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STRONG CONVERGENCE THEOREMS FOR FIXED POINT PROBLEMS AND GENERALIZED EQUILIBRIUM PROBLEMS OF THREE RELATIVELY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. The purpose of this paper is to introduce a new hybrid projection algorithm for finding a commom element of the set of common fixed points of three relatively quasi-nonexpansive mappings and the set of solutions of a generalized equilibrium problem in Banach space. Our results improve and extend the corresponding results announced by many others.

Keywords: Relatively quasi-nonexpansive mappings; Generalized equilibrium problem; Fixed points.

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1. Introduction

Let E be a real Banach space with the dual space of E^* and let $\langle \cdot, \cdot \rangle$ be the generalized duality pairing between E and E^*. Let C be a nonempty closed convex subset of E. We denote the sets of nonnegative integers and real numbers by N and R respectively. Let $A : C \to E^*$ be a nonlinear mapping and $f : C \times C \to R$ be a bifunction. The generalized

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equilibrium problem is to find $u \in C$, such that

$$f(u,y) + \langle Au, y-u\rangle \geq 0, \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by GEP.

Whenever $A \equiv 0$, problem (1.1) is equivalent to finding $u \in C$, such that

$$f(u,y) \geq 0, \forall y \in C. \tag{1.2}$$

The set of its solutions is denoted by EP.

Whenever $f \equiv 0$, problem (1.1) is equivalent to finding $u \in C$, such that

$$\langle Au, y-u\rangle \geq 0, \forall y \in C. \tag{1.3}$$

The set of its solutions is denoted by $VI(C,A)$.

We recall some definitions and results which will be needed in this paper. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$. Denote by $F(T)$ the set of fixed points of T, that is $F(T) = \{x \in C : Tx = x\}$. A mapping $A : C \to E^*$ is called $\alpha-$ inverse-strongly monotone, if there exits an $\alpha > 0$, such that $\langle Ax-Ay, x-y\rangle \geq \alpha\|Ax-Ay\|^2, \forall x, y \in C$. It is easy to see that if A is $\alpha-$ inverse-strongly monotone mapping, then it is $\frac{1}{\alpha}$-Lipschitzian, i.e. $\|Ax - Ay\| \leq \frac{1}{\alpha}\|x - y\|, \ \forall x, y \in C$.

The mapping $J : E \to 2^{E^*}$ defined by $Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}, \ \forall x \in E$ is called the normalized duality mapping. It is well-known that if E^* is uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E. We also defined the function ϕ as following

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \ \forall x, y \in E. \tag{1.4}$$

Following Alber[1], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = argmin_{y \in C}\phi(y,x), \ \forall x \in E.$$

It is clear that in Hilbert space H, (1.4) reduces to $\phi(x,y) = \|x-y\|^2$ and Π_C is the metric projection of H onto C.

Very recently, Takhashi and Zembayashi[2] proposed the following iteration for a relatively nonexpansive mapping:

$$\begin{cases} x_0 = x \in C,\ C_0 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ H_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, \end{cases} \tag{1.5}$$

and proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(S)\cap EP}x_0$.

In 2008,Qin et al.[3] introduced the following iterative for two closed relatively quasi-nonexpansive mappings in Banach space:

$$\begin{cases} x_0 \in E,\ C_1 = C, \\ y_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{1.6}$$

and proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap F(S)\cap EP}x_0$.

In 2009, K.Wattanawitoon and P.Kuman[4] introduced the following iterative for two closed relatively quasi-nonexpansive mappings in Banach space:

$$\begin{cases} x_0 \in E,\ C_1 = C, \\ y_n = J^{-1}(\delta_n Jx_n + (1-\delta_n)Jz_n), \\ z_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases} \tag{1.7}$$

and proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap F(S)\cap EP}x_0$.

In 2010, S.S. Chang[5]introduced the following iterative for two relatively nonexpansive mappings in Banach space:

$$\begin{cases} x_0 \in E,\ C_1 = C, \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)JSz_n), \\ z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \langle Au_n, y-u_n\rangle + \frac{1}{r_n}\langle y-u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ H_n = \{v \in C : \phi(v, u_n) \leq \beta_n\phi(v, x_n) + (1-\beta_n)\phi(v, z_n) \leq \phi(v, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n\cap W_n} x_0, \end{cases} \tag{1.8}$$

and proved that the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap F(S)\cap GEP} x_0$.

In this paper, motivated by K.Wattanawitoon and P.Kuman[4], we modified iterations of (1.7) to obtain strong convergence theorems for fixed point problems and generalized equilibrium problems of three relatively quasi-nonexpansive mappings in Banach spaces.

2. Preliminaries

Let C be a nonempty closed convex subset of E, and let T be a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F(T)}$. A mapping T from C into itself is said to be relatively nonexpansive if $\widetilde{F(T)} = F(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. T is said to be $\phi-$ nonexpansive, if $\phi(Tx, Ty) \leq \phi(x, y)$ for $x, y \in C$. T is said to be relatively quasi-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Remark2.1 The class of relatively quasi-nonexpansive is more general than the class of relatively nonexpansive mapping which requires the strong restriction: $\widetilde{F(T)} = F(T)$.

Lemma2.2(Kamimura and Takahashi[9]) Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.

Lemma2.3(Alber[1]) Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if

$$\langle x_0 - y, Jx - Jx_0\rangle \geq 0, \forall y \in C.$$

Lemma2.4(Alber[1]) Let E be a reflexive, strictly convex subset of a smooth Banach space and let C be a nonempty closed convex subset of E and let $x \in E$. Then

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(x, y), \forall y \in C.$$

Lemma2.5(Qin et al.[3]) Let E be a uniformly convex and smooth Banach space, let C be a closed convex subset of E, and let T be a closed and relatively quasi-ϕ-nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.

Lemma 2.6(Cho et al.[10]) Let E be a uniformly convex Banach space and $B_r(0)$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0, +\infty) \to [0, +\infty)$ with $g(0) = 0$ such that

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|),$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

Lemma 2.7(Kamimura and Takahashi[9]) Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and $g(\|x - y\|) \leq \phi(x, y)$ for all $x, y \in B_r$.

For solving the generalized equilibrium problem, let us assume a bifunction f satisfied the following conditions

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e. $f(x, y) + f(y, x) \leq 0 forall x, y \in C$;

(A3) for all $x, y, z \in C, limsup_{t\downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$;

(A4) for all $x \in C, f(x, \cdot)$ is convex and lower semicontinuous.

Lemma2.8 (Blum and Oetti [11])Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, let $f : C \times C \to R$ be a bifunction satisfying (A1)-(A4),and let $r > 0$ and $x \in E$, then there exists $z \in C$ such that

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \ \ \forall y \in C. \tag{2.4}$$

Lemma2.9(Takahashi and Zembayashi [2]) Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E, and let $f : C \times C \to R$ be a bifunction satisfying (A1)-(A4), for $r > 0$ and $x \in E$, define a mapping $T_r : E \to C$ as follows

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\}.$$

for all $x \in E$. Then

(i)T_r is single-valued;

(ii)T_r is a firmly nonexpansive-type mapping,i.e.,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle, \quad \forall x, y \in E;$$

(iii)$F(T_r) = \widetilde{F(T_r)} = EP$;

(iv)EP is a closed convex subset of C.

Lemma2.10(Takahashi and Zembayashi [2]) Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let $f : C \times C \to R$ be a bifunction satisfying (A1)-(A4), and $r > 0$. Then for $x \in E$ and $q \in F(T_r)$,

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Lemma2.11(S.S.Chang[5]) Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex of E. Let $A : C \to E^*$ be an $\alpha-$inverse-strongly monotone mapping, let f be a function from $C \times C \to R$ satisfying (A1)-(A4), and let $r > 0$. Then the following statements hold.

(I)for $x \in E$, there exists $u \in C$ such that

$$f(u, y) + \langle Au, y - u\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \forall y \in C;$$

(II)if E is additionally uniformly smooth and $K_r : E \to C$ is defined as

$$K_r(x) = \{u \in C : f(u, y) + \langle Au, y - u\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \forall y \in C\},$$

then the mapping K_r has the following properties

(i)K_r is single-valued.

(ii)K_r is a firmly nonexpansive-type mapping, i.e.,

$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle, \quad \forall x, y \in E,$$

(iii)$F(K_r) = \widetilde{F(K_r)} = GEP$,

(iv)GEP is a closed convex subset of C,

(v)$\phi(q, K_r x) + \phi(K_r x, x) \leq \phi(q, x), \forall q \in F(K_r)$.

3. Main results

Theorem3.1 Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $A : C \to E^*$ be an $\alpha-$ inverse-strong monotone mapping and let f be a bifunction from $C \times C \to R$ satisfying (A1)-(A4), let $T, S, R : C \to C$ be three closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap F(S) \cap F(R) \cap GEP \neq \emptyset$. $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ are the sequences generated by the following,

$$\begin{cases} x_0 \in E, \ C_1 = C, \ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\delta_n Jx_n + (1-\delta_n)JRz_n), \\ z_n = J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \delta_n \phi(z, x_n) + (1-\delta_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{3.1}$$

Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}and\{\delta_n\}$ are sequences in [0,1] satisfying the restrictions,

(a)$\alpha_n + \beta_n + \gamma_n = 1$;

(b)$\underline{lim}_{n\to\infty}\alpha_n\beta_n > 0, \underline{lim}_{n\to\infty}\alpha_n\gamma_n > 0, \underline{lim}_{n\to\infty}\delta_n(1-\delta_n) > 0$;

(c)$\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$and $\{u_n\}$ converge strongly to $z \in F$,where $z = \Pi_F x_0$.

Proof. We split the proof into six steps.

Step1. We show that C_n is closed and convex for all $n \geq 0$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in N$. For $z \in C_k$, one obtains that

$$\phi(z, u_k) \leq \delta_k \phi(z, x_k) + (1 - \delta_k)\phi(z, z_k),$$

is equivalent to

$$2\langle z, (1 - \delta_k)Jz_k + \delta_k Jx_k - Ju_k\rangle \leq (1 - \delta_k)\|z_k\|^2 - \|u_k\|^2 + \delta_k\|x_k\|^2,$$

and

$$\delta_k \phi(z, x_k) + (1 - \delta_k)\phi(z, z_k) \leq \phi(z, x_k),$$

is equivalent to

$$2\langle z, Jx_k - Jz_k\rangle \leq \|x_k\|^2 - \|z_k\|^2.$$

It implies that C_{k+1} is closed and convex. Then, for all $n \geq 0$, C_n is closed and convex. This show that $\Pi_{C_{n+1}} x_0$ is well defined. Notice that $u_n = K_{r_n} y_n$ for all $n \geq 1$.

Step 2. Let us show that $F \subset C_n$ for each $n \geq 0$.

$F \subset C_1 = C$ is obvious, suppose $F \subset C_k$ for some $k \in N$, then for any $w \in F \subset C_k$, one has,

$$\begin{aligned}
\phi(w, z_k) &= \phi(w, J^{-1}(\alpha_k Jx_k + \beta_k JTx_k + \gamma_k JSx_k)) \\
&= \|w\|^2 - 2\alpha_k\langle w, Jx_k\rangle - 2\beta_k\langle w, JTx_k\rangle - 2\gamma_k\langle w, JSx_k\rangle + \|\alpha_k Jx_k + \beta_k JTx_k + \gamma_k JSx_k\|^2 \\
&\leq \|w\|^2 - 2\alpha_k\langle w, Jx_k\rangle - 2\beta_k\langle w, JTx_k\rangle - 2\gamma_k\langle w, JSx_k\rangle \\
&\quad + \alpha_k\|Jx_k\|^2 + \beta_k\|JTx_k\|^2 + \gamma_k\|JSx_k\|^2 \\
&= \alpha_k \phi(w, x_k) + \beta_k \phi(w, Tx_k) + \gamma_k \phi(w, Sx_k) \\
&\leq \phi(w, x_k),
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
\phi(w,u_k) &= \phi(w,K_{r_k}y_k) \\
&\leq \phi(w,y_k) \\
&= \phi(w,J^{-1}(\delta_k Jx_k+(1-\delta_k)JRz_k)) \\
&\leq \|w\|^2-2\delta_k\langle w,Jx_k\rangle-2(1-\delta_k)\langle w,JRz_k\rangle+\delta_k\|x_k\|^2+(1-\delta_k)\|Rz_k\|^2 \\
&= \delta_k\phi(w,x_k)+(1-\delta_k)\phi(w,Rz_k) \\
&\leq \delta_k\phi(w,x_k)+(1-\delta_k)\phi(w,z_k) \\
&\leq \delta_k\phi(w,x_k)+(1-\delta_k)\phi(w,x_k) \\
&= \phi(w,x_k),
\end{aligned} \tag{3.3}$$

that is $w \in C_{k+1}$. This implies that $F \subset C_n$ for all $n \geq 0$.

Step 3. We claim that $\{x_n\}$ is bounded, and $lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$. Indeed, by the definition of $x_n = \Pi_{C_n}x_0$, from Lemma2.4 it follows that for each $w \in F$ and each $n \geq 1$,we obtain

$$\phi(x_n,x_0) = \phi(\Pi_{C_n}x_0,x_0) \leq \phi(w,x_0) - \phi(w,\Pi_{C_n}x_0) \leq \phi(w,x_0).$$

This implies that $\{\phi(x_n,x_0)\}$ is bounded, and so $\{x_n\},\{u_n\},\{z_n\},\{Tx_n\},\{Sx_n\},\{Rz_n\}$ are all bounded. Furthermore, noticing that $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we get $\phi(x_n,x_0) \leq \phi(x_{n+1},x_0)$, for all $n \geq 0$. Thus, $\phi(x_n,x_0)$ is nondecreasing, so the limit of $\phi(x_n,x_0)$ exists, from Lemma 2.4 we have

$$\phi(x_{n+1},x_n) = \phi(x_{n+1},\Pi_{C_n}x_0) \leq \phi(x_{n+1},x_0) - \phi(\Pi_{C_n}x_0,x_0) = \phi(x_{n+1},x_0) - \phi(x_n,x_0),$$

which leads to $lim_{n\to\infty}\phi(x_{n+1},x_n) = 0$, it follows that $lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

Step 4. We will prove that $\{x_n\}$ is a cauchy sequence.

By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = \Pi_{C_m}x_0 \in C_n$ for any positive integer $m \geq n$. It follows that,

$$\phi(x_m,x_n) = \phi(x_m,\Pi c_n x_0) \leq \phi(x_m,x_0) - \phi(\Pi c_n x_0,x_0) = \phi(x_m,x_0) - \phi(x_n,x_0),$$

letting $m,n \to \infty$, one has $\phi(x_m,x_n) \to 0$, it follows $lim_{n\to\infty}\|x_m - x_n\| = 0$. Hence $\{x_n\}$ is a Cauchy sequence. We can assume that $x_n \to p \in C$, as $n \to \infty$.

Step 5. We claim that $p \in F$.

In fact, for $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, from the definition of C_{n+1} we conclude that

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n),$$

and

$$\phi(x_{n+1}, z_n) \leq \phi(x_{n+1}, x_n).$$

It follows that $\phi(x_{n+1}, u_n) \to 0$ and $\phi(x_{n+1}, z_n) \to 0$. One has

$$lim_{n\to\infty}\|x_{n+1} - x_n\| = lim_{n\to\infty}\|x_{n+1} - u_n\| = lim_{n\to\infty}\|x_{n+1} - z_n\| = 0, \tag{3.4}$$

and so

$$lim_{n\to\infty}\|x_n - u_n\| = lim_{n\to\infty}\|x_n - z_n\| = lim_{n\to\infty}\|u_n - z_n\| = 0 \qquad (3.5).$$

Since E is uniformly smooth, and J is uniformly norm-norm continuous on bounded sets, we have

$$lim_{n\to\infty}\|Jx_n - Ju_n\| = lim_{n\to\infty}\|Jx_n - Jz_n\| = 0. \tag{3.6}$$

Let $r = sup_{n\geq 0}\{\|x_n\|, \|Tx_n\|, \|Sx_n\|, \|Rz_n\|\}$. From Lemma 2.9 and Lemma2.6, one has

$$\begin{aligned}
\phi(w, z_n) &= \phi(w, J^{-1}(\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n)) \\
&= \|w\|^2 - 2\alpha_n\langle w, Jx_n\rangle - 2\beta_n\langle w, JTx_n\rangle - 2\gamma_n\langle w, JSx_n\rangle + \|\alpha_n Jx_n + \beta_n JTx_n + \gamma_n JSx_n\|^2 \\
&\leq \|w\|^2 - 2\alpha_n\langle w, Jx_n\rangle - 2\beta_n\langle w, JTx_n\rangle - 2\gamma_n\langle w, JSx_n\rangle + \alpha_n\|Jx_n\|^2 + \beta_n\|JTx_n\|^2 \\
&\quad + \gamma_n\|JSx_n\|^2 - \alpha_n\beta_n g(\|JTx_n - Jx_n\|) \\
&= \alpha_n\phi(w, x_n) + \beta_n\phi(w, Tx_n) + \gamma_n\phi(w, Sx_n) - \alpha_n\beta_n g(\|JTx_n - Jx_n\|) \\
&\leq \phi(w, x_n) - \alpha_n\beta_n g(\|JTx_n - Jx_n\|).
\end{aligned} \tag{3.7}$$

Then

$$\alpha_n\beta_n g(\|JTx_n - Jx_n\|) \leq \phi(w, x_n) - \phi(w, z_n). \tag{3.8}$$

On the other hand,we have

$$\begin{aligned}
\phi(w, x_n) - \phi(w, z_n) &= \|x_n\|^2 - \|z_n\|^2 - 2\langle w, Jx_n - Jz_n\rangle \\
&= (\|x_n\| - \|z_n\|)(\|x_n\| + \|z_n\|) - 2\langle w, Jx_n - Jz_n\rangle \\
&\leq (\|x_n - z_n\|)(\|x_n\| + \|z_n\|) + 2\|w\|\|Jx_n - Jz_n\|.
\end{aligned} \tag{3.9}$$

It follows from (3.5)and(3.6),we have

$$lim_{n\to\infty}[\phi(w,x_n)-\phi(w,z_n)]=0, \tag{3.10}$$

hence,

$$lim_{n\to\infty}g(\|JTx_n-Jx_n\|)=0.$$

From the property of g that

$$lim_{n\to\infty}\|JTx_n-Jx_n\|=0.$$

Since J^{-1} is also uniformly norm-norm continuous on bounded sets, we see that

$$lim_{n\to\infty}\|Tx_n-x_n\|=0. \tag{3.11}$$

Similarly,

$$lim_{n\to\infty}\|Sx_n-x_n\|=0. \tag{3.12}$$

From Lemma 2.6, one also has

$$\begin{aligned}
\phi(w,y_n) &= \phi(w,J^{-1}(\delta_nJx_n+(1-\delta_n)JRz_n) \\
&= \|w\|^2-2\langle w,\delta_nJx_n+(1-\delta_n)JRz_n\rangle+\|\delta_nJx_n+(1-\delta_n)JRz_n\|^2 \\
&\leq \|w\|^2-2\delta_n\langle w,Jx_n\rangle-2(1-\delta_n)\langle w,JRz_n\rangle+\delta_n\|x_n\|^2+(1-\delta_n)\|Rz_n\|^2 \\
&\quad -\delta_n(1-\delta_n)g(\|JRz_n-Jx_n\|) \\
&= \delta_n\phi(w,x_n)+(1-\delta_n)\phi(w,Rz_n)-\delta_n(1-\delta_n)g(\|JRz_n-Jx_n\|) \\
&\leq \delta_n\phi(w,x_n)+(1-\delta_n)\phi(w,z_n)-\delta_n(1-\delta_n)g(\|JRz_n-Jx_n\|) \\
&\leq \phi(w,x_n)-\delta_n(1-\delta_n)g(\|JRz_n-Jx_n\|).
\end{aligned} \tag{3.13}$$

Hence

$$\delta_n(1-\delta_n)g[\|JRz_n-Jx_n\|]\leq\phi(w,x_n)-\phi(w,y_n). \tag{3.14}$$

On the other hand,we get

$$\begin{aligned}
\phi(w,x_n)-\phi(w,y_n) &= \|x_n\|^2-\|y_n\|^2-2\langle w,Jx_n-Jy_n\rangle \\
&= (\|x_n\|-\|y_n\|)(\|x_n\|+\|y_n\|)-2\langle w,Jx_n-Jy_n\rangle \\
&\leq (\|x_n-y_n\|)(\|x_n\|+\|y_n\|)+2\|w\|\|Jx_n-Jy_n\|.
\end{aligned} \tag{3.15}$$

From Lemma2.8, we have

$$\begin{aligned}
\phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n) \\
&\leq \phi(w, y_n) - \phi(w, K_{r_n} y_n) \\
&\leq \phi(w, x_n) - \phi(w, u_n) \\
&\leq \|x_n\|^2 - \|u_n\|^2 - 2\langle w, Jx_n - Ju_n\rangle \\
&\leq (\|x_n - u_n\|)(\|x_n\| + \|u_n\|) + 2\|w\|\|Jx_n - Ju_n\|.
\end{aligned} \tag{3.16}$$

Hence

$$lim_{n\to\infty}\phi(u_n, y_n) = 0,$$

and so,

$$lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.17}$$

Combining with (3.5), we conclude that

$$lim_{n\to\infty}\|x_n - y_n\| = 0, lim_{n\to\infty}\|Jx_n - Jy_n\| = 0. \tag{3.18}$$

It follows from (3.14),(3.15),(3.18), we obtain

$$g(\|JRz_n - Jx_n\|) \to 0, n \to \infty.$$

and so,

$$lim_{n\to\infty}\|Rz_n - x_n\| = 0.$$

Noticing (3.5), we have

$$lim_{n\to\infty}\|Rz_n - z_n\| = 0. \tag{3.19}$$

From the closedness of S, T and R, we get $p \in F$. Next, we show $p \in GEP = F(K_r)$. Since J is uniformly norm-to-norm continuous on bounded subsets of E, from (3.17) we have $lim_{n\to\infty}\|Ju_n - Jy_n\| = 0$. From $r_n \geq a > 0$, then

$$lim_{n\to\infty}\frac{\|Ju_n - Jy_n\|}{r_n} = 0. \tag{3.20}$$

Let $F(u, y) = f(u, y) + \langle Au, y - u\rangle$, for $u_n = K_{r_n} y_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C.$$

Therefore,

$$\|y-u_n\|\frac{\|Ju_n-Jy_n\|}{r_n}\geq\frac{1}{r_n}\langle y-u_n,Ju_n-Jy_n\rangle\geq -F(u_n,y)\geq F(y,u_n).$$

By taking the limit as $n\to\infty$ in the above inequality and from (A4) and (3.21),one has

$$F(y,p)\leq 0,\forall y\in C,$$

For all $0<t<1$ and $y\in C$,define $y_t=ty+(1-t)p$. Noticing that $y,p\in C$, then $y_t\in C$, which yields that $F(y_t,p)\leq 0$. From (A1) and (A4) that

$$0=F(y_t,y_t)\leq tF(y_t,y)+(1-t)F(y_t,p)\leq tF(y_t,y).$$

That is

$$F(y_t,y)\geq 0.$$

Let $t\downarrow 0$, we obtain $F(p,y)\geq 0,\forall y\in C$. This implies that $p\in GEP$. This shows that $p\in F$.

Step 6. We prove $p=\Pi_F x_0$.

In fact, by Lemma2.5,

$$\langle x_n-z,Jx_0-Jx_n\rangle\geq 0,\forall z\in C_n.$$

Since $F\subset C_n$ for all $n\geq 1$, we arrive at

$$\langle x_n-w,Jx_0-Jx_n\rangle\geq 0,\forall w\in F.$$

By taking the limit in the above inequality, one has

$$\langle p-w,Jx_0-Jp\rangle\geq 0,\forall w\in F.$$

At this point, in view of Lemma 2.3, we can get $p=\Pi_F x_0$. This completes the proof of theorem3.1.

Putting $A=0$ in Theorem3.1, we can get,

Corollary 3.2 Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C\times C\to R$ satisfying (A1)-(A4) and let $T,S,R:C\to C$ be three closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap F(S) \cap F(R) \cap EP \neq \emptyset$. Let $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ be the sequences generated by the following:

$$\begin{cases} x_0 \in E,\ C_1 = C,\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\delta_n J x_n + (1-\delta_n) JRz_n), \\ z_n = J^{-1}(\alpha_n J x_n + \beta_n JTx_n + \gamma_n JSx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \delta_n \phi(z, x_n) + (1-\delta_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{3.21}$$

Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}and\{\delta_n\}$ are sequences in [0,1] satisfying the restrictions:

(a)$\alpha_n + \beta_n + \gamma_n = 1$;

(b)$\underline{lim}_{n\to\infty} \alpha_n \beta_n > 0, \underline{lim}_{n\to\infty} \alpha_n \gamma_n > 0, \underline{lim}_{n\to\infty} \delta_n(1-\delta_n) > 0$;

(c)$\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$and $\{u_n\}$ converge strongly to $z \in F$,where $z = \Pi_F x_0$.

Putting $f = 0$ in Theorem3.1, we can obtain,

Corollary 3.3 Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $A : C \to E^*$ be an $\alpha-$ inverse- strong monotone mapping and let $T, S, R : C \to C$ be three closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap F(S) \cap F(R) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ be the sequences generated by the following:

$$\begin{cases} x_0 \in E,\ C_1 = C,\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\delta_n J x_n + (1-\delta_n) JRz_n), \\ z_n = J^{-1}(\alpha_n J x_n + \beta_n JTx_n + \gamma_n JSx_n), \\ u_n \in C \text{ such that } \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \delta_n \phi(z, x_n) + (1-\delta_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{3.22}$$

Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}and\{\delta_n\}$ are sequences in [0,1] satisfying the restrictions:

(a)$\alpha_n + \beta_n + \gamma_n = 1$;

(b)$\underline{lim}_{n\to\infty} \alpha_n \beta_n > 0, \underline{lim}_{n\to\infty} \alpha_n \gamma_n > 0, \underline{lim}_{n\to\infty} \delta_n(1-\delta_n) > 0$;

(c)$\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$and $\{u_n\}$ converge strongly to $z \in F$,where $z = \Pi_F x_0$.

Corollary 3.4 Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C \to R$ satisfying (A1)-(A4) and let $R : C \to C$ be a closed relatively quasi-nonexpansive mapping such that $F := F(R) \cap EP \neq \emptyset$. Let $\{x_n\}$ be the sequences generated by the following:

$$\begin{cases} x_0 \in E,\ C_1 = C,\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\delta_n J x_n + (1-\delta_n) J R x_n), \\ u_n \in C \text{ such that } \ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{3.23}$$

Suppose that δ_n are asequences in [0,1] satisfying the restrictions:

(a)$\underline{lim}_{n\to\infty} \delta_n(1-\delta_n) > 0$;

(b)$\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$and $\{u_n\}$ converge strongly to $z \in F$,where $z = \Pi_F x_0$.

Corollary 3.5 Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C \to R$ satisfying (A1)-(A4) and let $T, S : C \to C$ be two closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap F(S) \cap EP \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by the following:

$$\begin{cases} x_0 \in E,\ C_1 = C,\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S x_n), \\ u_n \in C \text{ such that } \ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{3.24}$$

Suppose that $\{\alpha_n\}, \{\beta_n\}and\{\gamma_n\}$ are three sequences in [0,1] satisfying the restrictions:

(a)$\alpha_n + \beta_n + \gamma_n = 1$;

(b)$\underline{lim}_{n\to\infty} \alpha_n\beta_n > 0, \underline{lim}_{n\to\infty} \alpha_n\gamma_n > 0$,

(c)$\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$and $\{u_n\}$ converge strongly to $z \in F$,where $z = \Pi_F x_0$.

Corollary 3.6 Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let f be a bifunction from $C \times C \to R$ satisfying (A1)-(A4) and let $T, S : C \to C$ be two closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap F(S) \cap EP \neq \emptyset$. Let $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ be the sequences generated by the following:

$$\begin{cases} x_0 \in E,\ C_1 = C,\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\delta_n J x_n + (1-\delta_n) J z_n), \\ z_n = J^{-1}(\alpha_n J x_n + \beta_n J T x_n + \gamma_n J S x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{3.25}$$

Suppose that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and$\{\delta_n\}$ are sequences in [0,1] satisfying the restrictions:

(a)$\alpha_n + \beta_n + \gamma_n = 1$;

(b)$\underline{lim}_{n\to\infty} \alpha_n \beta_n > 0, \underline{lim}_{n\to\infty} \alpha_n \gamma_n > 0, \underline{lim}_{n\to\infty} \delta_n (1-\delta_n) > 0$;

(c)$\{r_n\} \subset [a, \infty)$ for some $a > 0$.

Then $\{x_n\}$and $\{u_n\}$ converge strongly to $z \in F$,where $z = \Pi_F x_0$.

Corollary 3.7 Let C be a nonempty and closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $A : C \to E^*$ be an $\alpha-$ inverse-strong monotone mapping and f be a bifunction from $C \times C \to R$ satisfying (A1)-(A4) and let $T, R : C \to C$ be two closed relatively quasi-nonexpansive mappings such that $F := F(T) \cap F(R) \cap GEP \neq \emptyset$. Let $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ be the sequences generated by the following:

$$\begin{cases} x_0 \in E,\ C_1 = C,\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\delta_n J x_n + (1-\delta_n) J R z_n), \\ z_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ u_n \in C \text{ such that } f(u_n, y) + \langle A u_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \delta_n \phi(z, x_n) + (1-\delta_n)\phi(z, z_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{3.26}$$

Suppose that $\{\alpha_n\}$ and $\{\delta_n\}$ are two sequences in [0,1] satisfying the restrictions:

(a)$\underline{lim}_{n\to\infty}\alpha_n(1-\alpha_n)>0, \underline{lim}_{n\to\infty}\delta_n(1-\delta_n)>0$;

(b)$\{r_n\}\subset[a,\infty)$ for some $a>0$.

Then $\{x_n\}$and $\{u_n\}$ converge strongly to $z\in F$,where $z=\Pi_F x_0$.

Remark 3.8 From Corollary 3.4,3.5,3.6 and Corollary 3.7, we see Theorem 3.1 improve and extend the recent ones announced by W.Takahashi and K.Zembayyashi[2], Qin et al.[3],K.Wattanawitoon and P.Kumam [4] and S.S.Chang[5].

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