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THE NATURAL LIFT CURVES FOR THE SPHERICAL INDICATRICES OF SPACELIKE BERTRAND COUPLE IN MINKOWSKI 3-SPACE

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Abstract: In this paper, we introduce the natural lift curves for the spherical indicatrices of the Bertrand mates of spacelike curves on the tangent bundle $T\left(S_1^2\right)$ or $T\left(H_0^2\right)$ in Minkowski 3-space and we give some new characterizations for these curves. Additionally we illustrate two examples of our main results.

Keywords: Bertrand curve, natural lift curve, geodesic spray, spherical indicatrix.

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1. Introduction

In early course of classical differential geometry, students encounter with the subject of Bertrand curves discovered by J. Bertrand in 1850. Bertrand curves are one of the associated curve pairs for which at the corresponding points of the curves one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of the other curve. These special curves are very interesting and characterized as a kind of corresponding relation

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between two curves such that the curves have the common principal normal i.e., the Bertrand curve is a curve which shares the normal line with another curve. It is proved in most texts on the subject that the characteristic property of such a curve is the existence of a linear relation between the curvature and the torsion; the discussion appears as an application of the Frenet-Serret formulas. So, a circular helix is a Bertrand curve. Bertrand mates represent particular examples of offset curves [11] which are used in computer-aided design (CAD) and computer-aided manufacturing (CAM). For classical and basic treatments of Bertrand curves, we refer to [3], [6] and [12].

There are recent works about the Bertrand curves. Ekmekçi and İlarslan studied Nonnull Bertrand curves in the n-dimensional Lorentzian space. Straightforward modication of classical theory to spacelike or timelike curves in Minkowski 3-space is easily obtained, (see [1]). Izumiya and Takeuchi [16] have shown that cylindrical helices can be constructed from plane curves and Bertrand curves can be constructed from spherical curves. Also, the representation formulae for Bertrand curves were given by [8].

In differential geometry, especially the theory of space curves, the Darboux vector is the areal velocity vector of the Frenet frame of a space curve. It is named after Gaston Darboux who discovered it. In terms of the Frenet-Serret apparatus, the Darboux vector w can be expressed as $w = \tau t + \kappa b$. In addition, the concepts of the natural lift and the geodesic sprays have first been given by Thorpe (1979). On the other hand, Çalışkan et al. [4] have studied the natural lift curves and the geodesic sprays in Euclidean 3-space $\mathbb{R}^3$. Bilici et al. [7] have proposed the natural lift curves and the geodesic sprays for the spherical indicatrices of the involute-evolute curve couple in $\mathbb{R}^3$. Recently, Bilici [9] adapted this problem for the spherical indicatrices of the involutes of a timelike curve in Minkowski 3-space. However, this problem is not handled in other types of space curves.

Spherical images (indicatrices) are a well known concept in classical differential geometry of curves [1]. Kula and Yaylı [17] have studied spherical images of the tangent indicatrix and binormal indicatrix of a slant helix and they have shown that the spherical images are spherical helices. In [19] Süha et. all investigated tangent and trinormal spherical images of timelike curve lying on the pseudo hyperbolic space H_0^3 in Minkowski space-time. İyigün [20] defined the tangent spherical image of a unit speed timelike curve lying on the on the pseudo hyperbolic space H_0^2 in $\mathbb{R}_1^3$.

In this study, we carry tangents of the Bertrand mate β of a spacelike curve α to the center of the unit hypersphere H_0^2 and we obtain a spacelike curve $\beta_{T^*} = T^*$ on the unit hypersphere H_0^2. This curve is called the first spherical indicatrix or tangent indicatrix of β. One consider the principal normal indicatrix $\beta_{N^*} = N^*$ and the binormal indicatrix $\beta_{B^*} = B^*$ on the unit hypersphere S_1^2. Then the natural lift curves for the spherical indicatrices of the spacelike Bertrand curves are investigated in Minkowski 3-space and some new results are obtained.

2. Preliminaries

To meet the requirements in the next sections, the basic elements of the theory of curves and hypersurfaces in the Minkowski 3-space are briefly presented in this section. A more detailed information can be found in [10].

The Minkowski 3-space $\mathbb{R}_1^3$ is the real vector space $\mathbb{R}^3$ endowed with standard flat Lorentzian metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{R}_1^3$. A vector $V = (v_1, v_2, v_3) \in IR_1^3$ is said to be timelike if $g(V,V) < 0$, spacelike if $g(V,V) > 0$ or $V = 0$ and null (lightlike) if $g(V,V) = 0$ or $V \neq 0$ .Similarly, an arbitrary curve $\Gamma = \Gamma(s)$ in $\mathbb{R}_1^3$ can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors Γ' are respectively timelike, spacelike or null (lightlike), for every $t \in I \subset \mathbb{R}$. The pseudo-norm of an arbitrary vector $V \in \mathbb{R}_1^3$ is given by $\|V\| = \sqrt{|g(V,V)|}$. Γ is called a unit speed curve if the velocity vector V of Γ satisfies $\|V\| = 1$. A timelike vector V is said to be positive (resp. negative) if and only if $v_1 > 0$ (resp. $v_1 < 0$).

Let Γ be a unit speed spacelike curve with curvature κ and torsion τ. Denote by $\{t(s),n(s),b(s)\}$ the moving Frenet frame along the curve Γ in the space $\mathbb{R}_1^3$. Then *t*, *n* and b are the tangent, the principal normal and the binormal vector of the curve Γ, respectively.

The angle between two vectors in Minkowski 3-space is defined by [21]:

Definition 2.1:

Let X and Y be spacelike vektors in $\mathbb{R}_1^3$ that span a spacelike vector subspace, then we have $|g(X,Y)| \leq \|X\|\|Y\|$ and hence, there is a unique real number φ such that

$$g(X,Y) = \|X\|\|Y\| cos\varphi .$$

The reel number φ is called the Lorentzian spacelike angle between X and Y.

Definition 2.2:

Let X and Y be spacelike vectors in $\mathbb{R}_1^3$ that span a timelike vector subspace, then we have $g(X,Y) > \|X\|\|Y\|$ and hence, there is a unique pozitive real number φ such that

$$|g(X,Y)| = \|X\|\|Y\| cosh\varphi .$$

The reel number φ is called the Lorentzian timelike angle between X and Y.

Definition 2.3:

Let X be a spacelike vector and Y a positive timelike vector in $\mathbb{R}_1^3$, then there is a unique non-negative reel number φ such that

$$|g(X,Y)| = \|X\|\|Y\| sinh\varphi .$$

The reel number φ is called the Lorentzian timelike angle between X and Y.

Definition 2.4:

Let X and Y be pozitive (negative) timelike vectors in $\mathbb{R}_1^3$, then there is a unique non-negative real number φ such that

$$g(X,Y) \le \|X\|\|Y\| cosh\varphi .$$

The reel number φ is called the Lorentzian timelike angle between X and Y.

Case I. Let Γ be a unit speed spacelike curve with a spacelike binormal. For these Frenet vectors, we can write

$$T \times N = -B \ , \ N \times B = -T \ , \ B \times T = N ,$$

where '$\times$' is the Lorentzian cross product in space $\mathbb{R}_1^3$. Depending on the causal character of the curve Γ, the following Frenet formulae are given in [5].

$$\dot{T} = \kappa N, \qquad \dot{N} = \kappa T + \tau B, \qquad \dot{B} = \tau N$$

The Darboux vector for the spacelike curve with a spacelike binormal is defined by [11]:

$$w = -\tau T + \kappa B .$$

If *b* and *w* spacelike vectors that span a spacelike vector subspace then by the Definition1. we can write

$$\begin{cases} \kappa = \|w\| cos\,\varphi \\ \tau = \|w\| sin\,\varphi \end{cases} \quad , \quad \|w\|^2 = g(w,w) = \tau^2 + \kappa^2$$

Case II. Let Γ be a unit speed spacelike curve with a timelike binormal. For these Frenet vectors, we can write

$$T \times N = B \ , \ N \times B = -T \ , \ B \times T = -N ,$$

Depending on the causal character of the curve Γ, the following Frenet formulae are given in [5].

$$\dot{T} = \kappa N, \qquad \dot{N} = -\kappa T + \tau B, \qquad \dot{B} = \tau N$$

The Darboux vector for the spacelike curve with a timelike binormal is defined by [11]:

$$w = \tau T - \kappa B .$$

There are two cases corresponding to the causal characteristic of Darboux vector w.

i) If $|\kappa| < |\tau|$, then w is a timelike vector. In this situation, we have

$$\begin{cases} \kappa = \|w\| sinh\,\varphi \\ \tau = \|w\| cosh\,\varphi \end{cases} \quad , \quad \|w\|^2 = -g(w,w) = \tau^2 - \kappa^2$$

and the unit vector c of direction w is

$$c=\frac{1}{\|w\|}w=\sinh\varphi\, T-\cosh\varphi\, B .$$

ii) If $|\kappa|>|\tau|$, then w is a spacelike vector. In this situation, we can write

$$\begin{cases}\kappa=\|w\|\cosh\varphi\\ \tau=\|w\|\sinh\varphi\end{cases},\qquad \|w\|^2=g(w,w)=\kappa^2-\tau^2$$

and the unit vector c of direction w is

$$c=\frac{1}{\|w\|}w=\sinh\varphi\, T-\cosh\varphi\, B .$$

Proposition 2.5.

Let α be a timelike (or spacelike) curve with curvatures κ and τ. The curve α is a general helix if and only if $\frac{\tau}{\kappa}=costant$, [13].

Remark 2.6. *We can easily see from equations of the section Case I and Case II that: $\frac{\tau}{\kappa}=\tan\varphi$ and $\frac{\tau}{\kappa}=\tanh\varphi$ (or $\frac{\tau}{\kappa}=\coth\varphi$), if $\varphi=constant$ then α is a general helix.*

Lemma 2.7.

The natural lift $\overline{\alpha}$ of the curve α is an integral curve of the geodesic spray X if and only if α is a geodesic on M, [9].

Definition 2.8.

Let $\alpha=(\alpha(s);T(s),N(s),B(s))$ and $\beta=(\beta(s^*);T^*(s^*),N^*(s^*),B^*(s^*))$ be two regular non-null curves in $\mathbb{R}_1^3$. $\alpha(s)$ and $\beta(s^*)$ are called Bertrand curves if $N(s)$ and $N^*(s^*)$ are linearly dependent. In this situation, (α,β) is called a Bertrand couple in $\mathbb{R}_1^3$. (See [1] for the more details in the n-dimensional space)

Lemma 2.9.

Let α be a spacelike curve with a timelike binormal. In this situation, β is a spacelike with a timelike binormal Bertrand mate of α. The relations between the Frenet vectors of the (α,β) is as follow

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \begin{bmatrix} cosh\theta & 0 & sinh\theta \\ 0 & 1 & 0 \\ sinh\theta & 0 & cosh\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad g\left(T, T^*\right) = cosh\,\theta = constant, \text{ [8].}$$

Lemma 2.10.

Let β be a spacelike with a spacelike binormal Bertrand mate of a spacelike curve with a spacelike binormal α. The relations between the Frenet vectors of the (α,β) is as follow

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad g\left(T, T^*\right) = cos\,\theta = constant, \text{ [8].}$$

Definition 2.11.

Let S_1^2 and H_0^2 be hypersphere in $\mathbb{R}_1^3$. The Lorentzian sphere and hyperbolic sphere of radius 1 in IR_1^3 are given by

$$S_1^2 = \left\{ V = (v_1, v_2, v_3) \in \mathbb{R}_1^3 : g(V,V) = 1 \right\}$$

and

$$H_0^2 = \left\{ V = (v_1, v_2, v_3) \in \mathbb{R}_1^3 : g(V,V) = -1 \right\}$$

respectively, [10].

Definition 2.12.

Let *M* be a hypersurface in $\mathbb{R}_1^3$ equipped with a metric *g*. Let *TM* be the set $\bigcup\left\{T_p(M) : p \in M\right\}$ of all tangent vectors to *M*. Then each $v \in TM$ is in a unique $T_p(M)$, and the projection $\pi : TM \to M$ sends *v* to *p*. Thus $\pi^{-1}(p) = T_p(M)$. There is a natural way to make *TM* a manifold, called the *tangent bundle* of *M*.

A vector field $X \in \chi(M)$ is exactly a smooth section of *TM*, that is, a smooth function $X : M \to TM$ such that $\pi \circ X = id_M$, [9].

Definition 2.13.

Let *M* be a hypersurface in $\mathbb{R}_1^3$. A curve $\alpha : I \to M$ is an *integral curve* of $X \in \chi(M)$ provided $\dot{\alpha} = X_\alpha$; that is,

$$\frac{d}{dt}(\alpha(s)) = X(\alpha(s)) \quad \text{for all} \quad s \in I, \text{[10]}. \tag{1}$$

Definition 2.14.

For any parametrized curve $\alpha : I \to M$, the parametrized curve

$$\overline{\alpha} : I \to TM$$

given by

$$\overline{\alpha}(t) = (\alpha(s), \dot{\alpha}(s)) = \dot{\alpha}(s)\big|_{\alpha(s)} \tag{2}$$

is called the *natural lift* of α on *TM*. Thus, we can write

$$\frac{d\overline{\alpha}}{dt} = \frac{d}{dt}(\dot{\alpha}(s))\big|_{\alpha(s)} = D_{\dot{\alpha}(s)}\dot{\alpha}(s) \tag{3}$$

where *D* is the standart connection on $\mathbb{R}^3_1$, [9].

Definition 2.15.

For $v \in TM$, a smooth vector field $X \in \chi(TM)$ defined by

$$X(v) = \varepsilon\, g(v, S(v))\xi\big|_{\alpha(t)}, \quad \varepsilon = g(\xi, \xi) \tag{4}$$

is called the *geodesic spray* on the manifold *TM*, where ξ is the unit normal vector field of *M* and *S* is the shape operator of *M*, [9].

3. The Natural Lift Curves for the Spherical Indicatrices of Spacelike Bertrand Couple in Minkowski 3-Space

In this section we investigate the natural lift curves of the spherical indicatrices of Bertrand curves (α, β) as in Lemma 2.9. and Lemma 2.10.. Furthermore, some interesting theorems about the original curve were obtained depending on the assumption that the natural lift curves should be the integral curve of the geodesic spray on the tangent bundle $T(S^2_1)$ or $T(H^2_0)$.

Note that $\overline{D}$ and $\overline{\overline{D}}$ are Levi-Civita connections on S^2_1 and H^2_0, respectively. Then Gauss equations are given by the followings

$$D_X Y = \overline{D}_X Y + \varepsilon g(S(X), Y)\xi, \;\; D_X Y = \overline{\overline{D}}_X Y + \varepsilon g(S(X), Y)\xi, \;\; \varepsilon = g(\xi, \xi)$$

where ξ is a unit normal vector field of S_1^2 (or H_0^2) and $S = I_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is the shape operator of S_1^2 (or H_0^2).

i) The natural lift of the spherical indicatrix of the tangent vector of β

Let (α,β) be Bertrand curves as in Lemma 2.9. We will investigate the curve α to satisfy the condition that the natural lift curve of $\overline{\beta}_{T^*}$ is an integral curve of geodesic spray, where β_{T^*} is the tangent indicatrix of β. If the natural lift curve $\overline{\beta}_{T^*}$ is an integral curve of the geodesic spray, then by means of Lemma 2.9. we get,

$$\overline{D}_{\dot{\beta}_{T^*}} \dot{\beta}_{T^*} = 0, \tag{5}$$

where $\overline{D}$ is the connection on the hyperbolic unit sphere S_1^2 and the equation of tangent indicatrix is $\beta_{T^*} = T^*$. Thus from the Gauss equation we can write

$$D_{\dot{\beta}_{T^*}} \dot{\beta}_{T^*} = \overline{D}_{\dot{\beta}_{T^*}} \dot{\beta}_{T^*} + \varepsilon g\left(S\left(\dot{\beta}_{T^*}\right), \dot{\beta}_{T^*}\right) T^*, \ \varepsilon = g\left(T^*, T^*\right) = 1.$$

On the other hand, from the Lemma 2.9. straightforward computation gives

$$\dot{\beta}_{T^*} = t_{T^*} = \frac{d\beta_{T^*}}{ds} \frac{ds}{ds_{T^*}} = \left(\kappa cosh\theta + \tau sinh\theta\right) N \frac{ds}{ds_{T^*}}.$$

Moreover, we get

$$\frac{ds}{ds_{T^*}} = \frac{1}{\kappa \cos h\theta + \tau \sin h\theta}, \quad t_{T^*} = N,$$

$$D_{t_{T^*}} t_{T^*} = -\frac{\kappa}{\kappa \cos h\theta + \tau \sin h\theta} T + \frac{\tau}{\kappa \cos h\theta + \tau \sin h\theta} B$$

and $g\left(S\left(t_{T^*}\right), t_{T^*}\right) = 1$.

Using these in the Gauss equation, we immediately have

$$\overline{D}_{t_{T^*}} t_{T^*} = -\frac{\kappa}{\kappa \cos h\theta + \tau \sin h\theta} T + \frac{\tau}{\kappa \cos h\theta + \tau \sin h\theta} B - T^*.$$

From the Eq. (5) and Lemma 2.9. we get

$$\left(-\frac{\kappa}{\kappa \cos h\theta + \tau \sin h\theta} - cosh\theta\right) T + \left(\frac{\tau}{\kappa \cos h\theta + \tau \sin h\theta} - sinh\theta\right) B = 0.$$

Since *T*, *N*, *B* are linearly independent, we have

$$\begin{cases} \dfrac{\kappa}{\kappa cosh\theta + \tau sinh\theta} + ch\theta = 0 \\ \dfrac{\tau}{\kappa cosh\theta + \tau sinh\theta} - sh\theta = 0 \end{cases}$$

it follows that,

$$\kappa sinh\theta + \tau cosh\theta = 0, \tag{6}$$

$$\frac{\tau}{\kappa} = -tanh\theta. \tag{7}$$

So from the Eq. (7) and Remark 2.6. we can give the following result.

Result 3.1:

Let (α,β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the tangent indicatrix β_{T^*} of β is a geodesic on S_1^2.

Moreover from Lemma 2.7. and Result 3.1. we can give the following theorem to characterize the natural lift of the tangent indicatrix of β without proof.

Theorem 3.2:

Let (α,β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the natural lift $\overline{\beta}_{T^*}$ of the tangent indicatrix β_{T^*} of β is an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$.

ii) The natural lift of the spherical indicatrix of the principal normal vectors of β

Let β_{N^*} be the spherical indicatrix of principal normal vectors of β and $\overline{\beta}_{N^*}$ be the natural lift of the curve β_{N^*}. If $\overline{\beta}_{N^*}$ is an integral curve of the geodesic spray, then by means of Lemma 2.7. we get,

$$\overline{D}_{t_{N^*}} t_{N^*} = 0 \tag{8}$$

that is

$$D_{t_{N^*}} t_{N^*} = \overline{D}_{t_{N^*}} t_{N^*} + \varepsilon g\left(S\left(t_{N^*}\right), t_{N^*}\right) N^*, \ \varepsilon = g\left(N^*, N^*\right) = 1$$

On the other hand, from Lemma 2.9. and Case II. *i)* straightforward computation gives

$$\dot{\beta}_{N^*} = t_{N^*} = -sinh\varphi T + cosh\varphi B.$$

Moreover we get

$$D_{t_{N^*}} t_{N^*} = -\frac{\varphi' cosh\varphi}{\|W\|}T + \frac{-\kappa sinh\varphi + \tau cosh\varphi}{\|W\|}N + \frac{\varphi' sinh\varphi}{\|W\|}B$$

and $g\left(S\left(t_{N^*}\right), t_{N^*}\right) = 1$.

Using these in the Gauss equation, we immediately have

$$\overline{D}_{t_{N^*}} t_{N^*} = -\frac{\varphi' cosh\varphi}{\|W\|}T + \frac{\varphi' sinh\varphi}{\|W\|}B.$$

Since *T*, *N*, *B* are linearly independent, we have

$$\begin{cases} -\dfrac{\varphi' cosh\varphi}{\|W\|} = 0 \\ \dfrac{\varphi' sinh\varphi}{\|W\|} = 0 \end{cases},$$

it follows that,

$$\varphi' = 0, \tag{9}$$

$$\frac{\tau}{\kappa} = constant. \tag{10}$$

So from the Eq. (10) and Remark 2.6. we can give the following result.

Result 3.3:

Let (α, β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the principal normal indicatrix β_{N^*} of β is a geodesic on S_1^2.

Moreover from Lemma 2.7. and Result 3.3 we can give the following theorem to characterize the natural lift of the principal normal indicatrix of β without proof.

Theorem 3.4:

Let (α, β) be Bertrand curves as in Lemma 2.9. If α is a general helix, then the natural lift $\overline{\beta}_{N^*}$ of the principal normal indicatrix β_{N^*} of β is an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$.

iii) The natural lift of the spherical indicatrix of the binormal vectors of β

Let β_{B^*} be the spherical indicatrix of binormal vectors of β and $\overline{\beta}_{B^*}$ be the natural lift of the curve β_{B^*}. If $\overline{\beta}_{B^*}$ is an integral curve of the geodesic spray, then by means of Lemma 2.7. we get,

$$\overline{\overline{D}}_{t_{B^*}} t_{B^*} = 0 \tag{11}$$

that is

$$D_{t_{B^*}} t_{B^*} = \overline{\overline{D}}_{t_{B^*}} t_{B^*} + \varepsilon g\left(S\left(t_{B^*}\right), t_{B^*}\right) B^*, \ \ \varepsilon = g\left(B^*, B^*\right) = -1$$

On the other hand, from Lemma 2.9. straightforward computation gives

$$t_{B^*} = \left(\kappa sinh\theta + \tau cosh\theta\right) N \frac{ds}{ds_{B^*}}.$$

Moreover we get

$$\frac{ds}{ds_{B^*}} = \frac{1}{\kappa sinh\theta + \tau cosh\theta}, \ \ t_{B^*} = N,$$

$$D_{t_{B^*}} t_{B^*} = -\frac{\kappa}{\kappa sinh\theta + \tau cosh\theta} T + \frac{\tau}{\kappa sinh\theta + \tau cosh\theta} B$$

and $g\left(S\left(t_{B^*}\right), t_{B^*}\right) = -1$.

Using these in the Gauss equation, we immediately have

$$\overline{\overline{D}}_{t_{B^*}} t_{B^*} = \frac{\kappa}{\kappa sinh\theta + \tau cosh\theta} T - \frac{\tau}{\kappa sinh\theta + \tau cosh\theta} B + B^*$$

From the Eq. (11) and Lemma 2.9. we get

$$\left(-\frac{\kappa}{\kappa sinh\theta + \tau cosh\theta} - sinh\theta\right) T + \left(\frac{\tau}{\kappa sinh\theta + \tau cosh\theta} - cosh\theta\right) B = 0.$$

Since T, N, B are linearly independent, we have

$$\begin{cases} \dfrac{\kappa}{\kappa sinh\theta + \tau cosh\theta} + sinh\theta = 0 \\ \dfrac{\tau}{\kappa sinh\theta + \tau cosh\theta} - cosh\theta = 0 \end{cases}$$

it follows that,

$$\kappa cosh\theta + \tau sinh\theta = 0, \tag{12}$$

$$\frac{\tau}{\kappa} = -coth\theta \quad .$$

(13)

So from the Eq. (13) and Remark 2.6. we can give the following result.

Result 3.5:

Let (α,β) *be Bertrand curves as in Lemma 2.9. If* α *is a general helix, then the binormal indicatrix* β_{B^*} *of* β *is a geodesic on* H_0^2.

Moreover from Lemma 2.7. and Result 3.5 we can give the following theorem to characterize the natural lift of the binormal indicatrix of β without proof.

Theorem 3.6:

Let (α,β) *be Bertrand curves as in Lemma 2.9. If* α *is a general helix, then the natural lift* $\overline{\beta}_{B^*}$ *of the binormal indicatrix* β_{B^*} *of* β *is an integral curve of the geodesic spray on the tangent bundle* $T\left(H_0^2\right)$.

From the classification of all W-curves (i.e. a curves for which a curvature and a torsion are constants) in (Walrawe, 1995), we have following result with relation to curve α.

Result 3.7:

(1) *If the curve* α *with* $\kappa = constant > 0$, $\tau = 0$ *then* α *is a part of a circle,*

(2) *If the curve* α *with* $\kappa = constant > 0$, $\tau = constant \neq 0$ *and* $|\tau| > \kappa$ *then* α *is a part of a spacelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(\kappa sinh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa cosh\left(\sqrt{K}s\right)\right) \; K = \tau^2 - \kappa^2 .$$

(3) *If the curve* α *with* $\kappa = constant > 0$, $\tau = constant \neq 0$ *and* $|\tau| < \kappa$ *then* α *is a part of a spacelike circular helix,*

$$\alpha(s) = \frac{1}{K}\left(\sqrt{\tau^2 K}s, \kappa cos\left(\sqrt{K}s\right), \kappa sin\left(\sqrt{K}s\right)\right) \; , \; K = \kappa^2 - \tau^2 .$$

From Lemma 3.1 in Choi et al 2012, we can write the following result:

Result 3.8:

There is no spacelike general helix of spacelike curve with a timelike binormal in Minkowski 3*-space with condition* $|\tau| = |\kappa|$.

The method used for (α,β) Bertrand curves as in Lemma 2.9. , (α,β) Bertrand curves as in Lemma 2.10. applied in the following results and theorems are obtained.

Result 3.9:

Let (α,β) *be Bertrand curves as in Lemma 2.10. . If* α *is a general helix, then the tangent indicatrix* β_{T^*} *of* β *is a geodesic on* S_1^2.

Moreover from Lemma 2.7. and Result 3.9. we can give the following theorem to characterize the natural lift of the tangent indicatrix of β without proof.

Theorem 3.10:

Let (α,β) *be Bertrand curves as in Lemma 2.10. If* α *is a general helix, then the natural lift* $\overline{\beta}_{T^*}$ *of the tangent indicatrix* β_{T^*} *of* β *is an integral curve of the geodesic spray on the tangent bundle* $T\left(S_1^2\right)$.

Result 3.11:

Let (α,β) *be Bertrand curves as in Lemma 2.10.. If* α *is a general helix, then the principal normal indicatrix* β_{N^*} *of* β *is a geodesic on* H_0^2.

Moreover from Lemma 2.7. and Result 3.11. we can give the following theorem to characterize the natural lift of the principal normal indicatrix of β without proof.

Theorem 3.12:

Let (α,β) *be Bertrand curves as in Lemma 2.10.. If* α *is a general helix, then the natural lift* $\overline{\beta}_{N^*}$ *of the principal normal indicatrix* β_{N^*} *of* β *is an integral curve of the geodesic spray on the tangent bundle* $T\left(H_0^2\right)$.

Result 3.13:

Let (α,β) *be Bertrand curves as in Lemma 2.10.. If* α *is a general helix, then the binormal indicatrix* β_{B^*} *of* β *is a geodesic on* S_1^2.

Moreover from Lemma 2.7. and Result 3.13. we can give the following theorem to characterize the natural lift of the binormal indicatrix of β without proof.

Theorem 3.14:

Let (α,β) be Bertrand curves as in Lemma 2.10.. If α is a general helix, then the natural lift $\overline{\beta}_{B^*}$ of the binormal indicatrix β_{B^*} of β is an integral curve of the geodesic spray on the tangent bundle $T\left(S_1^2\right)$.

From Walrawe, (1995), we can write the following result:

Result 3.15:

(1) *If the curve α with $\kappa = constant > 0$, $\tau = 0$ then α is a part of a orthogonal hyperbola;*

(2) *If the curve α with $\kappa = constant > 0$, $\tau = constant \neq 0$ then α is a part of a spacelike hyperbolic helix,*

$$\alpha(s) = \frac{1}{K}\left(\kappa cosh\left(\sqrt{K}s\right), \sqrt{\tau^2 K}s, \kappa sinh\left(\sqrt{K}s\right)\right) \;, K = \kappa^2 + \tau^2.$$

Example 3.16:

Let $\alpha(s) = \left(\frac{\sqrt{3}}{3}s, \frac{2}{3}cos\left(\sqrt{3}\,s\right), \frac{2}{3}sin\left(\sqrt{3}\,s\right)\right)$ be a unit speed spacelike circular helix with

$$\begin{cases} T = \left(\frac{\sqrt{3}}{3}, -\frac{2\sqrt{3}}{3}sin\left(\sqrt{3}\,s\right), \frac{2\sqrt{3}}{3}cos\left(\sqrt{3}\,s\right)\right) \\ N = \left(0, -cos\left(\sqrt{3}\,s\right), -sin\left(\sqrt{3}\,s\right)\right) \\ B = \left(\frac{2\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}sin\left(\sqrt{3}\,s\right), \frac{\sqrt{3}}{3}cos\left(\sqrt{3}\,s\right)\right) \end{cases}, \quad \kappa = 2 \text{ and } \tau = 1.$$

In this situation, spacelike with timelike binormal Bertrand mate β for α can be given by the equation

$$\beta(s) = \left(\frac{\sqrt{3}}{3}s, \left(\frac{2}{3} - \lambda\right)cos\left(\sqrt{3}\,s\right), \left(\frac{2}{3} - \lambda\right)sin\left(\sqrt{3}\,s\right)\right), \; \lambda \in \mathrm{IR}.$$

For $\lambda = \frac{5}{3}$, we have

$$\beta(s) = \left(\frac{\sqrt{3}}{3}s, -cos\left(\sqrt{3}\,s\right), -sin\left(\sqrt{3}\,s\right)\right).$$

The straight forward calculations give the following spherical indicatrices and natural lift curves of spherical indicatrices for β,

$$\begin{cases} \beta_{T^*} = \left(\frac{1}{2\sqrt{2}}, \frac{3}{2\sqrt{2}} sin\left(\sqrt{3}\, s\right), -\frac{3}{2\sqrt{2}} cos\left(\sqrt{3}\, s\right) \right) \\ \beta_{N^*} = \left(0, cos\left(\sqrt{3}\, s\right), sin\left(\sqrt{3}\, s\right) \right) \\ \beta_{B^*} = \left(\frac{3}{2\sqrt{2}}, \frac{1}{2\sqrt{2}} sin\left(\sqrt{3}\, s\right), -\frac{1}{2\sqrt{2}} cos\left(\sqrt{3}\, s\right) \right) \end{cases}, \begin{cases} \overline{\beta}_{T^*} = \left(0, \frac{3\sqrt{3}}{2\sqrt{2}} cos\left(\sqrt{3}\, s\right), \frac{3\sqrt{3}}{2\sqrt{2}} sin\left(\sqrt{3}\, s\right) \right) \\ \overline{\beta}_{N^*} = \left(0, -\sqrt{3} sin\left(\sqrt{3}\, s\right), \sqrt{3} cos\left(\sqrt{3}\, s\right) \right) \\ \overline{\beta}_{B^*} = \left(0, \frac{\sqrt{3}}{2\sqrt{2}} cos\left(\sqrt{3}\, s\right), \frac{\sqrt{3}}{2\sqrt{2}} sin\left(\sqrt{3}\, s\right) \right), \end{cases}$$

respectively, (Figs. 1-6).

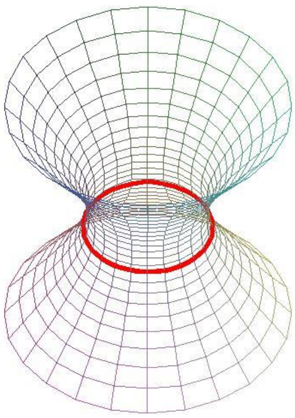

Figure 1. Tangent indicatrix for Bertrand mate of α on S_1^2

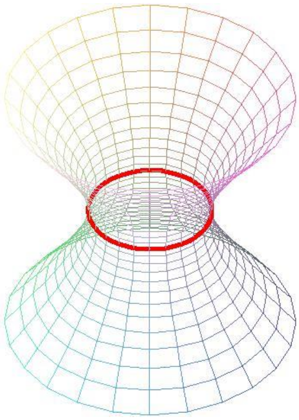

Figure 2. Principal normal indicatrix for Bertrand mate of α on S_1^2

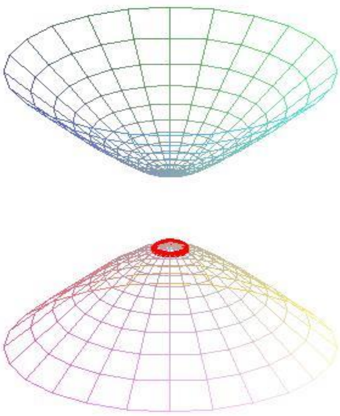

Figure 3. Binormal indicatrix for Bertrand mate of α on S_1^2

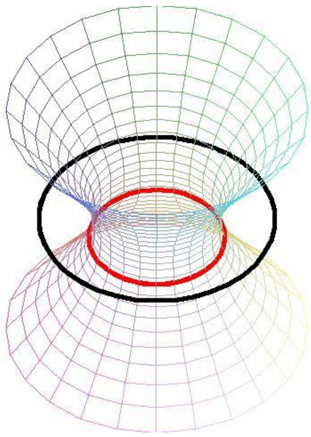

Figure 4. Tangent indicatrix for Bertrand mate of α and its natural lift

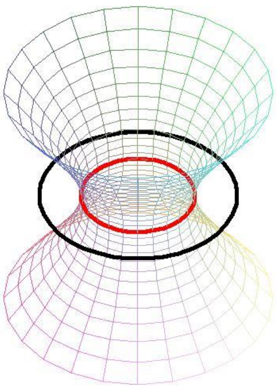

Figure 5. Principal normal indicatrix for Bertrand mate of α and its natural lift

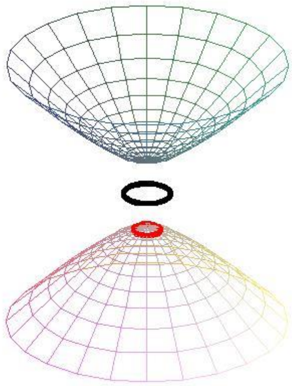

Figure 6. Binormal indicatrix for Bertrand mate of α and its natural lift

Example 3.17: Let $\alpha(s)=\left(\cosh\left(\frac{s}{\sqrt{2}}\right),\frac{s}{\sqrt{2}},\sinh\left(\frac{s}{\sqrt{2}}\right)\right)$ be a unit speed spacelike hyperbolic helix with

$$\begin{cases} T=\left(\frac{1}{\sqrt{2}}\sinh\left(\frac{s}{\sqrt{2}}\right),\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\cosh\left(\frac{s}{\sqrt{2}}\right)\right) \\ N=\left(\cosh\left(\frac{s}{\sqrt{2}}\right),0,\sinh\left(\frac{s}{\sqrt{2}}\right)\right) \\ B=\left(-\frac{1}{\sqrt{2}}\sinh\left(\frac{s}{\sqrt{2}}\right),\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\cosh\left(\frac{s}{\sqrt{2}}\right)\right) \end{cases}, \quad \kappa=\frac{1}{2} \text{ and } \tau=\frac{1}{2}.$$

In this situation, spacelike with spacelike binormal Bertrand mate β for α can be given by the equation

$$\beta(s)=\left((1+\lambda)\cosh\left(\frac{s}{\sqrt{2}}\right),\frac{s}{\sqrt{2}},(1+\lambda)\sinh\left(\frac{s}{\sqrt{2}}\right)\right),\ \lambda\in \mathrm{IR}.$$

For $\lambda=1$, we have

$$\beta(s)=\left(2\cosh\left(\frac{s}{\sqrt{2}}\right),\frac{s}{\sqrt{2}},2\sinh\left(\frac{s}{\sqrt{2}}\right)\right).$$

The straight forward calculations give the following spherical indicatrices and natural lift curves of spherical indicatrices for β,

$$\begin{cases} \beta_{T^*} = \left(\frac{2}{\sqrt{5}} sinh\left(\frac{s}{\sqrt{2}}\right), \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} cosh\left(\frac{s}{\sqrt{2}}\right) \right) \\ \beta_{N^*} = \left(cosh\left(\frac{s}{\sqrt{2}}\right), 0, sinh\left(\frac{s}{\sqrt{2}}\right) \right) \\ \beta_{B^*} = \left(\frac{1}{\sqrt{5}} sinh\left(\frac{s}{\sqrt{2}}\right), -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} cosh\left(\frac{s}{\sqrt{2}}\right) \right) \end{cases}, \begin{cases} \overline{\beta}_{T^*} = \left(\frac{2}{\sqrt{2}\sqrt{5}} cosh\left(\frac{s}{\sqrt{2}}\right), 0, \frac{2}{\sqrt{2}\sqrt{5}} sinh\left(\frac{s}{\sqrt{2}}\right) \right) \\ \overline{\beta}_{N^*} = \left(\frac{1}{\sqrt{2}} sinh\left(\frac{s}{\sqrt{2}}\right), 0, \frac{1}{\sqrt{2}} cosh\left(\frac{s}{\sqrt{2}}\right) \right) \\ \overline{\beta}_{B^*} = \left(\frac{1}{\sqrt{2}\sqrt{5}} cosh\left(\frac{s}{\sqrt{2}}\right), 0, \frac{1}{\sqrt{2}\sqrt{5}} sinh\left(\frac{s}{\sqrt{2}}\right) \right) \end{cases},$$

respectively, (Figs. 7-10).

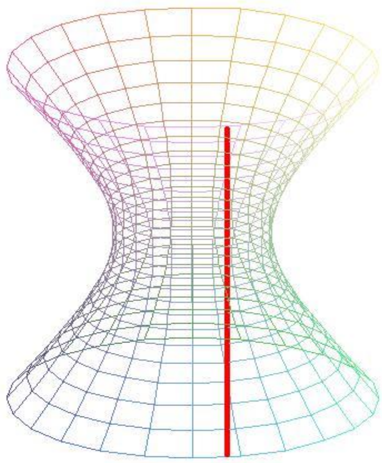

Figure 7. Tangent indicatrix for Bertrand mate of α on S_1^2

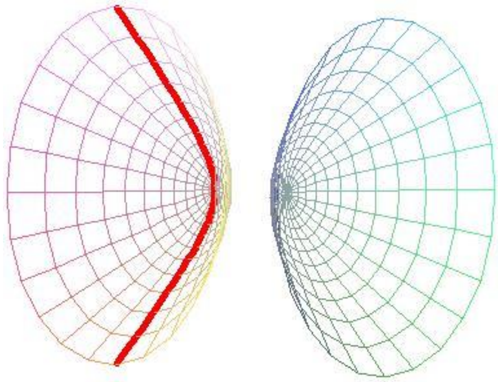

Figure 8. Principal normal indicatrix for Bertrand mate of α on H_0^2

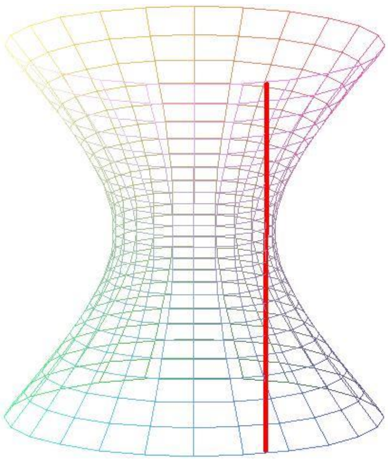

Figure 9. Binormal indicatrix for Bertrand mate of α on S_1^2

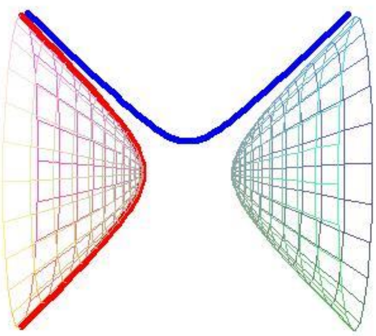

Figure 10. Principal normal indicatrix for Bertrand mate of α and its natural lift

Conflict of Interests

The author declares that there is no conflict of interests.

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