i_n \omega \cdot z} d^n y \right] e^{-i_n \omega \cdot (a-y)} d^n z \\ &= \int_{\mathbb{R}^n} \tilde{f}(y) \left[\int_{\mathbb{R}^n} e^{i_n \omega \cdot (a-y)} g_{\text{odd}}(z) e^{i_n \omega \cdot z} + \int_{\mathbb{R}^n} e^{-i_n \omega \cdot (a-y)} g_{\text{even}}(z) e^{i_n \omega \cdot z} \right] d^n z \\ &= \int_{\mathbb{R}^n} \tilde{f}(y) e^{i_n \omega \cdot (a-y)} d^n y \mathcal{F}\{g_{\text{odd}}\}(\omega) + \int_{\mathbb{R}^n} \tilde{f}(y) e^{-i_n \omega \cdot (a-y)} d^n y \{g_{\text{even}}\}(\omega). \end{aligned}$$

Applying Lemma 4.4 to the above expression we easily obtain

$$\begin{aligned} &\mathcal{F}\{f \circ \tau_a g\}(\omega) \\ &= \int_{\mathbb{R}^n} \tilde{f}(y) e^{-i_n \omega \cdot y} e^{i_n \omega \cdot a} d^n y \mathcal{F}\{g_{\text{odd}}\}(\omega) + \int_{\mathbb{R}^n} \tilde{f}(y) e^{i_n \omega \cdot y} e^{-i_n \omega \cdot a} d^n y \{g_{\text{even}}\}(\omega) \\ &= \int_{\mathbb{R}^n} (\widetilde{f_{\text{odd}}}(y) + \widetilde{f_{\text{even}}}(y)) e^{i_n \omega \cdot (a-y)} d^n y \mathcal{F}\{g_{\text{odd}}\}(\omega) \\ &\quad + \int_{\mathbb{R}^n} (\widetilde{f_{\text{odd}}}(y) + \widetilde{f_{\text{even}}}(y)) e^{-i_n \omega \cdot (a-y)} d^n y \{g_{\text{even}}\}(\omega) \\ &= (e^{-i_n \omega \cdot a} \{\mathcal{F}\{f_{\text{odd}}\}(\omega)\}^\sim + e^{i_n \omega \cdot a} \{\mathcal{F}\{f_{\text{even}}\}(-\omega)\}^\sim) \mathcal{F}\{g_{\text{odd}}\}(\omega) \\ &\quad + (e^{-i_n \omega \cdot a} \{\mathcal{F}\{f_{\text{odd}}\}(-\omega)\}^\sim + e^{i_n \omega \cdot a} \{\mathcal{F}\{f_{\text{even}}\}(\omega)\}^\sim) \{g_{\text{even}}\}(\omega). \end{aligned}$$

This proves the required result.

Theorem 4.8. *Given two Clifford-valued functions $f, g \in L^2(\mathbb{R}^n; Cl_{n,0})$, then*

$$\begin{aligned} &\mathcal{F}\{\tau_a f \circ g\}(\omega) = \{\mathcal{F}\{f_{\text{odd}}\}(\omega)\}^\sim + \{\mathcal{F}\{f_{\text{even}}\}(-\omega)\}^\sim e^{-i_n \omega \cdot a} \mathcal{F}\{g_{\text{odd}}\}(\omega) \\ (0.39) \quad &\quad + \{\mathcal{F}\{f_{\text{odd}}\}(-\omega)\}^\sim + \{\mathcal{F}\{f_{\text{even}}\}(\omega)\}^\sim e^{i_n \omega \cdot a} \{g_{\text{even}}\}(\omega). \end{aligned}$$

Proof. A direct computation shows that

$$\begin{aligned}
& \mathcal{F}\{\tau_a f \circ g\}(\omega) \\
& \stackrel{(0.19)}{=} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \widetilde{f(y-a)} g(y+x) d^n y e^{-i_n \omega \cdot x} d^n x \\
& = \int_{\mathbb{R}^n} \widetilde{f(y-a)} \left[\int_{\mathbb{R}^n} g(y+x) e^{-i_n \omega \cdot x} d^n x \right] d^n y \\
& = \int_{\mathbb{R}^n} \widetilde{f(y-a)} \left[\int_{\mathbb{R}^n} g(z) e^{-i_n \omega \cdot (z-y)} d^n z \right] d^n y \\
& = \int_{\mathbb{R}^n} \widetilde{f(y-a)} \left[\int_{\mathbb{R}^n} e^{-i_n \omega \cdot y} g_{\text{odd}}(z) e^{-i_n \omega \cdot z} d^n z + \int_{\mathbb{R}^n} e^{i_n \omega \cdot y} g_{\text{even}}(z) e^{-i_n \omega \cdot z} d^n z \right] d^n y \\
& = \int_{\mathbb{R}^n} \tilde{f}(y) e^{-i_n \omega \cdot y} d^n y e^{-i_n \omega \cdot a} \mathcal{F}\{g_{\text{odd}}\}(\omega) + \int_{\mathbb{R}^n} \tilde{f}(y) e^{i_n \omega \cdot y} d^n y e^{i_n \omega \cdot a} \mathcal{F}\{g_{\text{even}}\}(\omega).
\end{aligned}$$

Therefore, applying Lemma 4.4 completes the proof of (0.39).

In order to see the relation between the modulation property of the correlation theorem and the CFT we have the following result.

Theorem 4.9 Let $f, g \in L^2(\mathbb{R}^n; Cl_{n,0})$ be two Clifford-valued functions. Then we obtain

$$\begin{aligned}
(0.40) \quad \mathcal{F}\{f \circ M_{\omega_0} g\}(\omega) &= (\{\mathcal{F}\{f_{\text{odd}}\}(\omega)\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(-\omega)\}^{\sim}) \mathcal{F}\{g_{\text{odd}}\}(\omega + \omega_0) \\
&+ (\{\mathcal{F}\{f_{\text{odd}}\}(-\omega)\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(\omega)\}^{\sim}) \mathcal{F}\{g_{\text{even}}\}(\omega - \omega_0),
\end{aligned}$$

and

$$\begin{aligned}
(0.41) \quad \mathcal{F}\{M_{\omega_0} f \circ g\}(\omega) &= (\{\mathcal{F}\{f_{\text{odd}}\}(\omega + \omega_0)\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(-\omega - \omega_0)\}^{\sim}) \mathcal{F}\{g_{\text{odd}}\}(\omega) \\
&+ (\{\mathcal{F}\{f_{\text{odd}}\}(\omega_0 - \omega)\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(\omega - \omega_0)\}^{\sim}) \mathcal{F}\{g_{\text{even}}\}(\omega).
\end{aligned}$$

Proof. For the proof of (0.40), a direct computation gives

$$\begin{aligned}
\mathcal{F}\{f \circ M_{\omega_0} g\}(\omega) & \stackrel{(0.19)}{=} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{f}(y) e^{i_n \omega_0 \cdot (x+y)} g(x+y) d^n y e^{-i_n \omega \cdot x} d^n x \\
& = \int_{\mathbb{R}^n} \tilde{f}(y) \left[\int_{\mathbb{R}^n} e^{i_n \omega_0 \cdot (x+y)} g(x+y) e^{-i_n \omega \cdot x} d^n x \right] d^n y \\
& = \int_{\mathbb{R}^n} \tilde{f}(y) \left[\int_{\mathbb{R}^n} e^{i_n \omega_0 \cdot z} g(z) e^{-i_n \omega \cdot (z-y)} d^n z \right] d^n y
\end{aligned}$$

$$\begin{aligned}
 &= \int_{\mathbb{R}^n} \tilde{f}(y) \left[\int_{\mathbb{R}^n} e^{i_n \omega_0 \cdot z} g(z) e^{-i_n \omega \cdot z} e^{i_n \omega \cdot y} d^n z \right] d^n y \\
 &= \int_{\mathbb{R}^n} \tilde{f}(y) \left[\int_{\mathbb{R}^n} e^{-i_n \omega \cdot y} g_{\text{odd}}(z) e^{-i_n(\omega + \omega_0) \cdot z} d^n z \right. \\
 &\quad \left. + \int_{\mathbb{R}^n} e^{i_n \omega \cdot y} g_{\text{even}}(z) e^{-i_n(\omega - \omega_0) \cdot z} d^n z \right] d^n y \\
 &= \int_{\mathbb{R}^n} \tilde{f}(y) e^{-i_n \omega \cdot y} d^n y \mathcal{F}\{g_{\text{odd}}\}(\omega + \omega_0) \\
 &\quad + \int_{\mathbb{R}^n} \tilde{f}(y) e^{i_n \omega \cdot y} d^n y \mathcal{F}\{g_{\text{even}}\}(\omega - \omega_0) \\
 &= \mathcal{F}\{\tilde{f}\}(\omega) \mathcal{F}\{g_{\text{odd}}\}(\omega + \omega_0) + \mathcal{F}\{\tilde{f}\}(-\omega) \mathcal{F}\{g_{\text{even}}\}(\omega - \omega_0).
 \end{aligned}$$

An application of Lemma 4.4 proves (0.40). For the proof of (0.41), we have

$$\begin{aligned}
 &\mathcal{F}\{M_{\omega_0} f \circ g\}(\omega) \\
 &\stackrel{(0.19)}{=} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \{e^{i_n \omega_0 \cdot y} f(y)\} \sim g(x+y) d^n y e^{-i_n \omega \cdot x} d^n x \\
 &= \int_{\mathbb{R}^n} \{e^{i_n \omega_0 \cdot y} f(y)\} \sim \left[\int_{\mathbb{R}^n} g(x+y) e^{-i_n \omega \cdot x} d^n x \right] d^n y \\
 &= \int_{\mathbb{R}^n} \{e^{i_n \omega_0 \cdot y} f(y)\} \sim \left[\int_{\mathbb{R}^n} g(z) e^{-i_n \omega \cdot (z-y)} d^n z \right] d^n y \\
 &= \int_{\mathbb{R}^n} \{e^{i_n \omega_0 \cdot y} f(y)\} \sim \left[\int_{\mathbb{R}^n} g(z) e^{-i_n \omega \cdot z} e^{i_n \omega \cdot y} d^n z \right] d^n y \\
 &= \int_{\mathbb{R}^n} \{e^{i_n \omega_0 \cdot y} f(y)\} \sim \left[\int_{\mathbb{R}^n} e^{-i_n \omega \cdot y} g_{\text{odd}}(z) e^{-i_n \omega \cdot z} d^n z \right. \\
 &\quad \left. + \int_{\mathbb{R}^n} e^{i_n \omega \cdot y} g_{\text{even}}(z) e^{-i_n \omega \cdot z} d^n z \right] d^n y \\
 &= \int_{\mathbb{R}^n} \{e^{i_n(\omega + \omega_0) \cdot y} f(y)\} \sim d^n y \mathcal{F}\{g_{\text{odd}}\}(\omega) + \int_{\mathbb{R}^n} \{e^{i_n(\omega_0 - \omega) \cdot y} f(y)\} \sim d^n y \mathcal{F}\{g_{\text{even}}\}(\omega) \\
 &= \int_{\mathbb{R}^n} \{f_{\text{odd}}(y) e^{-i_n(\omega + \omega_0) \cdot y} + f_{\text{even}}(y) e^{i_n(\omega + \omega_0) \cdot y}\} \sim d^n y \mathcal{F}\{g_{\text{odd}}\}(\omega) \\
 &\quad + \int_{\mathbb{R}^n} \{f_{\text{odd}}(y) e^{-i_n(\omega_0 - \omega) \cdot y} + f_{\text{even}}(y) e^{i_n(\omega_0 - \omega) \cdot y}\} \sim d^n y \mathcal{F}\{g_{\text{even}}\}(\omega) \\
 &= (\{\mathcal{F}\{f_{\text{odd}}\}(\omega + \omega_0)\} \sim + \{\mathcal{F}\{f_{\text{even}}\}(-\omega - \omega_0)\} \sim) \mathcal{F}\{g_{\text{odd}}\}(\omega) \\
 &\quad + (\{\mathcal{F}\{f_{\text{odd}}\}(\omega_0 - \omega)\} \sim + \{\mathcal{F}\{f_{\text{even}}\}(\omega - \omega_0)\} \sim) \mathcal{F}\{g_{\text{even}}\}(\omega).
 \end{aligned}$$

This gives the desired result.

We state the special case of the above theorem in the following remark.

Remark 4.2. It should be remembered that if $f, g \in L^2(\mathbb{R}^n; Cl_{n,0})$ with $n = 3 \pmod{4}$, then Theorem 4.9 reduces to

$$\mathcal{F}\{f \circ M_{\omega_0} g\}(\omega) = \mathcal{F}\{f\}(\omega) \mathcal{F}\{g\}(\omega - \omega_0),$$

and

$$\mathcal{F}\{M_{\omega_0} f \circ g\}(\omega) = \mathcal{F}\{f\}(\omega - \omega_0) \mathcal{F}\{g\}(\omega).$$

We finally establish the time-frequency shift of the convolution theorem for the CFT.

Theorem 4.10. Let $f, g \in L^2(\mathbb{R}^n; Cl_{n,0})$ be Clifford-valued functions. Then we obtain

$$\begin{aligned} (0.42) \quad \mathcal{F}\{M_{\omega_0} \tau_a f \circ g\}(\omega) &= (e^{i_n(\omega_0 + \omega) \cdot a} \{\mathcal{F}\{f_{\text{odd}}\}(\omega_0 + \omega)\})^{\sim} \\ &+ e^{i_n(\omega - \omega_0) \cdot a} \{\mathcal{F}\{f_{\text{even}}\}(\omega - \omega_0)\})^{\sim} \mathcal{F}\{g_{\text{odd}}\}(\omega) \\ &+ (e^{i_n(\omega_0 - \omega) \cdot a} \{\mathcal{F}\{f_{\text{odd}}\}(\omega_0 - \omega)\})^{\sim} \\ &+ e^{i_n(\omega - \omega_0) \cdot a} \{\mathcal{F}\{f_{\text{even}}\}(\omega - \omega_0)\})^{\sim} \mathcal{F}\{g_{\text{even}}\}(\omega). \end{aligned}$$

Proof. Applying equations (0.17) and (0.19) we immediately obtain

$$\begin{aligned} &\mathcal{F}\{M_{\omega_0} \tau_a f \circ g\}(\omega) \\ &= \int_{\mathbb{R}^n} (M_{\omega_0} \tau_a f \circ g) e^{-i_n \omega \cdot x} d^n x \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \{e^{i_n \omega_0 \cdot y} f(y - a)\}^{\sim} g(x + y) d^n y e^{-i_n \omega \cdot x} d^n x \\ &= \int_{\mathbb{R}^n} \{e^{i_n \omega_0 \cdot y} f(y - a)\}^{\sim} \left[\int_{\mathbb{R}^n} g(x + y) e^{-i_n \omega \cdot x} d^n x \right] d^n y \\ &= \int_{\mathbb{R}^n} \{e^{i_n \omega_0 \cdot y} f(y - a)\}^{\sim} \left[\int_{\mathbb{R}^n} g(z) e^{-i_n \omega \cdot (z - y)} d^n z \right] d^n y \\ &= \int_{\mathbb{R}^n} \{e^{i_n \omega_0 \cdot y} f(y - a)\}^{\sim} \left[\int_{\mathbb{R}^n} g(z) e^{i_n \omega \cdot y} e^{-i_n \omega \cdot z} d^n z \right] d^n y. \end{aligned}$$

We decompose the multivectors f and g into $f_{\text{odd}} + f_{\text{even}}$ and $g_{\text{odd}} + g_{\text{even}}$, respectively. Therefore, we obtain

$$\begin{aligned}
 \mathcal{F}\{M_{\omega_0}\tau_a f \circ g\}(\omega) &= \int_{\mathbb{R}^n} \{e^{in\omega_0 \cdot y} f(y-a)\}^{\sim} \left[\int_{\mathbb{R}^n} e^{-in\omega \cdot y} g_{\text{odd}}(z) e^{-in\omega \cdot z} d^n z \right. \\
 &\quad \left. + \int_{\mathbb{R}^n} e^{in\omega \cdot y} g_{\text{even}}(z) e^{-in\omega \cdot z} d^n z \right] d^n y \\
 &= \left[\int_{\mathbb{R}^n} \{f_{\text{odd}}(y-a) e^{-in(\omega_0+\omega) \cdot y}\}^{\sim} d^n y \right. \\
 &\quad \left. + \int_{\mathbb{R}^n} \{f_{\text{even}}(y-a) e^{-in(\omega-\omega_0) \cdot y}\}^{\sim} d^n y \right] \mathcal{F}\{g_{\text{odd}}\}(\omega) \\
 &\quad + \left[\int_{\mathbb{R}^n} \{f_{\text{odd}}(y-a) e^{-in(\omega_0-\omega) \cdot y}\}^{\sim} d^n y \right. \\
 &\quad \left. + \int_{\mathbb{R}^n} \{f_{\text{even}}(y-a) e^{-in(\omega-\omega_0) \cdot y}\}^{\sim} d^n y \right] \mathcal{F}\{g_{\text{even}}\}(\omega) \\
 &= \left[e^{in(\omega_0+\omega) \cdot a} \int_{\mathbb{R}^n} \{f_{\text{odd}}(y) e^{-in(\omega_0+\omega) \cdot y}\}^{\sim} d^n y \right. \\
 &\quad \left. + e^{in(\omega-\omega_0) \cdot a} \int_{\mathbb{R}^n} \{f_{\text{even}}(y) e^{-in(\omega-\omega_0) \cdot y}\}^{\sim} d^n y \right] \mathcal{F}\{g_{\text{odd}}\}(\omega) \\
 &\quad + \left[e^{in(\omega_0-\omega) \cdot a} \int_{\mathbb{R}^n} \{f_{\text{odd}}(y) e^{-in(\omega_0-\omega) \cdot y}\}^{\sim} d^n y \right. \\
 &\quad \left. + e^{in(\omega-\omega_0) \cdot a} \int_{\mathbb{R}^n} \{f_{\text{even}}(y) e^{-in(\omega-\omega_0) \cdot y}\}^{\sim} d^n y \right] \mathcal{F}\{g_{\text{even}}\}(\omega).
 \end{aligned}$$

which was to be proved.

Corollary 4.11. *It is not difficult to see that for $g \in L^2(\mathbb{R}^n; Cl_{n,0})$ with $n = 3 \pmod{4}$, Theorem 4.10 reduces to*

$$\begin{aligned}
 (0.43) \quad \mathcal{F}\{M_{\omega_0}\tau_a f \circ g\}(\omega) &= (e^{in(\omega_0-\omega) \cdot a} \{\mathcal{F}\{f_{\text{odd}}\}(\omega_0-\omega)\}^{\sim} \\
 &\quad + e^{in(\omega-\omega_0) \cdot a} \{\mathcal{F}\{f_{\text{even}}\}(\omega-\omega_0)\}^{\sim}) \mathcal{F}\{g\}(\omega),
 \end{aligned}$$

and if $f \in L^2(\mathbb{R}^n; Cl_{n,0})$ with $n = 3 \pmod{4}$, then

$$\begin{aligned}
 (0.44) \quad \mathcal{F}\{M_{\omega_0}\tau_a f \circ g\}(\omega) &= (e^{in(\omega-\omega_0) \cdot a} \{\mathcal{F}\{f\}(\omega-\omega_0)\}^{\sim} \mathcal{F}\{g_{\text{odd}}\}(\omega) \\
 &\quad + e^{in(\omega-\omega_0) \cdot a} \{\mathcal{F}\{f\}(\omega-\omega_0)\}^{\sim}) \mathcal{F}\{g_{\text{even}}\}(\omega).
 \end{aligned}$$

Theorem 4.12. For two Clifford-valued functions $f, g \in L^2(\mathbb{R}^n; Cl_{n,0})$, we have

$$\begin{aligned} & \mathcal{F}\{f \circ \tau_a M_{\omega_0} g\}(\omega) \\ &= (\{\mathcal{F}\{f_{\text{odd}}\}(-\omega - \omega_0)\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(\omega + \omega_0)\}^{\sim}) e^{i_n(\omega - \omega_0) \cdot a} \mathcal{F}\{g_{\text{odd}}\}(\omega) \\ & \quad + (\{\mathcal{F}\{f_{\text{odd}}\}(\omega - \omega_0)\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(\omega_0 - \omega)\}^{\sim}) e^{-i_n(\omega + \omega_0) \cdot a} \mathcal{F}\{g_{\text{even}}\}(\omega). \end{aligned}$$

Proof. By the definition of the CFT (0.19) and the Clifford convolution (0.22), we have

$$\begin{aligned} & \mathcal{F}\{f \circ \tau_a M_{\omega_0} g\}(\omega) \\ & \stackrel{(0.19)}{=} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{f}(y) e^{i_n \omega_0 \cdot (y-a)} g(x-y-a) d^n y e^{-i_n \omega \cdot x} d^n x \\ &= \int_{\mathbb{R}^n} \tilde{f}(y) e^{i_n \omega_0 \cdot (y-a)} \left[\int_{\mathbb{R}^n} g(x-y-a) e^{-i_n \omega \cdot x} d^n x \right] d^n y \\ &= \int_{\mathbb{R}^n} \tilde{f}(y) e^{i_n \omega_0 \cdot (y-a)} \left[\int_{\mathbb{R}^n} g(z-a) e^{-i_n \omega \cdot (y+z)} d^n z \right] d^n y \\ &= \int_{\mathbb{R}^n} \tilde{f}(y) e^{i_n \omega_0 \cdot (y-a)} \left[\int_{\mathbb{R}^n} g(z-a) e^{-i_n \omega \cdot y} e^{-i_n \omega \cdot z} d^n z \right] d^n y \\ &= \int_{\mathbb{R}^n} \tilde{f}(y) e^{i_n \omega_0 \cdot (y-a)} \left[\int_{\mathbb{R}^n} e^{i_n \omega \cdot y} g_{\text{odd}}(z-a) e^{-i_n \omega \cdot z} d^n z \right. \\ & \quad \left. + \int_{\mathbb{R}^n} e^{-i_n \omega \cdot y} g_{\text{even}}(z-a) e^{-i_n \omega \cdot z} d^n z \right] d^n y \\ &= \int_{\mathbb{R}^n} \tilde{f}(y) e^{-i_n(-\omega - \omega_0) \cdot y} d^n y e^{i_n(\omega - \omega_0) \cdot a} \mathcal{F}\{g_{\text{odd}}\}(\omega) \\ & \quad + \int_{\mathbb{R}^n} \tilde{f}(y) e^{-i_n(\omega - \omega_0) \cdot y} d^n y e^{-i_n(\omega + \omega_0) \cdot a} \mathcal{F}\{g_{\text{even}}\}(\omega) \\ &= (\{\mathcal{F}\{f_{\text{odd}}\}(-\omega - \omega_0)\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(\omega + \omega_0)\}^{\sim}) e^{i_n(\omega - \omega_0) \cdot a} \mathcal{F}\{g_{\text{odd}}\}(\omega) \\ & \quad + (\{\mathcal{F}\{f_{\text{odd}}\}(\omega - \omega_0)\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(\omega_0 - \omega)\}^{\sim}) e^{-i_n(\omega + \omega_0) \cdot a} \mathcal{F}\{g_{\text{even}}\}(\omega). \end{aligned}$$

Thus, the proof is complete.

Corollary 4.13. It is straightforward to check that for $g \in L^2(\mathbb{R}^n; Cl_{n,0})$ with $n = 3 \pmod{4}$,

Theorem 4.10 reduces to

$$\mathcal{F}\{f \circ \tau_a M_{\omega_0} g\}(\omega) = (\{\mathcal{F}\{f_{\text{odd}}\}(\omega - \omega_0)\}^{\sim} + \{\mathcal{F}\{f_{\text{even}}\}(\omega_0 - \omega)\}^{\sim}) e^{-i_n(\omega + \omega_0) \cdot a} \mathcal{F}\{g\}(\omega).$$

Conflict of Interests

The author declares that there is no conflict of interests.

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