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$*-G-$FRAMES IN HILBERT C^*-MODULES

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Abstract. In this paper, we introduce $*-g-$frames and study the operators associated with a give $*-g-$frames. We also show many useful properties with corresponding notions, g-frames and $*-$frames in Hilbert C*-modules.

Keywords: g-frame; $*$-frame operator; g-frame operator; C*-module; C*-algebra.

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1. Introduction and preliminaries

Frames for Hilbert spaces were introduced in 1952 by Duffin and Schaefer [8]. They abstracted the fundamental notion of Gabor [13] to study signal processing. Many generalizations of frames were introduced, *e.g.*, frames of subspaces [2], Pseudo-frames [18], oblique frames [6], G-frames [15], $*$ frames [1] in Hilbert spaces.

In 2000, Frank-Larson [11] introduced the notion of frames in Hilbert C^*-modules as a generalization of frames in Hilbert spaces. Recentely, A. Khosravi and B. Khosravi [15] introduced the $g-$frames theory in Hilbert C^*-modules, and Alijani, and Dehghan [1] introduced

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the $*-$frames theories in Hilbert C^*-modules. In this note, we introduce the $*-g-$frames which are generalizations of $g-$frames in Hilbert C^*-modules.

The content of the present note is as follow: we continue this introductory section with review of the definitions and basic properties of C^*-algebras, Hilbert C^*-modules, $*-$frame and g-frames. In Section 2, we introduce $*-g-$frame and present examples of such $*-g-$frame. Similar to the ordinary frames, $g-$frames and $*-$frames, we introduce the pre$-*-g-$frame transform and the $*-g-$frames operator. For information about Hilbert C^*-module, we refer authors to [10, 14, 17] and about $*-$frame, $g-$frame we refer authors to [1, 15]. Our reference for C^*-algebras as [7].

Let A be a unitary C^*-algebra and $a \in A$. The nonzero element a is called strictly nonzero if zero does not belong $\sigma(a)$, and a is said to be strictly positive if it is strictly nonzero and positive. If a is positive, there is a positive element $b \in A$ such that $b^2 = a$. The relation $' \leq'$ given by

$a \leq$ b if and only if $b - a$ is positive defines a partial ordering in A. Let be $a, b, c \in A$, we have

(i) if $a \leq b$, then $cac^* \leq cbc^*$. And if c commutes with a and b, then $ca \leq cb$ for $0 \leq c$.

(ii) $0 \leq a \leq b$ implies $||a|| \leq ||b||, ab \geq 0, a+b \geq 0$, and $a^t \leq b^t$ for $t \in (0,1)$

(iii) if $0 \leq a \leq b$ and a, b invertible elements then $0 \leq b^{-1} \leq a^{-1}$.

Now, we are going to introduce some of elementary definitions and the basic properties of Hilbert C^*-modules.

Definition 1.1. Let A be a C^*- algebra, a pre-Hilbert A-module is a left A-module X equipped with a sesquilinear map $\langle .,. \rangle : X \times X \to A$ satisfying

(1) $\langle x, x \rangle \geq 0; \langle x, x \rangle = 0$ if and only if $x = 0$ for all x in X,

(2) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all x, y, z in X, α, β in $\mathbb{C}$,

(3) $\langle x, y \rangle = \langle y, x \rangle^*$ for all x, y in X,

(4) $\langle ax, y \rangle = a \langle x, y \rangle$ for all x, y in X, a in A.

The map $\langle .,. \rangle$ is called an A-valued inner product of X, and for $x \in X$, we define $||x|| = ||\langle x, x \rangle||^{\frac{1}{2}}$ is a norm on X, where the latter norm denotes that in the C^*-algebra A . This norm makes X into a left normed module over A. A pre-Hilbert module X is called a Hilbert A-module if it is complete with respect to its norm. Examples of Hilbert C^*-modules are as follows:

(I) Every Hilbert space is a Hilbert C^*-module.

(II) Every C^*-algebra A is a Hilbert A -module via $\langle a,b\rangle = ab^* \ \ (a,b \in A)$.

(III) Let $\{Y_i, i \in I\}$ be a sequence of $A-$modules and

$\oplus Y_i = \{x = (x_i) : i \in Y_i, \sum_{i\in I} \langle x_i, x_i\rangle \ \textit{is norm convergent in } A\}$. Then $\oplus Y_i$ is a Hilbert $A-$modules with A- valued inner product $\langle (x_i)_{i\in I}, (y_i)_{i\in I}\rangle = \sum_{i\in I} \langle x_i, y_i\rangle$, point wise operations and the norm defined by $||x|| = ||\langle x,x\rangle||^{\frac{1}{2}}$.

Notice that the inner product structure of a C^*-algebra is essentially more complicated than complex numbers. One may define an A -valued norm $|.|$ by $|x| = \langle x,x\rangle^{\frac{1}{2}}$. Clearly, $\|x\| = \||x|\|$ for each $x \in X$.

It is known that $|.|$ does not satisfy the triangle inequality in general. Throughout this paper I and J be finite or countable index, sets, X and Y are countably or finitely generated Hilbert A-modules and $\{(Y_i) : i \in I\}$ is a sequence of closed sub-modules of Y. For each $i \in I, End^*(X,Y_i)$ is the collection of all adjointable A-linear maps from X to Y_i and $End^*(X,X)$ is denoted by $End^*(X)$.

Definition 1.2 [15] A sequence $\{\Lambda_i \in End^*(X,Y_i) : i \in I\}$ is called $g-$frame in X with respect to $\{Y_i : i \in I\}$ if there exist constant reel $C,D > 0$ such that for every $x \in X$,

$$C\langle x,x\rangle \leq \sum_{i\in I} \langle \Lambda_i(x), \Lambda_i(x)\rangle \leq D\langle x,x\rangle, \forall x \in X. \tag{1.1}$$

The elements C and D are called the lower and upper $g-$ frame bounds respectively of $\{\Lambda_i, i \in I\}$ with respect to $\{Y_i : i \in I\}$.

Definition 1.3. [1] A sequence $\{x_i : i \in I\}$ of X is called a $*-$frame for X if there exist two strictly nonzero elements C,D in A such that for every $x \in X$,

$$C\langle x,x\rangle C^* \leq \sum_{i\in I} \langle x,x_i\rangle \langle x_i,x\rangle \leq D\langle x,x\rangle D^*. \tag{1.2}$$

The elements C and D are called the lower and upper $*-$ frame bounds respectively.

Throughout the paper we need the following lemma.

Lemma 1.4. [17] *Let X and Y two Hilbert $A-$modules and $T \in End^*(X,Y)$. Then*

(i) if T is injective and T has closed range, then the adjointable map T^*T is invertible and $||(T^*T)^{-1}||^{-1} \leq T^*T \leq ||T||^2$.

(ii) if T is surjective, then the adjointable map TT^* is invertible and $||(TT^*)^{-1}||^{-1} \leq TT^* \leq ||T||^2$.

2. Main results

Alijani and Dehghan in [1] introduced $*$-frames and A. Khosravi and B. Khosravi in [15] introduced g-frames for Hilbert $C*$-modules. Our next definition and example are generalizations of (3.1) and (3.2) in [15].

Definition 2.1. A sequence $\{\Lambda_i \in End^*(X,Y_i) : i \in I\}$ is called $*-g-$frame with respect to $\{Y_i : i \in I\}$ if there exist C,D strictly nonzero of A such that for every $x \in X$,

$$C\langle x,x\rangle C^* \leq \sum_{i\in I}\langle \Lambda_i(x),\Lambda_i(x)\rangle \leq D\langle x,x\rangle D^*. \tag{2.1}$$

The elements C and D are called the lower and upper $*-g-$ frame bounds respectively in X with respect to $\{Y_i, i \in I\}$. Since A is not a partial ordered set, lower and upper $*-g-$ frame bounds may not have order and the optimal bounds may not exist.
If $\lambda = C = D$, then the $*-g-$ frame is said to be a λ-tight $*-g-$ frame and if $C = D = 1_A$, it is called a Parseval $*-g-$ frame or a normalized $*-g-$ frame. The $*-g-$ frame is standard if for every $x \in X$, the sum in (2.1) converges in norm.

Example 2.2. Let $\{x_i, i \in I\}$ be a $*-$frame of X with lower and upper, C and D, respectively. For each $i \in I$, we define $T_{x_i} : X \to A$, by $T_{x_i}(x) = \langle x, x_i\rangle$ for all $x \in X$. As example in [15], T_{x_i} is adjointable and $T^*_{x_i}(a) = ax_i$ for each $a \in A$. And we have,

$$C\langle x,x\rangle C^* \leq \sum_{i\in I}\langle x,x_i\rangle\langle x_i,x\rangle \leq D\langle x,x\rangle D^*, \forall x \in X.$$

Then

$$C\langle x,x\rangle C^* \leq \sum_{i\in I}\langle T_{x_i}(x),T_{x_i}(x)\rangle \leq D\langle x,x\rangle D^*,$$

for all $x \in X$. So, $\{T_{x_i}, i \in I\}$ is a $*-g$ -frame with bounds C and D, respectively, in X with respect to A.

Now we studies the corresponding operators of a $*-g-$frame.

Theorem 2.3. *Let $\{\Lambda_i \in End^*(X,Y_i) : i \in I\}$ be $*-g-$frame with lower and upper, C and D, respectively. The $*-g-$frame transform or pre $-*-g-$frame operator $T : X \longrightarrow \oplus Y_i$ defined by $T(x) = (\Lambda_i(x))_{i\in I}$ is injective, closed range adjointable A-module map and $||T|| \leq ||D||$. The adjointable T^* is surjective and it given by: $T^*(y) = \sum_{i\in I} \Lambda_i^*(y_i)$ where $y = (y_i)_{i\in I} \in \oplus Y_i$.*

Proof. Let x be a vector of X. We have

$$||T(x)||^2 = ||\langle(\Lambda_i(x))_{i\in I},(\Lambda_i(x))_{i\in I}\rangle||, \qquad (2.2)$$

and by definition of the norm in $\oplus Y_i$, we have

$$\begin{aligned} ||\langle(\Lambda_i(x))_{i\in I},(\Lambda_i(x))_{i\in I}\rangle|| &= ||\sum_{i\in I}\langle\Lambda_i(x),\Lambda_i(x)\rangle|| \\ &\leq ||D\langle\Lambda_i x,\Lambda_i x\rangle D^*|| \\ &= ||D||^2||x||^2, \end{aligned}$$

for all $x \in X$. Then

$$||T(x)||^2 \leq ||D||^2||x||^2, \quad \forall x \in X. \qquad (2.3)$$

So T is well defined and $||T|| \leq ||D||$. Thus $\Lambda_i \in End^*(X,Y_i), T$ is a linear A-module map.
We now show that R_T is closed. Let $\{Tx_n\}$ be a sequence in the range of T such that

$$\lim_{n\to\infty} Tx_n = y.$$

The definition of $*-g-$frame concludes that

$$C\langle x_n - x_m, x_n - x_m\rangle C^* \leq \sum_{i\in I}\langle\Lambda_i(x_n - x_m),\Lambda_i(x_n - x_m)\rangle \leq D\langle x_n - x_m, x_n - x_m\rangle D^*, \qquad (2.4),$$

which is equivalent to

$$C\langle x_n - x_m, x_n - x_m\rangle C^* \leq \langle T(x_n - x_m), T(x_n - x_m)\rangle \leq D\langle x_n - x_m, x_n - x_m\rangle D^*.$$

Hence, we have

$$||C\langle x_n - x_m, x_n - x_m\rangle C^*|| \leq ||T(x_n - x_m)||^2.$$

Since

$$\lim_{n\to\infty} T(x_n - x_m) = 0; \ \lim_{n\to\infty} C\langle x_n - x_m, x_n - x_m\rangle C^* = 0,$$

we have

$$||\langle x_n-x_m,x_n-x_m\rangle|| \leq ||C^{-1}||^2||C\langle x_n-x_m,x_n-x_m\rangle C^*||.$$

Hence, there exists $x\in X$, such that $\lim_{n\to\infty}x_n=x$,

$$\begin{aligned}||T(x_n-x)||^2 &= ||\langle(\Lambda_i(x_n-x))_{i\in I},(\Lambda_i(x_n-x))_{i\in I}\rangle||\\ &= ||\sum\langle\Lambda_i(x_n-x),\Lambda_i(x_n-x)\rangle||\\ &\leq ||D||^2||x_n-x||.\end{aligned}$$

Thus $\lim_{n\to\infty}Tx_n=y$, so range of T is closed.

We show that T is injective: Suppose that $x\in X$ and $Tx=0$. We have

$$\begin{aligned}||\langle x,x\rangle|| &= ||C^{-1}C\langle x,x\rangle C^*(C^*)^{-1}||\\ &\leq ||C^{-1}||^2||C\langle x,x\rangle C^*||\\ &\leq ||C^{-1}||^2||\sum_{i\in I}\langle\Lambda_i(x),\Lambda_i(x)\rangle||\\ &= ||C^{-1}||^2||\langle(\Lambda_i(x))_{i\in I},(\Lambda_i(x))_{i\in I}\rangle||\\ &= ||C^{-1}||^2||Tx||^2.\end{aligned}$$

Thus $x=0$, and T is injective.

We determine T^*: Let be $x\in X$ and $(y_i)\in\oplus Y_i$. We have $\langle Tx,(y_i)_{i\in I}\rangle=\langle(\Lambda_i(x))_{i\in I},(y_i)_{i\in I}\rangle$.

And by definition of the norm in $\oplus Y_i$, we have

$$\langle(\Lambda_i(x))_{i\in I},(y_i)_{i\in I}\rangle=\sum_{i\in I}\langle\Lambda_i x,y_i\rangle.$$

Then

$$\langle Tx,(y_i)_{i\in I}\rangle=\sum_{i\in I}\langle x,\Lambda_i^*y_i\rangle.$$

So $T^*((y_i)_{i\in I})=\sum_{i\in I}\Lambda_i^*y_i$. By injectivity of T, the operator T^* has closed range and $X=R_{T^*}$.

This completes the proof.

Now we define $*-g-$ frame operator and studies some of its properties.

Definition 2.4. Let $\{\Lambda_i\in End^*(X,V_i):i\in I\}$ be a $*-g-$frame with lower and upper $*-g-$frame, C and D. Then its $*-g-$ frame operator S is defined by: $S(x)=T^*T(x)=\sum_{i\in I}\Lambda_i^*\Lambda_i(x)$, $(\forall x\in X)$.

Theorem 2.5. *Let $\{\Lambda_i \in End^*(X,Y_i) : i \in I\}$ be a $*-g-$frame with lower and upper $*-g-$frame; C and D, respectively, and with The $*-g-$ frame operator S. Then S, positive , invertible, adjointable and $||C^{-1}||^{-2} \leq ||S|| \leq ||D||^2$.*

Proof. By Lemma 1.4, and Theorem 2.3, S is invertible, positive and self-adjointable map. The definition of $*-g-$frame and of the operator S, concludes that

$$\langle Sx, x\rangle = \sum_{i\in I} \langle \Lambda_i(x)), \Lambda_i(x))\rangle \leq D\langle x,x\rangle D^*$$

and

$$\langle x,x\rangle \leq C^{-1}\sum_{i\in I} \langle \Lambda_i(x)), \Lambda_i(x))\rangle (C^*)^{-1}, \forall x \in X.$$

Then

$$||C^{-1}||^{-2}|||| \langle x,x\rangle || \leq || \langle Sx,x\rangle || \leq ||D||^2|| \langle x,x\rangle ||$$

for all $x \in X$. So

$$||C^{-1}||^{-2} \leq ||S|| \leq ||D||^2.$$

This completes the proof.

Now we gave a generalization for Theorem 3.3 in [15].

Theorem 2.6. *Let for every $i \in I, \Lambda_i \in End^*(X,Y_i)$ and $\{y_{i,j}, j \in I_i\}$ be a $*$-frame for Y_i with frame bounds C_i, D_i, such there exist C, D strictly nonzero of A such that*

$$CaC^* \leq C_iaC_i^* \text{ and } D_iaD_i^* \leq DaD^*, \tag{2.5}$$

for all positive a of A.

Then the following conditions are equivalent

(i) $\{\Lambda_i^*(y_{i,j}), j \in I_i\}$ is a $*$-frame for X.

(ii) $\{\Lambda_i, i \in I\}$ is a $*-g$-frame for X.

Proof. Let $i \in I$, since $\{y_{i,j}, j \in I_i\}$ is a $*$-frame for Y_i with bounds C_i, D_i, we have

$$C_i\langle \Lambda_ix, \Lambda_ix\rangle C_i^* \leq \sum_{j\in I_i} \langle \Lambda_ix, y_{i,j}\rangle \langle y_{i,j}, \Lambda_ix\rangle \leq D_i\langle \Lambda_ix, \Lambda_ix\rangle D_i^*, \tag{2.6}$$

for all $x \in X$. So

$$C_i\langle \Lambda_ix, \Lambda_ix\rangle C_i^* \leq \sum_{j\in I_i} \langle x, \Lambda_i^*y_{i,j}\rangle \langle \Lambda_i^*y_{i,j}, x\rangle \leq D_i\langle \Lambda_ix, \Lambda_ix\rangle D_i^*, \forall x \in X. \tag{2.7}$$

And by using the conditions (2.5), we deduce

$$C\langle\Lambda_i x,\Lambda_i x\rangle C^* \leq \sum_{j\in I_i}\left\langle x,\Lambda_i^* y_{i,j}\right\rangle\left\langle\Lambda_i^* y_{i,j},x\right\rangle \leq D\langle\Lambda_i x,\Lambda_i x\rangle D^*, \forall x\in X. \tag{2.8}$$

Then

$$\begin{aligned} C\sum_i\langle\Lambda_i x,\Lambda_i x\rangle C^* \leq & \sum_i\sum_{j\in I_i}\left\langle x,\Lambda_i^* y_{i,j}\right\rangle\left\langle\Lambda_i^* y_{i,j},x\right\rangle \\ \leq & D\sum_i\langle\Lambda_i x,\Lambda_i x\rangle D^*, \forall x\in X. \end{aligned} \tag{2.9}$$

And, if we suppose the condition $\left\{\Lambda_i^*(y_{i,j}), j\in I_i\right\}$ is a $*$-frame for X with frame bounds $C^{'},D^{'}$, we have

$$C^{'}\langle x,x\rangle C^{'*} \leq \sum_i\sum_{j\in I_i}\left\langle x,\Lambda_i^* y_{i,j}\right\rangle\left\langle\Lambda_i^* y_{i,j},x\right\rangle \leq D^{'}\langle x,x\rangle D^{'*}, \forall x\in X. \tag{2.10}$$

So, by combining (2.9) and (2.10), we get

$$C\sum_i\langle\Lambda_i x,\Lambda_i x\rangle C^* \leq D^{'}\langle x,x\rangle D^{'*}$$

and

$$C^{'}\langle x,x\rangle C^{'*} \leq D\sum_i\langle\Lambda_i x,\Lambda_i x\rangle D^*, \forall x\in X.$$

Then

$$D^{-1}C^{'}\langle x,x\rangle C^{'*}(D^*)^{-1} \leq \sum_i\langle\Lambda_i x,\Lambda_i x\rangle \leq C^{-1}D^{'}\langle x,x\rangle (D^{'*})(C^*)^{-1}, \forall x\in X.$$

So $(i)\Rightarrow(ii)$. Next, we show the Converse. Similarly we have

$$C\sum_i\langle\Lambda_i x,\Lambda_i x\rangle C^* \leq \sum_i\sum_{j\in I_i}\left\langle x,\Lambda_i^* y_{i,j}\right\rangle\left\langle\Lambda_i^* y_{i,j},x\right\rangle \leq\leq D\sum\langle\Lambda_i x,\Lambda_i x\rangle D^*, \forall x\in X. \tag{2.11}$$

If we suppose $\{\Lambda_i, i\in I\}$ is a $*-g-$ frame, with frame bounds $C^{'},D^{'}$, so

$$D\sum_i\langle\Lambda_i x,\Lambda_i x\rangle D^* \leq DD^{'}\langle x,x\rangle D^{'*}D^*$$

and

$$CC^{'}\langle x,x\rangle C^{'*}C^* \leq C\sum_i\langle\Lambda_i x,\Lambda_i x\rangle C^*, \forall x\in X.$$

Then

$$CC^{'}\langle x,x\rangle C^{'*}C^* \leq \sum_i\sum_{j\in I_i}\left\langle x,\Lambda_i^* y_{i,j}\right\rangle\left\langle\Lambda_i^* y_{i,j},x\right\rangle \leq DD^{'}\langle x,x\rangle D^{'*}D^*, \forall x\in X.$$

This completes the proof.

The next result is analog to Corollary 3.4 in [15].

Corollary 2.7. *Let for every $i \in I, \Lambda_i \in End^*(X, Y_i)$ and $\left\{x_{i,j}, j \in I_i\right\}$ and be a Parseval $*$-frames for Y_i. Then we have*

(i) $\{\Lambda_i, i \in I\}$ is a $* - g-$frame (resp. $* - g-$Bessel sequence, tight $* - g-$frame) for X if only if $\left\{\Lambda_i^* x_{i,j}, i \in I, j \in I_i\right\}$ is a $*-$frames (resp. Bessel sequence, tight frame) for X

(ii) The $* - g$- frame operator of $\{\Lambda_i, i \in I\}$ is the $*-$ frame operator of $\left\{\Lambda_i^* x_{i,j}, i \in I, j \in I_i\right\}$

Proof. (i) Follow from the theorem 2.6.

(ii) Letting $x \in X$ and $y \in Y$, we have

$$\begin{aligned} \langle \Lambda_i^* y, x\rangle = & \langle y, \Lambda_i x\rangle \\ = & \sum \left\langle y, x_{i,j}\right\rangle \left\langle x_{i,j}, \Lambda_i x\right\rangle \\ = & \sum \left\langle y, x_{i,j}\right\rangle \left\langle \Lambda_i^* x_{i,j}, x\right\rangle \\ = & \sum \left\langle \left\langle y, x_{i,j}\right\rangle \Lambda_i^* x_{i,j}, x\right\rangle. \end{aligned}$$

Then $\Lambda_i^* y = \sum_j \left\langle y, x_{i,j}\right\rangle \Lambda_i^* x_{i,j}$. So

$$\begin{aligned} \sum_i \Lambda_i^* \Lambda_i x = & \sum_i \sum_j \left\langle \Lambda_i x, x_{i,j}\right\rangle \Lambda_i^* x_{i,j}. \\ = & \sum_i \sum_j \left\langle x, \Lambda_i^* x_{i,j}\right\rangle \Lambda_i^* x_{i,j}. \end{aligned}$$

Since, the $*$ frame operator of $\left\{\Lambda_i^* x_{i,j}, i \in I, j \in I_i\right\}$, is defined by: $S^{'}(x) = \sum_i \sum_j \left\langle x, \Lambda_i^* x_{i,j}\right\rangle \Lambda_i^* x_{i,j}$, see [1]. Then the $* - g$- frame operator of $\{\Lambda_i, i \in I\}$ is the $*-$ frame operator of $\left\{\Lambda_i^* x_{i,j}, i \in I, j \in I_i\right\}$. This completes the proof.

Now we gave a generalization of Theorem 3.5 in [15].

Theorem 2.8. *Let $\{\Lambda_i \in End^*(X, Y_i) : i \in I\}$ be a $* - g$- frame with lower and upper $* - g-$frame; C and D, respectively and with The $* - g$- frame operator S, and M be a Hilbert $A-$module and let $T \in End^*(M, X)$ be invertible. Then $\{\Lambda_i T \in End^*(M, Y_i, i \in I\})$ is a $* - g-$frame with $* - g-$ frame operator $T^* S T$ with bounds $||T^{-1}||^{-1} C, ||T|| D$.*

Proof. We have

$$C \langle Tx, Tx\rangle C^* \leq \sum_{i \in I} \langle \Lambda_i Tx, \Lambda_i Tx\rangle \leq D \langle Tx, Tx\rangle D^*, \tag{2.12}$$

for all $x \in M$. Using Lemma 1.4, we have $||(TT^*)^{-1}||^{-1}\langle x,x\rangle \leq \langle Tx,Tx\rangle$. for all $x \in M$. Or $||T^{-1}||^{-2} \leq ||(TT^*)^{-1}||^{-1}$. This implies

$$C||T^{-1}||^{-2}\langle x,x\rangle C^* \leq C\langle Tx,Tx\rangle C^*. \tag{2.13}$$

for all $x \in M$. And we know that $\langle Tx,Tx\rangle = \langle T^*Tx,x\rangle \leq ||T^*T||\langle x,x\rangle$. for all $x \in M$. This implies that

$$D\langle Tx,Tx\rangle D^* \leq D||T||\langle x,x\rangle (D||T||)^* \tag{2.14}$$

for all $x \in M$. Using (2.12), (2.13), (2.14) we have

$$||T^{-1}||C\langle x,x\rangle \ (||T^{-1}||C)^* \leq \sum_{i\in I}\langle \Lambda_i Tx, \Lambda_i Tx\rangle \leq ||T||D\langle x,x\rangle (||T||D)^*. \tag{2.15}$$

for all $x \in M$. So $\{\Lambda_i T \in End^*(M,Y_i, i \in I\})$ is a $*-g-$frame with bounds $||T^{-1}||C, D||T||$. Moreover for every $x \in M$, we have

$$T^*ST(x) = T^*\sum_{i\in I}\Lambda_j^*\Lambda_j T(x) = \sum_{i\in I} T*\Lambda_j^*\Lambda_j T(x) = \sum_{i\in I}(\Lambda_j T)^*\Lambda_j T(x).$$

for all $x \in M$. This completes the proof.

The next result is a generalization of Corollary 3.6 in [1].

Corollary 2.9. *Let $\{\Lambda_i \in End^*(X,Y_i) : i \in I\}$ be a $*-g$- frame with lower and upper $*-g-$frame; C and D, respectively and with The $*-g$- frame operator S. Then $\{\Lambda_i S^{-1} \in End^*(X,Y_i) : i \in I\}$ is a $*-g$- frame with lower and upper $*-g-$frame; $||D||^{-2}$ and $||C^{-1}||^2$, respectively, $*-g$- frame operator S^{-1} and for every $x \in X, x = \sum \Lambda_i S^{-1}\Lambda_i^* = \sum(\Lambda_i S^{-1})^*\Lambda_i$*

Proof. By taking $M = X$ and $T = S^{-1}$ in Theorem (2.8), it follow that $\{\Lambda_i S^{-1} \in End^*(X,Y_i) : i \in I\}$ is a $*-g$- frame where the $*-g$- frame operator is S^{-1} and, we have
$||S||^{-1}\langle x,x\rangle \leq \langle S^{-1}x,x\rangle = \sum\langle \Lambda_i S^{-1}x, \Lambda_i S^{-1}x\rangle \leq ||S^{-1}||\langle x,x\rangle, \forall x \in X$.
Or, we have $||S^{-1}|| \leq ||C^{-1}||^2$ and $||S|| \leq ||D||^2$, so $\{\Lambda_i S^{-1} \in End^*(X,Y_i) : i \in I\}$ is a $*-g$-frame with lower and upper $*-g-$frame; $||D||^{-2}$ and $||C^{-1}||^2, \forall x \in X$. Moreover since for every $i \in I, (\Lambda_i S^{-1})^* = S^{-1}\Lambda_i^*$ and for every $x \in X, x = S^{-1}Sx = SS^{-1}x$, then $x = \sum \Lambda_i S^{-1}\Lambda_i^* x = \sum(\Lambda_i S^{-1})^*\Lambda_i x$. This complete the proof.

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REFERENCES

[1] A. Alijani, M. Dehghan, $*$-frames in Hilbert $C*$-modules, U. P. B. Sci. Bull. Series A 2011.

[2] M. S. Asgari, A. Khorsavi, Frames and Bases of subspaces in Hilbert spaces, J. Math. Anal. Appl. 308 (2005), 541-553.

[3] D. Bakic, B. Guljas, Hilbert $C*$-modules over $C*$-algebras of compact operators, Acta Sci. Math, 68 (2002), 249-269.

[4] N. Bounader, A. Chahbi, Selberg type inequalities in Hilbert C*-modules, Int. J. Math. Anal. 7 (2013), 385-391.

[5] P. G. Casazza, G. Kutyniok, Frames of subspaces, Contemporary Math. 2004.

[6] O. Christensen, Y. C. Eldar, Oblique dual frames and shift-invariant spaces, Appl. Comput. Harmon. Anal. 17 (2004), 48-68.

[7] F. R. Davidson, C^*-algebra by example. Fields Ins. Monog. 1996.

[8] R. J. Duffin, A. C. Schaeffer, A class of nonharmonic fourier series, Trans. Amer. Math. Soc. 72 (1952), 341-366.

[9] M. Frank, Self-duality and C^*-reflexivity of Hilbert C^*-modules. Zeitschr Anal. Anw. 9 (1990), 165-176.

[10] M. Frank, Geometrical aspects of Hilbert C^*-modules, Positivity 1998.

[11] M. Frank, D. R. Larson, A-module frame concept for Hilbert C^*-modules, functinal and harmonic analysis of wavelets, Contempt. Math. 247 (2000), 207-233.

[12] M. Frank, D. R. Larson, Frames in Hilbert C^*- modules and C^*-algebras, J. Oper. Theory 48 (2002), 273-314.

[13] D. Gabor, Theory of communications, J. Elec. Eng. 93 (1946), 429-457.

[14] I. Kaplansky, Modules over operator algebras, Amer. J. Math. 75 (1953), 839-858.

[15] A. Khorsavi, B. Khorsavi, Fusion frames and g-frames in Hilbert C^*-modules, Int. J. Wavelet, Multiresolution and Information Processing 6 (2008), 433-446.

[16] A. Khorsavi, K. Musazadeh, Fusion frames and $g-$frames, J. Math. Anal. Appl. 343 (2008), 1068-1083.

[17] E. C. Lance, Hilbert C^*-modules: A Toolkit for Operator Algebraists, London Math. Soc. Lecture Series 1995.

[18] S. Li, H. Ogawa, Pseudoframes for subspaces with applications, J. Fourier Anal. Appl. 10 (2004), 409-431.

[19] W. Sun, G-frame and G-Riesz Bases, J. Math. Appl. 32 (2006), 437-456.