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ON THE RANK 1 DECOMPOSITIONS OF SYMMETRIC TENSORS

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Abstract. Here we study the uniqueness of a representation of a homogeneous polynomial as a sum of a small number of powers of linear forms (equivalently, a representation of a symmetric tensor as a sum of powers) or (when it is not unique) describe all such additive decompositions. We require a linear upper bound for the number of addenda with respect to the degree of the polynomial and, for some results, assumptions like linearly general position.

Keywords: Waring problem; Polynomial decomposition; Symmetric tensor rank; Symmetric rank; Symmetric tensors.

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1. Introduction

Let $\mathbb{K}$ be an algebraically closed base field with characteristic zero. For any finite subset A of a projective space let $\langle A\rangle$ denote its linear span. Fix an integer $m \ge 1$. For any integer $d \ge 1$ let $\nu_d : \mathbb{P}^m \to \mathbb{P}^N$, $N := \binom{m+d}{m} - 1$, denote the order d Veronese embedding of $\mathbb{P}^m$. Set $X_{m,d} := \nu_d(\mathbb{P}^m)$. For any $P \in \mathbb{P}^N$ the symmetric rank $sr(P)$ of P is the minimal cardinality of a finite set $S \subset X_{m,d}$ such that $P \in \langle S\rangle$. Up to a scalar the point P represents a homogeneous degree d polynomial $f \in \mathbb{K}[x_0, \dots, x_m]$ and $sr(P)$

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is the minimal integer s such that $f = \sum_{i=1}^{s} \ell_i^d$ with each $\ell_i \in \mathbb{K}[x_0, \dots, x_m]_1$ a linear form. Dually, f may be seen as a symmetric tensor τ and $sr(P)$ is the minimal number of rank 1 symmetric tensors with τ as their sum. Similarly, a finite set $S \subset \mathbb{P}^N$ such that $P \in \langle S \rangle$ corresponds to a decomposition $f = \sum_{Q \in S} \ell_Q^d$, where ℓ_Q^d is associated to the unique $O \in \mathbb{P}^m$ such that $Q = \nu_d(O)$. There are many practical problems which use the symmetric tensor rank and several general mathematical works on it ([10], [14], [9], [4], [7], [13], [16], [15], [3], [8], [6] and references therein). If $sr(P)$ is very low, then there is a unique set $A \subset \mathbb{P}^N$ computing $sr(P)$, i.e. with $P \in \langle A \rangle$ and $\sharp(A) = sr(P)$ ([6], Theorem 1.2.6, [2], Theorem 2). In this paper we study a similar situation for larger (but not very large) values of the symmetric rank. We ask for sets $A, S \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(A) \rangle \cap \langle \nu_d(S) \rangle$ and $A \neq S$. Without loss of generality we assume that A and S are " minimal ", i.e. we assume $P \notin \langle A' \rangle$ for any $A' \subsetneq A$ and $P \notin \langle S' \rangle$ for any $S' \subsetneq S$. For any $P \in \mathbb{P}^N$ let $\mathcal{S}(P)$ denote the set of all $B \subset \mathbb{P}^m$ such that $\nu_d(B)$ computes $sr(P)$, i.e., the set of all $B \subset \mathbb{P}^m$ such that $\sharp(B) = sr(P)$ and $P \in \langle \nu_d(B) \rangle$. Notice that $P \notin \langle \nu_d(B') \rangle$ for any $B \in \mathcal{S}(P)$ and any $B' \subsetneq B$. The set $\mathcal{S}(P)$ is a constructible subset of $\mathbb{P}^m$. As usual for constructible sets $\dim(\mathcal{S}(P))$ denotes the maximal dimension of a quasi-projective variety contained in $\mathcal{S}(P)$. This integer is the maximal dimension of an irreducible component of the Zariski closure of $\mathcal{S}(P)$ in $\mathbb{P}^m$.

Let $E \subset \mathbb{P}^r$ be a finite set. The set E is said to be *in linearly general position* if $\dim(\langle F \rangle) = \min\{\sharp(F) - 1, r\}$ for every $F \subseteq E$. We prove the following results.

Theorem 1.1. *Fix integers $d > m \geq 2$ and subsets S, A of $\mathbb{P}^m$ such that $\sharp(A) \geq m+1$, $\sharp(S) \geq m+1$, $\sharp(S) + \sharp(A) \leq md + 1$ and both S and A are in linearly general position in $\mathbb{P}^m$. Then $\langle \nu_d(A) \rangle \cap \langle \nu_d(S) \rangle = \langle \nu_d(A \cap S) \rangle$.*

Theorem 1.2. *Fix integers $m \geq 4$ and $d \geq 2m+1$. Fix $S \subset \mathbb{P}^m$ such that $\sharp(S) \leq (3d+1)/2$ and S is in linearly general position in $\mathbb{P}^m$. Fix any $P \in \langle \nu_d(S) \rangle$ such that $P \notin \langle \nu_d(S') \rangle$ for any $S' \subsetneq S$. Then $sr(P) = \sharp(S)$ and $\mathcal{S}(P) = \{S\}$.*

Theorem 1.1 shows that $\langle \nu_d(A \cap S) \rangle$ is the set of all $P \in \mathbb{P}^N$ which may be described both as a sum over the points of $\nu_d(A)$ and as a sum over the points of $\nu_d(S)$, when $\sharp(A)$ and $\sharp(S)$ are low. It obviously implies $sr(P) \leq \sharp(S \cap A)$ for every $P \in \langle A \rangle \cap \langle S \rangle$. Theorem

1.1 is sharp (see Example 2.7). Theorem 1.2 is a " partial improvement " of [2], Theorem 2 (it assumes less on $\sharp(S)$, but more on the shape of S).

To state our next result we introduce the following cases. Fix integers $m \ge 2$ and $d \ge 2$. We fix $P \in \mathbb{P}^N$ and assume the existence of finite sets $A, S \subset \mathbb{P}^m$ such that $S \ne A$, $P \in \langle \nu_d(A)\rangle \cap \langle \nu_d(S)\rangle$, $P \notin \langle \nu_d(A')\rangle$ for any $A' \subsetneq A$ and $P \notin \langle \nu_d(S')\rangle$ for any $S' \subsetneq S$.

(A) We say that (A, S, P) is as in case A if there is a line $D \subset \mathbb{P}^m$ such that $\sharp((A \cup S) \cap D) \ge d+2$, $\sharp(A \cap D) \le d+1$, $\sharp(S \cap D) \le d+1$, $A \setminus A \cap D = S \setminus S \cap D$, $\nu_d(A \setminus A \cap D)$ is linearly independent, and $\langle \nu_d(A \setminus A \cap D)\rangle \cap \langle \nu_d(D)\rangle = \emptyset$.

(B) We say that (A, S, P) is as in case B if $\sharp(A) + \sharp(S) = 2d+2$, $A \cap S = \emptyset$ and there are a plane $U \subseteq \mathbb{P}^m$ and a smooth conic $C \subset U$ such that $A \cup S \subset C$.

(C) We say that (A, S, P) is as in case C if there are a plane $U \subseteq \mathbb{P}^m$ and lines $L_1, L_2 \subset U$ such that $L_1 \ne L_2$, $A \cup S \subset L_1 \cup L_2$, $L_1 \cap L_2 \notin A \cup S$, $A \cap S = \emptyset$, and $\sharp((A \cup S) \cap L_1) = \sharp((A \cup S) \cap L_2) = d+1$.

Notice that in case A we assume neither $A \cap S \cap D = \emptyset$ nor $\sharp(D \cap (A \cup S)) = d+2$.

Proposition 1.3. *Fix integers $m \ge 2$ and $d \ge 3$. Fix $A, S \subset \mathbb{P}^m$ such that $\sharp(A) + \sharp(S) \le 2d+2$. Assume the existence of $P \in \langle \nu_d(A)\rangle \cap \langle \nu_d(S)\rangle$ such that $P \notin \langle \nu_d(A')\rangle$ for any $A' \subsetneq A$ and $P \notin \langle \nu_d(S')\rangle$ for any $S' \subsetneq S$. Then:*

(a) (A, S, P) is either as in case A or as in case B or as in case C.

(b) If (A, S, P) is either as in case B or as in case C, then $\{P\} = \langle \nu_d(A)\rangle \cap \langle \nu_d(S)\rangle$.

Part (b) of Proposition 1.3 shows that in cases B and C the pair (A, S) uniquely determines P.

Proposition 1.4. *Assume $d \ge 5$ and fix a triple (A, S, P) as in case A with respect to the line D. Set $E := A \setminus A \cap D$. Assume $\sharp(A) + \sharp(S) \le 2d+2$.*

(a) There is a unique $P_1 \in \langle \nu_d(D \cap A)\rangle \cap \langle \{P\} \cup \nu_d(E)\rangle$ and $sr(P) = sr(P_1) + \sharp(E)$. Set $\Gamma := \{E \sqcup \beta\}_{\beta \in \mathcal{S}(P_1)}$. We have $\Gamma \subseteq \mathcal{S}(P)$ and equality holds, unless $\sharp(A) = \sharp(B) = sr(P) = d+1$.

(b) Take another $(\widetilde{A}, \widetilde{S}, P)$ as in case A with respect to the same line D and with $\sharp(\widetilde{A}) + \sharp(\widetilde{S}) \le 2d+2$. Then $\sharp(\widetilde{A} \setminus \widetilde{A} \cap D) = \sharp(E)$.

(c) Take another $(\overline{A}, \overline{S}, P)$ *as in case A with respect to some line* $\overline{D}$*. If* $\sharp(\overline{A})+\sharp(\overline{S}) \le 2d+2$*,* $\sharp(A)+\sharp(\overline{A}) \le 2d+1$ *and* $2 \le \sharp(A \cap D) \le d$*, then* $\overline{D} = D$*.*

For an example which shows the necessity of some assumptions in part (c) of Proposition 1.4, see Example 3.6.

The integer $sr(P_1)$ appearing in Proposition 1.4 is also the symmetric rank of P_1 with respect to the rational normal curve $\nu_d(D)$ ([14], Proposition 3.1, or [15], Theorem 2.1). Hence, knowing P_1 one can use several known algorithms to compute the integer $sr(P_1)$ ([8], [15], Theorem 4.1, [3], §3).

Proposition 1.5. *Assume* $d \ge 3$ *and* (A, S, P) *as in case B with respect to the smooth conic* C*. Then:*

(a) We have $sr(P) = \min\{\sharp(A), \sharp(S)\}$ *and* $\{P\} = \langle \nu_d(A)\rangle \cap \langle \nu_d(S)\rangle$*.*

(b) If $\sharp(A) \neq \sharp(S)$*, say* $\sharp(A) < \sharp(S)$*, then* A *is the only element of* $\mathcal{S}(P)$*.*

(d) If $\sharp(A) = \sharp(S) = d+1$*, then* $\mathcal{S}(P)$ *is one-dimensional, every* $B \in \mathcal{S}(P)$ *is contained in* C *and any two different elements of* $\mathcal{S}(P)$ *are disjoint.*

Proposition 1.6. *Assume* $d \ge 5$ *and fix* (A, S, P) *as in case C with respect to the reducible conic* $L_1 \cup L_2$*. Set* $\{Q\} := L_1 \cap L_2$*. We have* $\{P\} = \langle \nu_d(A)\rangle \cap \langle \nu_d(S)\rangle$*. Set* $A_i := A \cap L_i$ *and* $S_i := S \cap L_i$*. Either* $sr(P)$ *is computed by* A *or by* S *or by* $A_1 \cup S_2 \cup \{Q\}$ *or by* $A_2 \cup S_1 \cup \{Q\}$*. If* $sr(P) < \min\{\sharp(A), \sharp(S)\}$*, then* $\mathcal{S}(P) \subseteq \{A_1 \cup S_2 \cup \{Q\}, A_2 \cup S_1 \cup \{Q\}\}$*.*

The existence of a curve as in (A), (B) or (C) (respectively a line, a smooth conic and a reducible conic) would easily follow from the main result of [1]. In the range $\sharp(A)+\sharp(S) < 3d$ the existence of a suitable curve follows from [11], Theorem 3.8. We will use [11], Theorem 3.8, to shorten the proof. We prefer to present here a proof which not use [1], but the main point of this paper is the analysis of the pairs (A, S) associated to a given P and of the computation of $sr(P)$ (Propositions 1.4, 1.5, 1.6)..

2. The proofs of Theorems 1.1 and 1.2

Grassmann's formula and the linear normality of Veronese varieties immediately give the following lemma.

Lemma 2.1. *For all finite subsets A, S of $\mathbb{P}^m$ such that $h^1(\mathbb{P}^m, \mathcal{I}_A(d)) = h^1(\mathbb{P}^m, \mathcal{I}_S(d)) = 0$ we have*

$$\dim(\langle \nu_d(A)\rangle \cap \langle \nu_d(S)\rangle) = \dim(\langle \nu_d(A \cap S)\rangle) + h^1(\mathbb{P}^m, \mathcal{I}_{A\cup S}(d)).$$

Lemma 2.2. *Fix finite subsets A, S of $\mathbb{P}^m$ such that $h^1(\mathbb{P}^m, \mathcal{I}_A(d)) = h^1(\mathbb{P}^m, \mathcal{I}_S(d)) = 0$ and a proper linear subspace M of $\mathbb{P}^m$. Set $F := (A\cup S)\setminus (A\cup S)\cap M$ and $E := (S\cap A)\setminus (S\cap A\cap M)$. If $h^1(\mathbb{P}^m, \mathcal{I}_F(d-1)) = 0$, then $\langle \nu_d(A)\rangle \cap \langle \nu_d(S)\rangle$ is the linear span of $\langle E\rangle$ and of $\langle \nu_d(A\cap M)\rangle \cap \langle \nu_d(S\cap M)\rangle$ and its dimension is $\sharp(E) + \dim(\langle \nu_d(A\cap M)\rangle \cap \langle \nu_d(S\cap M)\rangle)$.*

Proof. Since $E \subsetneq A$ we have $h^1(\mathbb{P}^m, \mathcal{I}_E(d)) = 0$. Hence $\dim(\langle \nu_d(E)\rangle = \sharp(E) - 1$. Take a general hyperplane H of $\mathbb{P}^m$ containing M. Since $A \cup S$ is finite, we have $(A \cup S) \cap H = (A \cup S) \cap M$. From the residual exact sequence

$$0 \to \mathcal{I}_F(d-1) \to \mathcal{I}_{S\cup A}(d) \to \mathcal{I}_{(S\cup A)\cap H}(d) \to 0 \tag{1}$$

we get $h^1(\mathbb{P}^m, \mathcal{I}_{S\cup A}(d)) = h^1(H, \mathcal{I}_{(S\cup A)\cap M}(d))$. Hence $\dim(\langle \nu_d(A)\rangle \cap \langle \nu_d(S)\rangle) - \dim(\langle \nu_d(A\cap A)\rangle) = \dim(\langle \nu_d(S \cap M)\rangle \cap \langle \nu_d(S \cap M)\rangle - \dim(\langle \nu_d(A \cap S \cap M)\rangle)$ (Lemma 2.1). We have $S\cap A = (S\cap A\cap M) \sqcup E$. Since $E \subsetneq F$ and $h^1(\mathbb{P}^m, \mathcal{I}_F(d-1)) = 0$, the exact sequence (1) also gives $\dim(\langle \nu_d(A)\rangle \cap \langle \nu_d(S)\rangle) = \sharp(E) + \dim(\langle \nu_d(A\cap M)\rangle \cap \langle \nu_d(S\cap M)\rangle)$ and that $\langle \nu_d(E)\rangle$ and $\langle \nu_d(A \cap M)\rangle \cap \langle \nu_d(S \cap M)\rangle$ are supplementary linear subspaces of $\langle \nu_d(S)\rangle \cap \langle \nu_d(S)\rangle$. This completes the proof.

We will often call (1) (or similar exact sequences) the Castelnuovo's sequence. Let $Z \subset \mathbb{P}^m$ be a zero-dimensional scheme. For any hyperplane $H \subset \mathbb{P}^m$ the residual scheme $\mathrm{Res}_H(Z)$ of Z with to H is the closed subscheme of $\mathbb{P}^m$ with $\mathcal{I}_Z : \mathcal{I}_H$ as its ideal sheaf. We have $\mathrm{Res}_H(Z) \subseteq Z$, $\deg(Z) = \deg(\mathrm{Res}_H(Z)) + \deg(Z \cap H)$ and for any $t \in \mathbb{Z}$ there is a Castelnuovo's sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{H(Z)}}(t-1) \to \mathcal{I}_Z(t) \to \mathcal{I}_{Z\cap H,H}(t) \to 0.$$

If Z is reduced, i.e. if Z is a finite set, then $\mathrm{Res}_H(Z) = Z \setminus Z \cap H$.

Lemma 2.3. *Fix integers $m \geq 2$, $d \geq 3$ and sets $S, A \subset \mathbb{P}^m$ such that $h^1(\mathbb{P}^m, \mathcal{I}_A(d)) = h^1(\mathbb{P}^m, \mathcal{I}_S(d)) = 0$, $\sharp(A \cup S) \leq 2d + 1$ and $\langle \nu_d(A)\rangle \cap \langle \nu_d(S)\rangle \neq \langle \nu_d(A \cap S)\rangle$. Then there is* *a line* $D \subset \mathbb{P}^m$ *such that* $\sharp((A \cup S) \cap D) \geq d+2$ *and, taking* $E := (A \cap S) \setminus (A \cap S \cap D)$, $\langle \nu_d(E) \rangle$ *and* $\langle \nu_d(A \cap D) \rangle \cap \langle \nu_d(S \cap D) \rangle$ *are supplementary subspaces of* $\langle \nu_d(A) \rangle \cap \langle \nu_d(S) \rangle$ *and* $\dim(\nu_d(E)\rangle = \sharp(E) - 1$.

Proof. Since $h^1(\mathbb{P}^m, \mathcal{I}_{A \cup S}(d)) > 0$ (Lemma 2.1), there is a line $D \subset \mathbb{P}^m$ such that $\sharp(D \cap (A \cup S)) \geq d+2$ ([3], Lemma 34). Set $E := (A \cup S) \setminus (A \cup S) \cap D$. Since $\sharp(E) \leq d-1$, we have $h^1(\mathbb{P}^m, \mathcal{I}_E(d-1)) = 0$ ([3], Lemma 3.4). Hence $\langle \nu_d(A) \rangle \cap \langle \nu_d(S) \rangle$ is the linear span of $\langle \nu_d(E) \rangle$ and of $\langle \nu_d(A \cap D) \rangle \cap \langle \nu_d(S \cap D) \rangle$ (Lemma 2.2). Since $h^1(\mathbb{P}^m, \mathcal{I}_E(d)) \leq h^1(\mathbb{P}^m, \mathcal{I}_E(d-1)) = 0$, we have $\dim(\nu_d(E)\rangle = \sharp(E) - 1$. Use Lemma 2.2. This completes the proof.

We need the following obvious lemma.

Lemma 2.4. *Fix a linearly independent subset* $F' \subset \mathbb{P}^r$. *Then the linear system* $|\mathcal{I}_{F'}(2)|$ *has no base point outside* F', *i.e.* $h^1(\mathcal{I}_{F' \cup \{P\}}(2)) = 0$ *for every* $P \in \mathbb{P}^r \setminus F'$.

Lemma 2.5. *Fix integers* $r \geq 1$ *and* $t \geq 3$ *and subsets* E, F *of* $\mathbb{P}^r$ *such that both* E *and* F *are linearly independent. Then* $h^1(\mathcal{I}_{E \cup F}(t)) = 0$.

Proof. If $r = 1$, then the lemma is true. Hence we may assume $r \geq 2$ and use induction on r. Enlarging if necessary E we may assume $\sharp(E) = r+1$. Let H be a hyperplane spanned by r points of E. Set $E' := E \setminus E \cap H$ and $F' := F \setminus F \cap H$. Since both E and F are linearly independent, both $E \cap H$ and $F \cap H$ are linearly independent. Hence the inductive assumption gives $h^1(H, \mathcal{I}_{(E \cup F) \cap H}(t)) = 0$. Since $\sharp(E' \cup F') \leq \sharp(F') + 1$ and F' is linearly independent, it is sufficient to apply Lemma 2.4. This completes the proof.

Lemma 2.6. *Fix a finite set* $E \subset \mathbb{P}^r$ *such that* $h^1(\mathcal{I}_E(2)) > 0$. *Then there is a linear subspace* $U \subseteq \mathbb{P}^r$ *such that* $\sharp(E \cap U) \geq \dim(U) + 3$.

Proof. We use induction on r, the case $r = 1$ being obvious. Assume $r \geq 2$. Let $H \subset \mathbb{P}^r$ be a hyperplane such that $\sharp(E \cap H)$ is maximal. First assume $h^1(H, \mathcal{I}_{H \cap E}(2)) > 0$. By the inductive assumption there is a linear subspace $U \subseteq H$ such that $\sharp(E \cap U) \geq \dim(U) + 2$. Now assume $h^1(H, \mathcal{I}_{H \cap E}(2)) = 0$. By the Castelnuovo's sequence (1) with $d = 2$ and $E = A \cup S$ we have $h^1(\mathcal{I}_{E \setminus E \cap H}(1)) > 0$. Hence $\sharp(E \setminus E \cap H) \geq 3$. Since we took E with $\sharp(E \cap H)$ maximal and E is not contained in H, $E \cap H$ spans H. Therefore $\sharp(E \cap H) \geq r$. Hence $\sharp(E) \geq r+3$. Hence we may take $\mathbb{P}^r$ as U. This completes the proof.

Example.2.7. *Let $C \subset \mathbb{P}^m$ be a rational normal curve. Fix finite subsets A, S of C such that $A \neq \emptyset$, $S \neq \emptyset$, $A \cap S = \emptyset$ and $\sharp(A) + \sharp(B) = md + 2$. Since $h^0(C, \mathcal{O}_C(d)) = md + 1$, $h + 0(C, \mathcal{I}_{A\cup S}(d)) = 0$, and $h^1(C, \mathcal{I}_E(d)) = 0$ for every $E \subset C$ such that $\sharp(E) \leq md + 1$, Lemma 2.1 gives that $\langle \nu_d(A)\rangle \cap \langle \nu_d(A)\rangle$ is a unique point, P, and $\langle \nu_d(A)\rangle \cap \langle \nu_d(S')\rangle = \langle \nu_d(A')\rangle \cap \langle \nu_d(S)\rangle$ for any $A' \subsetneq A$ and any $S' \subsetneq S$.*

Proof of Theorem 1.1. Assume $\langle \nu_d(A)\rangle \cap \langle \nu_d(S)\rangle \neq \langle \nu_d(A \cap S)\rangle$. Since S and A are in linearly general position in $\mathbb{P}^m$ and $\sharp(A) \leq md + 1$, $\sharp(S) \leq md + 1$, we have $h^1(\mathbb{P}^m, \mathcal{I}_A(d)) = h^1(\mathbb{P}^m, \mathcal{I}_S(d)) = 0$ ([12], Theorem 3.2). Hence our assumption is equivalent to $h^1(\mathbb{P}^m, \mathcal{I}_{A\cup S}(d)) > 0$ (Lemma 2.1). $\sharp(A \cup S) \leq dm + 1$, the set $A \cup S$ is not in linearly general position ([12], Theorem 3.2). Set $W_0 := A \cup S$. Let $M_1 \subset \mathbb{P}^m$ be a hyperplane such that $\sharp(W_0 \cap M_1)$ is maximal. Set $W_1 := W_0 \setminus (W_0 \cap M_1)$. Fix an integer $i \geq 2$ and assume to have defined the sets W_j and the hyperplane $M_j \subset \mathbb{P}^m$ for all $j < i$. Let $M_i \subset \mathbb{P}^m$ be a hyperplane such that $\sharp(M_i \cap W_{i-1})$ is maximal. Set $W_i := W_{i-1} \setminus (W_{i-1} \cap M_i)$, $w_i := \sharp(W_i)$ and $b_i = \sharp(M_i \cap W_{i-1})$. Hence $w_0 = \sharp(A \cup S)$, $w_{i-1} = w_i + b_i$ for all $i > 0$, and $b_i \geq b_j$ for all $i \geq j$. Since $h^1(\mathbb{P}^m, \mathcal{I}_{A\cup S}(d)) > 0$ (Lemma 2.1), there is an integer $i \geq 1$ such that $h^1(M_i, \mathcal{I}_{M_i \cap W_{i-1}}(d + 1 - i)) > 0$. Call k the minimal such integer. Notice that if $b_j \leq m - 1$, then $b_i = 0$ for all $i > j$. Hence $b_i = 0$ for all $i > \lceil w_0/m \rceil$. Hence $b_{d+2} = 0$ and $b_{d+1} \leq 1$. Since $h^1(\mathbb{P}^m, \mathcal{I}_E) = 0$ if $\sharp(E) \leq 1$, we have $k \leq d$. Since both A and S are in linearly general position, then $\sharp(A \cap M_k) \leq m$, $\sharp(S \cap M_k) \leq m$ and both $A \cap M_k$ and $S \cap M_k$ are linearly independent in M_k. Lemma 2.4 with $r = m - 1$, $E = A \cap M_k$ and $F = S \cap M_k$ gives $k \geq d - 1$. Since $A \cup S$ is not in linearly general position, we have $b_1 \geq m + 1$. Since $b_i \geq m$ if $b_{i+1} > 0$, we have $b_i \geq m$ for $2 \leq i \leq k - 2$. Hence $\sharp(A \cup S) \geq m + 1 + (k - 2)m + b_k$. Fix an integer $i \geq 1$ such that $b_{i+1} > 0$. Since M_i contains the maximal number of points of W_{i-1}, either W_{i-1} is in linearly general position in $\mathbb{P}^m$ or $b_i \geq m + 1$. If W_{i-1} is in linearly general position in $\mathbb{P}^m$, then all its subsets W_j, $j \geq i$, are in linearly general position in $\mathbb{P}^m$. Hence either $M_k \cap W_{k-1}$ is in linearly general position in M_k or $b_i \geq m + 1$ for all $i \in \{1, \dots, k - 1\}$.

(a) Here we assume that $M_k \cap W_{k-1}$ is in linearly general position in M_k. Since $h^1(M_k, \mathcal{I}_{W_{k-1} \cap M_k}(d + 1 - k)) > 0$, we get $b_k \geq (m - 1)(d + 1 - k) + 2$ ([12], Theorem

3.2). First assume $k = d-1$. Since $b_{d-1} \geq 2m$ and $b_i \geq b_{d-1}$ for all $i \leq d-1$, we get $\sharp(A \cup S) \geq 2m(d-1) > md+1$, a contradiction. For $k = d$ we get $b_d \geq m+1$ and hence $\sharp(A \cup S) \geq (m+1)d$, a contradiction.

(b) In this step we assume that $M_k \cap W_{k-1}$ is not in linearly general position in M_k.

(b1) First assume $k = d$. Since $M_d \cap W_{d-1}$ is not in linearly general position, we have $b_d \geq 3$. Hence $\sharp(A \cup S) \geq (m+1)(d-1)+3 > md+1$ (since $d > m$).

(b2) Now assume $k = d-1$. Hence $h^1(M_{d-1}, \mathcal{I}_{M_{d-1} \cap W_{d-2}}(2)) > 0$. Applying Lemma 2.6 with $r = m-1$ and $E = M_{d-1} \cap W_{d-2}$ we get the linear subspace $U \subseteq M_{d-1}$ such that $\sharp((A \cup S) \cap U) \geq \dim(U)+3$. Since b_1 is at least the maximal integer $\sharp(F \cap (A \cup S))$, where F is a hyperplane containing U, we have $b_1 \geq m+3$. If there is linear subspace V such that $\sharp(V \cap W_1) \geq \dim(V)+3$, then $b_2 \geq m+3$ (or $b_3 = 0$). If there is no such linear subspace then we may take the hyperplanes so that W_{d-1} has no linear subspace U as above. And so on. Hence we get $b_i \geq m+3$ for $1 \leq i \leq d-2$. Hence $\sharp(A \cup S) \geq (m+3)(d-2)+b_{d-1}$. Since $b_{d-1} \geq 4$ and $d > m$ we get $\sharp(A \cup S) \geq md+2$, a contradiction. □

Proof of Theorem 1.2. Take $A \subset \mathbb{P}^m$ such that $\nu_d(A)$ computes $sr(P)$. If $sr(P) = \sharp(S)$, then assume $A \neq S$. It is sufficient to prove that these assumptions give a contradiction. We have $\sharp(A \cup S) \leq 3d+1$ with strict inequality if d is even. Set $W := A \cup S$ and $\rho_0 := \sharp(W)$. We assumed $P \notin \langle \nu_d(S') \rangle$ for any $S' \subsetneq S$. Since $\nu_d(A)$ computes $sr(P)$, then $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$. Hence $h^1(\mathbb{P}^m, \mathcal{I}_W(d)) > 0$ ([2], Lemma 1). If $\sharp(S) \leq d+1$, then the statement is a particular case of [2], Theorem 2. Hence we may assume $\sharp(S) \geq d+2$.

(a) Let $H_1 \subset \mathbb{P}^m$ be a hyperplane such that $\rho_1 := \sharp(W \cap H_1)$ is maximal. Set $W_0 := W$ and $W_1 := W_0 \setminus W_0 \cap H_1$. For every integer $i \geq 2$ define inductively the subsets W_i of W, the hyperplane $H_i \subset \mathbb{P}^m$ and the integer ρ_i in the following way. Fix an integer $i \geq 2$ and assume that W_{i-1} is defined. Let $H_i \subset \mathbb{P}^m$ be any hyperplane such that $\rho_i := \sharp(W_{i-1} \cap H_i)$ is maximal. Set $W_i := W_{i-1} \setminus W_{i-1} \cap H_i$. Hence $W_{i+1} \subseteq W_i$ for all i, $\sharp(W_i) = \rho_0 - \sum_{h=1}^{i} \rho_h$ for all $i \geq 1$. The maximality condition implies that the sequence $\{\rho_i\}_{i \geq 1}$ is non-increasing and $\rho_0 \geq \sum_{i \geq 1} \rho_i$. Hence $W_{i+1} = W_i \Leftrightarrow \rho_i = 0 \Leftrightarrow \rho_h = 0$ for all $h \geq i$. Since $W_i = W_{i-1} \setminus W_{i-1} \cap H_i$, for all integers t, i with $i \geq 1$ we have the following exact sequence of sheaves (often called the Castelnuovo's sequence)

$$0 \to \mathcal{I}_{W_i}(t-1) \to \mathcal{I}_{W_{i-1}}(t) \to \mathcal{I}_{W_{i-1}\cap H_i, H_i}(t) \to 0 \tag{2}$$

Since $W_i = \emptyset$ for all $i \gg 0$ (say for all $i \geq \rho_0$)) and $h^1(\mathbb{P}^n, \mathcal{I}_W(d)) > 0$, there is an integer $i \geq 1$ such that $h^1(H_i, \mathcal{I}_{W_{i-1}\cap H_i, H_i}(d+1-i)) > 0$. Call i_0 the minimal such integer. Since $\rho_0 \leq 3d+1$ and $h^1(\mathbb{P}^m, \mathcal{I}_W(d)) > 0$, W is not in linearly general position ([12], Theorem 3.2). Hence $\rho_1 \geq m+1$. By the maximality of each ρ_i we get that either $W_{i-1} \cap H_i$ spans H_i (and hence $\rho_{i-1} \geq m$) or $W_{i-1} \subset H_i$ and hence $\rho_j = 0$ for all $j \geq i_0$. Since $\sharp(A \cup S) \leq 3d+1 < m(d-1)$, we have $i_0 \leq d$. Hence $d+1-i_0 > 0$. By [3], Lemma 34, we have $\rho_{i_0} \geq d+3-i_0$ and equality holds if and only if $W_{i_0-1} \cap H_i$ is contained in a line. Since the sequence $\{\rho_i\}_{i\geq 1}$ is non-increasing, we get $i_0(d+3-i_0) \leq \rho_0$. Since $\rho_0 \leq 3d+1$ and the function $t \mapsto t(d+3-t)$ is strictly increasing for $t < (d+3)/2$ and strictly decreasing for $t > (d+3)/2$, we get that either $i_0 \in \{1,2,3\}$ or $i_0 \geq d-3$ (for $t = 4$ we need $d \geq 5$).

(b) Here we assume $i_0 = 1$ and $\rho_1 \leq 2d+1$. There is a line $L \subset H_1$ such that $\sharp(W \cap L) \geq d+2$ ([3], Lemma 34). Since S is in linearly general position, we have $\sharp(S \cap L) \leq 2$. Hence $\sharp(A \cap L) \geq d$. Set $S' := S \setminus L$ and $A' := A \setminus S \cap L$. Since $P \in \langle \nu_d(A) \rangle$ and $P \notin \langle A \setminus L \cap A \rangle$, the set $\langle \{P\} \cup \nu_d(A \setminus A \cap L) \rangle \cap \langle \nu_d(A) \rangle$ is a unique point; call P_1 this point. Since $P \in \langle \nu_d(A \setminus A \cap L) \cup \{P_1\} \rangle$, $P_1 \in \langle \nu_d(A \cap L) \rangle$, and A computes $sr(P)$, the set $\nu_d(A \cap L)$ computes $sr(P_1)$. Since $\nu_d(A \cap L) \subset \nu_d(L)$, then $P_1 \in \langle \nu_d(L) \rangle$ and $A \cap L$ computes the symmetric rank of P_1 with respect to the rational normal curve $\nu_d(L)$ ([14], Proposition 3.1, [15]). Hence $\sharp(A \cap L) \leq d$ ([8], [15], Theorem 4.1, [3], Theorem 34). Since we knew the opposite inequality, we get $\sharp(A \cap L) = d$. Hence P_1 has border rank 2 ([8], [15], Theorem 4.1, [3], Theorem 34). Hence there is a degree two 0-dimensional scheme $Z \subset L$ such that $P_1 \in \langle \nu_d(Z) \rangle$ ([6], Lemma 2.1.5, or [3], Proposition 11). Hence $P \in \langle \nu_d(Z \cup (A')) \rangle$. Since $\sharp(A) \leq \sharp(S) \leq 3d+1$, we get $\deg(Z \cup A') + \sharp(S) \leq 3d+1+2-d \leq 2d+3$. If $\deg(Z \cup A') + \sharp(S) \leq 2d+1$ (e.g., if $\sharp(A) + \sharp(S) \leq 3d-1$), then we may repeat the proof of [2], Theorem 1, applied to $\mathcal{Z} := \nu_d(Z \cup A')$ and to $\mathcal{S} := \nu_d(S)$, and obtain a contradiction, because no line contains at least $\lceil (d+2)/2 \rceil$ points of S. Hence we could assume $\sharp(A) + \sharp(S) \geq 3d$. First assume $h^1(\mathcal{I}_{A' \cup S'}(d-1)) = 0$. For a general hyperplane M

containing L we have $\mathrm{Res}_M(Z \cup A' \cup S) = A' \cup S'$. From the Castelnuovo's sequence with respect to M we get that $\langle \nu_d(Z \cup A') \rangle \cap \langle \nu_d(S) \rangle$ is the linear span of $\langle \nu_d(Z) \rangle \cap \langle \nu_d(S \cap L) \rangle$ and of $\nu_d(A' \cap S') \rangle$. Since $S \cap L$ is reduced, either $Z_{red} \in S \cap L$ or $\sharp(S \cap L) \geq d$ or $\langle \nu_d(Z) \rangle \cap \langle \nu_d(S \cap L) \rangle = \emptyset$ ([8]). Since S is in linearly general position and $d > 2$, we have $\sharp(S \cap L) < d$. Now assume $Z_{red} \subset S \cap L$; we get $\langle \nu_d(Z) \rangle \cap \langle \nu_d(S \cap L) \rangle = \{\nu_d(Z_{red})\}$; hence $P \in \langle Z_{red} \cup S' \rangle$ with $Z_{red} \subset S$; since $P \notin \langle \nu_d(E) \rangle$ for any $E \subsetneq S$, we get $S \cap L = Z_{red}$. Hence $\sharp(A \cap L) \geq d + 1$, a contradiction. Similarly, if $\langle \nu_d(Z) \rangle \cap \langle \nu_d(S \cap L) \rangle = \emptyset$ we get $P \in \langle \nu_d(S') \rangle$ and hence $\sharp(A \cap L) \geq d + 2$, a contradiction.

Now assume $h^1(\mathcal{I}_{A' \cup S'}(d-1)) > 0$. Since $\sharp(A' \cup S') \leq \sharp(A \cup S) - d - 2 \leq 2(d-1) + 1$, there is a line $R \subset \mathbb{P}^m$ such that $\sharp(R \cap (A' \cup S')) \geq d + 1$. Since S' is in linearly general position, we have $\sharp(S' \cap R) \leq 2$. Hence $\sharp(A') \geq d - 1$. Hence $\sharp(A) \geq 2d - 1$, a contradiction.

(c) Here we assume $i_0 = 1$ and $\rho_1 \geq 2d + 2$. Since S is in linearly general position, we have $\sharp(S \cap H_1) \leq m$. Hence $\sharp(A \cap H_1) \geq 2d + 2 - m$. Since $d \geq 2m + 1$, we have $2d + 2 - m > (3d + 1)/2$. Hence $\sharp(A) > (3d + 1)/2$, a contradiction.

(d) Here we assume $i_0 = 2$. Hence $\rho_2 \geq d + 1$ ([3], Lemma 34). Since the sequence $\{\rho_j\}_{j \geq 1}$ is non-increasing and $2(2d - 1) > 3d + 1 \geq \rho_0$, we get $\rho_2 \leq 2d - 1$. Hence there is a line $L_1 \subset H_2$ such that $\sharp(W_1 \cap L_1) \geq d + 1$. If $\sharp(S) \geq 2m + 1$, then $\rho_3 \geq \sharp(S) - 2m > 0$, because S is in linearly general position. Hence $W_1 \cap H_2$ spans H_2. Hence $\rho_2 \geq \deg(W_1 \cap L) + m - 2 \geq m + d - 1$. Since $\rho_1 \geq \rho_2$ and $\sharp(S \cap H_1) \leq m$, we also get $\sharp(A \cap (H_1 \cup H_2)) \geq 2d - 2$, a contradiction. Now assume $\sharp(S) \leq 2m$. Since $d > 2m$, the theorem in this case is a particular case of [2], Theorem 2.

(e) Here we assume $i_0 = 3$. Since the sequence $\{\rho_j\}_{j \geq 1}$ is non-increasing and $3(d+1) > 3d + 1$, we get that $W_2 \cap H_3$ is the union of d collinear points, say on a line L_3, and hence $\rho_j = 0$ for all $j > 3$. We get $\rho_0 = 3d + \epsilon$ with $\epsilon \in \{0, 1\}$, $\rho_1 = d + \epsilon$, $\rho_2 = d$ and $\rho_3 = d$. Instead of H_1 we take a hyperplane M_1 containing L_3 and at least $m - 2$ other points of W. Since $m \geq 4$, we get a contradiction.

(f) Here we assume $i_0 \geq d - 3$. Recall that the sequence $\{\rho_i\}_{i \geq 1}$ is non-increasing and that $\rho_i \geq m$ if $\rho_{i+1} > 0$. Since $A \cup S$ is not in linearly general position, we have $\rho_1 \geq m + 1$.

(f1) If $i_0 \geq d+1$ we get $\rho_0 \geq m+1+m(d-1)+1$, a contradiction.

(f2) Now assume $i_0 = d$. Since $h^1(H_d, \mathcal{I}_{W_d}(1)) > 0$, we get $\rho_d \geq 3$. Hence $\rho_0 \geq m+1+m(d-2)+3$. Since $m \geq 4$, we get $\rho_0 > 3d+1$, a contradiction.

(f3) Now assume $i_0 = d-1$. We have $\rho_{d-1} \geq 4$ and either $\rho_{d-1} \geq 6$ or $W_{d-2} \cap H_{d-1}$ contains 4 collinear points ([3], Lemma 34). If $\rho_{d-1} \geq 6$ we get $\rho_0 \geq (m+1)+(d-3)m+6$; we have $(m+1)+(d-3)m+6 \geq 3d+2$ if and only if $m \geq 4$ and $(m-3)d \geq 2m-5$ (true under our assumptions $d \geq 2m+1$ and $m \geq 4$). If $\rho_{d-1} \leq 5$, then $W_{d-2} \cap H_{d-1}$ contains 4 collinear points. Hence (as in step (b2) of the proof of Theorem 1.1) we easily get $\rho_i \geq m+2$ for all $i \leq d-2$. Hence $\rho_0 \geq (m+2)(d-2)+4 \geq 3d+2$.

(f4) Now assume $i_0 = d-2$. We have $\rho_{d-2} \geq 5$ and either $\rho_{d-2} \geq 8$ or $W_{d-3} \cap H_{d-2}$ contains 5 collinear points ([3], Lemma 34). If $\rho_{d-2} \geq 8$ we get $\rho_0 \geq (m+1)+(d-4)m+8$; we have $(m+1)+(d-4)m+8 \geq 3d+2$ if and only if $(m-3)d \geq 3m-7$ (true under our assumptions $m \geq 4$ and $d \geq 2m+1$). If $\rho_{d-2} \leq 7$, then W_{d-3} contains 5 collinear points. As above we get $\rho_i \geq m+3$ for all $i \leq d-3$. Hence $\rho_0 \geq 5+(d-2)(m+3)$. We have $5+(d-2)(m+3) \geq 3d+2$ if and only if $md-2m \geq 3$ (true under our assumptions).

(f5) Now assume $i_0 = d-3$. We have $\rho_{d-3} \geq 6$ and either $\rho_{d-3} \geq 10$ or $W_{d-4} \cap H_{d-3}$ contains 6 collinear points ([3], Lemma 34). If $\rho_{d-3} \geq 10$ we get $\rho_0 \geq (m+1)+(d-5)m+10$; we have $(m+1)+(d-5)m+10 \geq 3d+2$ if and only if $(m-3)d \geq 4m-9$ (true under our assumptions). If $\rho_{d-3} \leq 9$, then $W_{d-4} \cap H_{d-3}$ contains 6 collinear points. As above get $\rho_i \geq m+4$ for all $i \leq d-4$. Hence $\rho_0 \geq (m+4)(d-4)+6$. We have $(m+4)(d-4)+6 \geq 3d+2$ if and only if $m(d-4) \geq 12-d$ (true under our assumptions). □

3. The proofs of Propositions 1.3, 1.4, 1.5, 1.6

Lemma 3.1. *Fix an integer $d > 0$ and finite sets $A, S \subset \mathbb{P}^m$, $m \geq 2$, such that $\sharp(A) + \sharp(S) \leq 2d+2$ and there is a line $D \subset \mathbb{P}^m$ such that $\sharp((A \cup S) \cap D) \geq d+2$. Assume $\langle \nu_d(A) \rangle \cap \langle \nu_d(S) \rangle \neq \langle \nu_d(A \cap S) \rangle$ and the existence of $P \in \langle \nu_d(A) \rangle \cap \langle \nu_d(S) \rangle$ such that $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$ and $P \notin \langle \nu_d(S') \rangle$ for any $S' \subsetneq S$. Then $A \backslash A \cap D = S \backslash A \cap D$, i.e., (A, S, P) is as in case A.*

Proof. Since $P \notin \langle \nu_d(E) \rangle$ for any $E \subsetneq A$, the set $\nu_d(A)$ is linearly independent. For the same reason $\nu_d(S)$ is linearly independent. Hence $\sharp(A \cap D) \le d+1$ and $\sharp(S \cap D) \le d+1$. Hence $(S \setminus S \cap A) \cap D \neq \emptyset$. Set $A' := A \setminus A \cap D$ and $S' := S \setminus S \cap D$. Since $\sharp((A \cup S) \cap D) \ge d+2$, we have $\sharp(A' \cup S') \le d$. Hence $h^1(\mathcal{I}_{A' \cup S'}(d-1)) = 0$. Hence $\nu_d(A' \cup S')$ is linearly independent. Let $H \subset \mathbb{P}^m$ be a general hyperplane containing D. Since $A \cup S$ is finite and H is general, we have $A' = A \setminus A \cap H$ and $S' = S \setminus S \cap H$. Since $(A \cup S) \cap H = (A \cup S) \cap D$ and the restriction map $H^0(\mathcal{O}_{\mathbb{P}^m}(d)) \to H^0(D, \mathcal{O}_D(d))$ is surjective, the Castelnuovo's sequence (1) with $A' \cup S'$ instead of F gives $h^1(\mathcal{I}_{A \cup S}(d)) = h^1(D, \mathcal{I}_{(A \cup S) \cap D}(d))$. Lemma 2.2 gives that $\langle \nu_d(A) \rangle \cap \langle \nu_d(S) \rangle$ is spanned by its supplementary subspaces $\langle \nu_d(A \cap D) \rangle \cap \langle \nu_d(S \cap D) \rangle$ and $\langle \nu_d(A' \cap S') \rangle$. Since $P \notin \langle \nu_d(E) \rangle$ for any $E \subsetneq A$, we get $A' \cap S' = A'$. For the same reason we get $A' \cap S' = S'$. Hence $A' = S'$. This completes the proof.

Lemma 3.2. *Fix an integer $d \ge 2$, a smooth conic $C \subset \mathbb{P}^m$, $m \ge 2$, and sets $A, S \subset C$ such that $S \cap A = \emptyset$ and $\sharp(A) + \sharp(S) = 2d+2$. Then $\langle \nu_d(A) \rangle \cap \langle \nu_d(S) \rangle$ is a single point (call it P), and $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$, $P \notin \langle \nu_d(S') \rangle$ for any $S' \subsetneq S$.*

(i) If $\sharp(A) \le d$, then $sr(A) = \sharp(A)$ and $\mathcal{S}(P) = \{A\}$.

(ii) If $\sharp(A) = d+1$, then $sr(P) = d+1$ and $\dim(\mathcal{S}(d,P)) \ge 1$; if we assume $d \ge 5$, then $\dim(\mathcal{S}(d,P)) = 1$ and every $B \in \mathcal{S}(d,P)$ is contained in C.

Proof. Since $\dim(\langle \nu_d(C) \rangle) = 2d$ and $h^1(\mathcal{I}_E(d)) = 0$ for any $E \subseteq C$ (use that C is arithmetically normal), we get $\langle \nu_d(A) \rangle \cap \langle \nu_d(S) \rangle$ is a single point (call it P), and $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$, $P \notin \langle \nu_d(S') \rangle$ for any $S' \subsetneq S$.

(a) Assume $\sharp(A) \le d$ and the existence of $B \in \mathcal{S}(P)$ such that $B \neq A$. Hence $h^1(\mathcal{I}_{A \cup B}(d)) > 0$ ([2], Lemma 1). Since $\sharp(A) + \sharp(B) \le 2d+1$, there is a line $D \subset \mathbb{P}^m$ such that $\sharp((A \cup B) \cap D) \ge d+2$. Lemma 3.3 gives $A \setminus A \cap D = B \setminus B \cap D$. Since $\sharp(A \cap D) \le 2$, we get $\sharp(B) \ge \sharp(B \cap D) + 1 \ge d+1$, a contradiction.

(b) Now assume $\sharp(A) = d+1$. As in step (a) we get a contradiction assuming $sr(P) \le d$. Hence $sr(P) = d+1$. Since $\nu_d(C)$ is a degree $2d$ rational normal curve in $\langle \nu_d(C) \rangle$, it is well-known that the set of all $E \subset C$ computing the symmetric rank of P with respect to $\nu_d(C)$ is one-dimensional. Now assume $d \ge 5$. Take any $B \in \mathcal{S}(P)$ and assume that B is not contained in C. By [14], Proposition 3.1, B spans a plane $U \subseteq \mathbb{P}^m$

and U is the plane spanned by C. Hence in order to obtain a contradiction we may assume $m = 2$. Set $W := B \cup S$. Since $\sharp(W \cap C) \le 2d+1$, we have $h^1(C, \mathcal{I}_{W\cap C}(d)) = 0$. Hence in order to obtain a contradiction it is sufficient to prove $h^1(U, \mathcal{I}_{W\setminus W\cap C}(d-2)) = 0$ (use a Castelnuovo's sequence and [2], Lemma 1). Since $S \subset C$, we have $\sharp(W \setminus W \cap C) \le d+1 \le 2(d-2)+1$. Hence if $h^1(U, \mathcal{I}_{W\setminus W\cap C}(d-2)) > 0$, then there is a line $D \subset U$ such that $\sharp(D \cap B \setminus D \cap B \cap C) \ge d$. Since $\sharp(B) \le d+1$, we have $h^1(U, \mathcal{I}_{B\cap D}(d)) = 0$. Since $\sharp(W \cap C) \le d+2 \le 2(d-2)+1$, we have $h^1(C, \mathcal{I}_{(W\setminus D)\cap C}(d-2)) = 0$. Since $W \setminus W \cap (C \cup D))$ is at most one point, we have $h^1(U, \mathcal{I}_{W\setminus(W\cap C\cup D)}(d-4)) = 0$. A Castelnuovo's exact sequence gives $h^1(U, \mathcal{I}_{W\setminus W\cap C}(d-2)) = 0$. This completes the proof.

Proof of Proposition 1.5. By Lemma 3.2 it only remains to prove that if $sr(P) = d+1$, $B, B_1 \in \mathcal{S}(P)$ and $B \ne B_1$, then $B \cap B_1 = \emptyset$. Assume $B \cap B_1 \ne \emptyset$. Hence $\sharp(B \cup B_1) \le 2d+1$. Since $B \cup B_1 \subset C$, we get $h^1(\mathbb{P}^m, \mathcal{I}_{B\cup B_1}(d)) = 0$, contradicting [2], Lemma 1. □

Lemma 3.3. *Fix $A, S \subset \mathbb{P}^m$, $m \ge 2$, such that $\sharp(A \cup S) \le 2d+2$ and $A \cup S$ is not in linearly general position in $\langle A \cup S\rangle$. Assume the existence of $P \in \langle \nu_d(A)\rangle \cap \langle \nu_d(S)\rangle$ such that $P \notin \langle \nu_d(A')\rangle$ for any $A' \subsetneq A$ and $P \notin \langle \nu_d(S')\rangle$ for any $S' \subsetneq S$. Then (A, S, P) is either as in case A or as in case C.*

Proof. First assume $m = 2$. We repeat the proof of Theorem 1.2. Set $W_0 := A \cup S$ and let $L_1 \subset \mathbb{P}^2$ be any line such that $\sharp(W_0 \cap L_1)$ is maximal. Set $W_1 := W_0 \setminus L_1 \cap W_0$. Define inductively the line L_i, $i \ge 1$, as one of the lines such that $b_i := \sharp(L_i \cap W_{i-1})$ is maximal and set $W_i := W_{i-1} \setminus L_i \cap W_{i-1}$. Notice that if $b_i \le 1$, then $b_j = 0$ for all $j > i$. Since W_0 is not in linearly general position, we have $b_1 \ge 3$. Hence $b_i = 0$ for $i \ge d+1$, $b_{d+1} \le 1$ and $b_{d+1} = 1$ if and only if $b_i = 2$ for $2 \le i \le d$. Let k be the minimal integer i such that $h^1(L_i, \mathcal{I}_{W_{i-1}\cap L_i}(d+1-i)) > 0$, i.e. such that $b_i \ge d+3-i$ (k exists by [2], Lemma 1). If $k = 1$, i.e. if $b_1 \ge d+2$, then (A, S, P) is in case A by Lemma 3.1. Assume $k \ge 2$. Since $b_{d+1} \le 1$ and $b_i = 0$ for all $i \ge d+2$, we have $k \le d$. Hence $\sharp(W_0) \ge k(d+3-k) \ge 2(d+1)$ and the last equality holds if and only if $k = 2$. Assume $k = 2$. Hence $b_2 \ge d+1$. Since $\sharp(A \cup S) \le 2d+2$, we get $b_1 = b_2 = d+1$ and $b_3 = 0$. Hence $W_1 \subset L_2$. Since $b_2 = b_1$, we must have $L_2 \cap W_1 = L_2 \cap (A \cup S)$, i.e. $L_1 \cap L_2 \notin (A \cup S)$. Hence (A, S, P) is as in case C with respect to the reducible conic $L_1 \cup L_2$.

Now assume $m > 2$. We repeat the same proof starting from a hyperplane $H_1 \subset \mathbb{P}^m$ such that $\sharp((A \cup S) \cap H_1)$ is maximal. If $A \cup S \subset H_1$, we conclude by induction on m. Now assume $(A \cup S) \cap H_1 \neq H_1$. Hence $\sharp((A \cup S) \cap H_1)) \leq 2d + 1$. First assume $h^1(H_1, \mathcal{I}_{(A \cup S) \cap H_1}(d)) > 0$. By [3], Lemma 34, we have $\sharp((A \cup S) \cap H_1) \geq d + 2$ and there is a line $D \subset H_1$ such that $D \cap (A \cup S) \geq d + 2$. Lemma 3.1 gives that (A, S, P) is as in case A. Now assume $h^1(H_1, \mathcal{I}_{(A \cup S) \cap H_1}(d)) = 0$. We continue as in the case $m = 2$ using hyperplanes H_i instead of lines L_i. Now the inequality $b_k \geq d + 3 - k$ does not follow from the cohomology of line bundles on $L_k \cong \mathbb{P}^1$, but from [3], Lemma 34.This completes the proof.

Lemma 3.4. *Fix an integer $d \geq 2$. Fix lines L_1, L_2 of $\mathbb{P}^2$ and set $\{Q\} := L_1 \cap L_2$. Fix sets A, S such that $A \cap S = \emptyset$, $Q \notin (A \cup S)$, $A \cup S \subset L_1 \cup L_2$, and $\sharp((A \cup S) \cap L_1) = \sharp((A \cup S) \cap L_2) = d+1$. Then $\langle \nu_d(A) \rangle \cap \langle \nu_d(S) \rangle$ is a single point (call it P), and $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$, $P \notin \langle \nu_d(S') \rangle$ for any $S' \subsetneq S$.*

Proof. Since $L_1 \cup L_2$ is a reducible conic, we have $\dim(\langle \nu_d(L_1 \cup L_2) \rangle) = 2d$. Since $\sharp(A \cap S) \cap L_i) \geq d + 1$, we have $\langle \nu_d(L_i) \rangle \subset \langle \nu_d(A \cup S) \rangle$. Hence $\dim(\langle \nu_d(A \cup S) \rangle) = 2d$. Since $A \cap S = \emptyset$ and $\sharp(A \cup S) = 2d + 2$, we get $h^1(\mathbb{P}^2, \mathcal{I}_{A \cup S}(d)) = 1$ and that $\langle \nu_d(A) \rangle \cap \langle \nu_d(S) \rangle$ is a single point (call it P). Fix $A' \subsetneq A$. Since $\sharp(A' \cup S) \leq 2d + 1$ and no line contains at least $d + 2$ points of $A' \cup S$, [3], Lemma 34, gives $h^1(\mathbb{P}^2, \mathcal{I}_{A' \cup S}(d)) = 0$, i.e. $\langle \nu_d(A') \rangle \cap \langle \nu_d(S) \rangle = \langle \nu_d(A' \cap S) \rangle = \emptyset$. Hence $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$. Similarly, $P \notin \langle \nu_d(S') \rangle$ for any $S' \subsetneq S$. This completes the proof.

Notice that in the statement of Lemma 3.4 we allow the case $S \subset L_i$, i.e., $A \subset L_{2-i}$.

Proof of Proposition 1.3. By Lemma 3.3 to prove part (a) we may assume that $A \cup S$ is in linearly general position in $U := \langle A \cup S \rangle$. Since $\sharp(A \cup S) < 3d$ and $A \cup S$ is linearly independent in U, [11], theorem 3.8, gives the existence of a smooth plane conic C such that $\sharp(C \cap (A \cup S)) \geq 2d+2$. Hence $A \cup S \subset C$ and $A \cap S = \emptyset$. Hence (A, S, P) is as in case B. Part (b) in case C is true by Lemma 3.4. The proof of part (b) in case B is similar, but easier, because any $E \subset C$ with $\sharp(E) \leq 2d-1$ satisfies $h^1(\mathbb{P}^m, \mathcal{I}_E(d)) = 0$. □

Lemma 3.5. *Fix a line $D \subset \mathbb{P}^m$, $m \geq 2$, and a finite set $B \subset \mathbb{P}^m$ such that $\sharp(B \setminus B \cap D) \leq d$. Then $\langle \nu_d(B) \rangle \cap \langle \nu_d(D) \rangle = \langle \nu_d(B \cap D) \rangle$.*

Proof. Fix a general hyperplane $H \subset \mathbb{P}^m$ containing D. Since B is finite and H is general, we have $B \cap H = B \cap D$. Since $\sharp((B \setminus B \cap D) \leq d-1$, we have $h^1(\mathcal{I}_{B \setminus B \cap D}(d-1)) = 0$. Hence a Castelnuovo's sequence and linear algebra gives $\langle \nu_d(B \setminus B \cap D) \rangle \cap \langle \nu_d(D) \rangle = \emptyset$. Hence $\langle \nu_d(B) \rangle \cap \langle \nu_d(D) \rangle = \langle \nu_d(B \cap D) \rangle$. This completes the proof.

Proof of Proposition 1.4. Since $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$, $\nu_d(A)$ is linearly independent. For the same reason $\nu_d(S)$ is linearly independent. Since (A, S, P) is as in case A with respect to the line D, we have $E = S \setminus D \cap S$. Since $P \in \langle \nu_d(A) \rangle$ and $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$, the set $\langle \nu_d(E) \cup \{P\} \rangle \cap \langle \nu_d(A \cap D) \rangle$ is a single point and we called it P_1. Lemma 3.5 gives $\langle \nu_d(E) \rangle \cap \langle \nu_d(D) \rangle = \emptyset$. Hence $\langle \nu_d(E) \cup \{P\} \rangle \cap \langle \nu_d(D) \rangle$ is at most one point. Hence $\langle \nu_d(E) \cup \{P\} \rangle \cap \langle \nu_d(D) \rangle = \{P_1\}$. Taking S instead of A we get $\langle \nu_d(E) \cup \{P\} \rangle \cap \langle \nu_d(S \cap D) \rangle = \{P_1\}$.

(i) In this step we check part (c). Assume $D \neq \overline{D}$. Notice that $D \cup \overline{D}$ is contained in a quadric hypersurface (even if $m \geq 3$ and $D \cap \overline{D} = \emptyset$). Set $G := \overline{A} \setminus \overline{A} \cap \overline{D}$. Using $\overline{A}$, $\overline{S}$, $\overline{D}$, and G instead of A, S, D, and E, we get that $\langle \{P\} \cup G \rangle \cap \langle \nu_d(\overline{D}) \rangle$ is a single point. Call it P_3. Since $\sharp(E \cup G) \leq d-1$, we have $h^1(\mathcal{I}_{E \cup G}(d-2)) = 0$. Hence a Castelnuovo's exact sequence and the fact that $D \cup \overline{D}$ is contained in a quadric hypersurface give $\langle \nu_d(E \cup G) \rangle \cap \langle \nu_d(D \cup \overline{D}) \rangle = \emptyset$. Hence $\langle \{P\} \cup \nu_d(E \cup G) \rangle \cap \langle \nu_d(D \cup \overline{D}) \rangle)$ is at most one point. Hence $P_3 = P_1$ and $\langle \{P\} \cup \nu_d(E \cup G) \rangle \cap \langle \nu_d(D \cup \overline{D}) \rangle = \{P_1\}$. Hence $P_1 \in \langle \nu_d(D) \rangle \cap \langle \nu_d(\overline{D}) \rangle$. Since $d \geq 2$, we have $\langle \nu_d(D) \rangle \cap \langle \nu_d(\overline{D}) \rangle = \nu_d(D \cap \overline{D})$. Hence $D \cap \overline{D} \neq \emptyset$ and $\{P_1\} = \nu_d(D \cap \overline{D})$. Hence $sr(P_1) = 1$. Recall that $P_1 \in \langle \nu_d(A \cap D) \rangle$. Since any $d+1$ points of $\nu_d(D)$ are linearly independent, we get that either $P_1 \in A \cap D$ or $\sharp(A \cap D) \geq d+1$. Notice that if $P_1 \in \nu_d(A \cap D)$, then $A \cap D$ is the only point, Q', such that $\nu_d(Q') = P_1$, because $P \in \langle \{P_1\} \cup \nu_d(E) \rangle$ and $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$. Hence the assumption $2 \leq \sharp(A \cap D) \leq d$ made in part (c) is not satisfied.

(ii) In this step we check part (a). Obviously, $sr(P) \leq sr(P_1) + \sharp(E)$. Fix $B \in \mathcal{S}(P)$ and $B_1 \in \mathcal{S}(P_1)$. By a parsimony lemma we have $B_1 \subset D$ ([14], Proposition 3.1, [15], theorem 2.1). Set $M := E \cup B_1$. We have $P \in \langle \nu_d(M) \rangle$. Let M' be a minimal subset of M such that $P \in \langle M \rangle$.

Claim: We have $M' = M$.

Proof of the Claim: Assume $M' \neq M$. Hence either there is $E' \subsetneq E$ such that $P \in \langle \nu_d(E' \cup B_1) \rangle$ or there is $B' \subsetneq B_1$ such that $P \in \langle \nu_d(E \cup B') \rangle$. First assume the existence of E'. Since $B_1 \subset D$ and $P \notin \langle \nu_d(E) \rangle$, we get $\langle \{P\} \cup \nu_d(E') \rangle \cap \langle \nu_d(D) \rangle \neq \emptyset$. Since $\{P_1\} = \langle \{P\} \cup \nu_d(E) \rangle \cap \langle \nu_d(D) \rangle$, we get $\langle \{P\} \cup \nu_d(E') \rangle \cap \langle \nu_d(D) \rangle = \{P_1\}$. Since $P_1 \in \langle \nu_d(A \cap D) \rangle$, we get $P \in \langle \nu_d(E' \cup (A \cap D) \rangle$. Since $E' \cup (A \cap D) \subsetneq E$, we obtained a contradiction. Now assume the existence of $B' \subsetneq B_1$ such that $P \in \langle \nu_d(E \cup B') \rangle$. Since $\langle \{P\} \cup \nu_d(E) \rangle \cap \langle \nu_d(D) \rangle = \{P_1\}$, we get $P_1 \in \langle \nu_d(B' \cup E) \rangle$. Taking B' minimal and applying [2], Lemma 1, to P_1 we get $h^1(\mathcal{I}_{E \cup B_1 \cup B}(d)) > 0$. Since $E \cup B_1 \cup B = E \cup B$ and $\sharp(E \cup B) \leq 2d + 1$, there is a line $T \subset \mathbb{P}^m$ such that $\sharp(T \cap (E \cup B)) \geq d + 2$. Since $\sharp(E) \leq d - 1$ and $B \subset D$, we have $T = D$. Since $D \cap E = \emptyset$ and $\sharp(B) < d + 2$, we get a contradiction.

Assume $M \neq B$. Since $P \notin \langle \nu_d(M_1) \rangle$ for any $M_1 \subsetneq M$ by the Claim and B has the same property, [2], Lemma 1, gives $h^1(\mathcal{I}_{M \cup B}(d)) > 0$. Since $B_1 \in \mathcal{S}(P_1)$ and $P_1 \in \langle \nu_d(A \cap D) \rangle \cap \langle \nu_d(A \cap S) \rangle$, we have $\sharp(M) \leq \min\{\sharp(A), \sharp(S)\}$. Since $B \in \mathcal{S}(P)$ and $P \in \langle \nu_d(M) \rangle$, we have $\sharp(B) \leq \sharp(M)$. Hence $\sharp(M \cup B) \leq 2d + 2$.

(ii.1) Here we assume $\sharp(M \cup B) \leq 2d + 1$. Since $h^1(\mathcal{I}_{M \cup B}(d)) > 0$, there is a line $T \subset \mathbb{P}^m$ such that $\sharp(T \cap (M \cup B)) \geq d + 2$, $\nu_d(M \cup B \setminus (M \cup B) \cap T)$ is linearly independent and $\langle \nu_d(M \cup B \setminus (M \cup B) \cap T) \rangle \cap \langle \nu_d(T) \rangle = \emptyset$. Lemma 3.1 gives $M \setminus M \cap T = B \setminus B \cap T$. Hence $\sharp(B \cap T) \leq \sharp(M \cap T)$. Assume for the moment $T = D$. Since $M \setminus M \cap T = B \setminus B \cap T$, we get $E \subseteq B$, say $B = E \sqcup B_2$ with $\sharp(B_2) \leq \sharp(B_1)$ and $B_2 \subset D$. Since $\dim(\langle \nu_d(E \cup D) \rangle) = d + \sharp(E)$ and $B_2 \subset D$, we have $\langle \nu_d(B) \rangle \cap \langle \nu_d(D) \rangle = \langle \nu_d(B_2) \rangle$ (Grassmann's formula). Since $P_1 \in \langle \nu_d(E) \cup \{P\} \rangle$, $\langle \nu_d(E) \cup \{P\} \rangle \subseteq \langle \nu_d(B) \rangle$ and $P_1 \in \langle \nu_d(B) \rangle$, we get $P_1 \in \langle \nu_d(B_2) \rangle$. Since $\sharp(B_2) \leq \sharp(B_1) = sr(P_1)$, we get $B_2 \in \mathcal{S}(P)$. Hence $B \in \Gamma$. Now assume $T \neq D$. Since (B, M, P) is in case A with respect to the line T, step (i) gives a contradiction, unless either $B \cap T$ is a single point or $\sharp(B \cap T) \geq d + 1$. First assume $\sharp(B \cap T) = 1$. Hence $\sharp(M \cap T) \geq d + 1$. Since $\sharp(M \cap D \cap T) \leq 1$ and $\sharp(E) \leq d$, this is absurd. Now assume $\sharp(B \cap T) \ge d+1$. Since $\sharp(B) \le \sharp(M) \le \min\{\sharp(A), \sharp(S)\}$, we get $\sharp(A) = \sharp(S) = \sharp(M) = \sharp(B) = d+1$ and $B \subset T$. Hence $P \in \langle \nu_d(T) \rangle$. Hence $sr(P) \le d$ ([8], [15], Theorem 4.1, or [3], §3). Hence $\sharp(B) \le d$, a contradiction.

(ii.2) Here we assume $\sharp(B \cup M) = 2d+2$. Since $\sharp(B) \le \sharp(M) \le \min\{\sharp(A), \sharp(S)\}$, we have $\sharp(A) = \sharp(S) = \sharp(M) = \sharp(B) = d+1$, and $M \cap B = \emptyset$. Since $\sharp(M) = \sharp(B)$, we get $M \in \mathcal{S}(P)$.

(iii) Now we check part (b). Set $F := \widetilde{A} \setminus \widetilde{A} \cap D$. Since $\sharp(A) + \sharp(S) \le 2d+2$, we have $\sharp(E) \le d/2$. Similarly we get $\sharp(F) \le d/2$. Hence $\sharp(E \cup F) \le d$. We saw at the beginning of the proof that $\langle \{P\} \cup \nu_d(F) \rangle \cap \langle \nu_d(D) \rangle$ is a unique point. We call it P_2. We saw in step (ii) that $sr(P) = sr(P_2) + \sharp(F)$. Since $\sharp(E \cup F) \le d$, Lemma 3.5 gives $\dim(\langle \nu_d(E \cup F) \rangle) = \sharp(E \cup F)$ and $\langle \nu_d(E \cup F) \rangle \cap \langle \nu_d(D) \rangle = \emptyset$. Hence $\langle \nu_d(E \cup F) \cup \{P\} \rangle \cap \langle \nu_d(D) \rangle$ is at most one point. Therefore $P_2 = P_1$. Hence $\sharp(F) = \sharp(E)$. □

Example 3.6. *Fix integers m, d, e such that $m \ge 2$, $d \ge 2$ and $0 \le e \le d-1$. Fix a line $D \subset \mathbb{P}^m$, $P_1 \in D$, $S_1 \subset D \setminus \{P_1\}$ such that $\sharp(S_1) = d+1$ and $E \subset \mathbb{P}^m$ such that $\sharp(E) = e$ (if $e = 0$ we just take $P = P_1$). Set $A := \{P_1\} \cup E$ and $S = S_1 \cup E$. Since Obviously (A, S, P) is as in case A with respect to the line D. Take a general line $\overline{D} \subset \mathbb{P}^m$ containing P_1 and $\overline{S}_1 \subset \overline{D} \setminus \{P_1\}$ with $\sharp(\overline{S}_1) = d+1$. We also assume $\overline{S}_1 \cap E = \emptyset$. Set $\overline{A} := A$ and $\overline{S} := E \sqcup \overline{S}_1$. The triple $(\overline{A}, \overline{S}, P)$ is as in case A with respect to the line $\overline{D} \ne D$.*

Lemma 3.7. *Assume $d \ge 5$. Take (A, S, P) as in case C with respect to the lines L_1 and L_2. Assume $S \subset L_1$. Set $\{Q\} := L_1 \cap L_2$ and $B := \{Q\} \cup A_1$. Then $sr(P) = \min\{\sharp(S), 2+d-\sharp(S)\}$. If $\sharp(S) < (d+2)/2$, then $\mathcal{S}(P) = \{S\}$. If $\sharp(S) > (d+2)/2$, then $\mathcal{S}(P) = \{B\}$. If $\sharp(S) = (d+2)/2$, then $sr(P) = \sharp(S)$, $\mathcal{S}(P)$ is one-dimensional and every element of $\mathcal{S}(P)$ is contained in L_1.*

Proof. Since $S \subset L_1$, we have $P \in \langle \nu_d(L_1) \rangle$. By a parsimony lemma ([14], Proposition 3.1, or [5], Theorem 2.1, for a generalization of the non-symmetric one), every element of $\mathcal{S}(P)$ is contained in L_1. Since $\sharp(A \cap L_2) = d+1$, we have $\langle \nu_d(A \cap L_2) \rangle = \langle \nu_d(L_2) \rangle$. Since $\langle \nu_d(L_1) \rangle \cap \langle \nu_d(L_2) \rangle = \{\nu_d(Q)\}$ and $\nu_d(A)$ is linearly independent, we get $\langle \nu_d(A) \rangle \cap \langle \nu_d(L_1) \rangle = \langle \nu_d(B) \rangle$. Hence $P \in \langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle$. Since $Q \notin (A \cup S)$, we have $\sharp(S) +$

$\sharp(B) = d+2$. Since any $d+1$ points of $\nu_d(L_1)$ are linearly independent, all the statements are obvious consequences of Sylvester's theorem ([8], [15], Theorem 4.1, [3], Theorem 23). This completes the proof.

Proof of Proposition 1.6. Assume $sr(P) < \min\{\sharp(A), \sharp(S)\}$ and fix $P \in \mathcal{S}(P)$. Fix any $E \subset A \cup B$ such that $\sharp(E) = 2d+1$. Since $\sharp(E) \le 2d+1$ and $\sharp(R \cap E) \le d+1$ for every line $R \subset \mathbb{P}^m$, then $h^1(\mathcal{I}_E(d)) = 0$ ([3], Lemma 34). Hence $\dim(\langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle) \le 1 + \dim(\langle A \cap B \rangle) = 1 - 1$. Hence $\langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle = \{P\}$. Assume $sr(P) < \min\{\sharp(A), \sharp(B)\}$ and take $B \in \mathcal{S}(P)$. Since $P \notin \langle \nu_d(A') \rangle$ for any $A' \subsetneq A$, we have $B \nsubseteq A$. Since $\sharp(A \cup B) \le 2d+1$ and $h^1(\mathcal{I}_{A \cup B}(d)) > 0$ ([2], Lemma 1), there is a line D such that $\sharp(D \cap (A \cup B)) \ge d+2$. Lemma 3.1 gives $B \setminus B \cap D = A \setminus A \cap D$. For the same reason there is a line R such that $B \setminus B \cap R = S \setminus A \cap R$.

(a) First assume $R = D$. Since $A \cap S = \emptyset$ and $A \setminus A \cap D = B \setminus B \cap D = S \setminus S \cap D$, we get $A \cup S \subset D$, contradicting the assumption $\sharp((A \cup S) \cap L_i) = d+1$ for all i.

(b) Now assume $R \ne D$ and $\{L_1, L_2\} \ne \{D, R\}$. First assume $D \notin \{L_1, L_2\}$. Therefore $\sharp(D \cap (L_1 \cup L_2)) \le 2$. Since $A \subset L_1 \cup L_2$, we get $\sharp(A \cap D) \le 2$. Hence $\sharp(B \cap D) \ge d$. Since $\sharp(B) < \min\{\sharp(A), \sharp(S)\} \le d+1$, we get $\sharp(A) = \sharp(S) = d+1$, $sr(P) = \sharp(B) = d$, and $B = B \cap D$, i.e. $B \subset D$. Assume for the moment $R \in \{L_1, L_2\}$, say $R = L_1$. Since $B \subset D$, $D \ne L_1$ and $\sharp((B \cup S) \cap D) \ge d+2$, we get $S \subset L_1$. We analyzed this case in Lemma 3.7. Now assume $R \notin \{L_1, L_2\}$. Hence $\sharp(R \cap S) \le 2$. Hence $\sharp(R \cap B) \ge d > 1$. Since $B \subset D$ and $R \ne D$, we get a contradiction.

(c) Now assume $R \ne D$ and $\{L_1, L_2\} = \{D, R\}$, say $L_1 = D$ and $L_2 = R$. Set $B_i := B \cap L_i$, $i = 1, 2$. Since $A \setminus A \cap D = B \setminus B \cap D$, we get $A_2 = B \setminus (B \cap B_1)$. Hence $B \subset L_1 \cup L_2$. Since $S_1 = S \setminus S \cap R = B \setminus B_2$, we get that either $B = S_1 \cup A_2$ or $B = S_1 \cup A_2 \cup \{Q\}$. We have $\sharp(A_1) + \sharp(S_1) = d+1$. Since $\sharp(A_1) + \sharp(B_1) \ge d+2$, we get $B = S_1 \cup A_2 \cup \{Q\}$. Similarly, if $L_1 = R$ and $L_2 = D$, then we get $B = S_2 \cup A_1 \cup \{Q\}$. $\square$

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