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FIXED POINTS OF P-CYCLIC KANNAN TYPE CONTRACTIONS IN PROBABILISTIC SPACES

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Abstract. Probabilistic fixed point theory has developed substantially during the last two decades. In this paper we define a new contraction in probabilistic metric spaces which are spaces in which distribution functions play the role of the metric. We have shown that under two separate conditions the mapping has a fixed point. Two illustrative examples are given.

Keywords: Menger space; Cauchy sequence; fixed point; Φ-function; Ψ-function.

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1. Introduction

Fixed point theory is one of the most active branches of modern analysis. In 1922, S. Banach proved the well known Banach contraction mapping principle in metric spaces. Sehgal and Bharucha-Reid [34] generalized this concept to probabilistic metric spaces in 1972. Probabilistic metric spaces are probabilistic generalization of metric spaces. The inherent flexibility of these spaces allows us to extend the contraction mapping principle in more than one inequivalent ways. Hicks [13] established another generalization of

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contraction mapping principles in probabilistic metric spaces. This extension is known as C-contraction. Subsequently, fixed point theory in probabilistic metric spaces has developed in extensive way. Hadzic and Pap have given a comprehensive survey of this development upto 2001 in [12]. Some of the references in Menger spaces of more recent times may be noted in [24, 27, 28] and [29].

Khan, Swaleh and Sessa [23] introduced a new type of contraction in metric spaces in 1984. They had used a control function to prove their result. This control function is known as 'altering distance function'. After this paper many authors have used altering distance function to get fixed point results. Here we mention a few of these results in [26, 30] and [31]. Recently, Choudhury and Das [1] extend the concept of altering distance function in the context of Menger spaces. The idea of control function in Menger spaces opened the possibility of proving new fixed point and coincidence point results. Some of the recent results where this control function has been used in Menger spaces may be seen in [2, 3, 9, 10] and [25].

Another type of fixed point results appeared in the literature due to R. Kannan [15, 16]. Now we give the definition of Kannan type mappings.

Definition 1.1. [15, 16] *Let (X, d) be a metric space and f be a mapping on X. The mapping f is called a Kannan type mapping if there exists $0 \leq \alpha < \frac{1}{2}$ such that*

$$d(fx, fy) \leq \alpha[d(x, fx) + d(y, fy)] \text{ for all } x, y \in X. \tag{1.1}$$

Kannan proved that every mappings satisfying (1.1) have a unique fixed point. This type of mappings are a class of contractive mappings which are different from the Banach contraction. A difference between Banach contraction mappings and Kannan type mappings is that Banach contraction mappings are always continuous but Kannan type mappings are not necessarily continuous. After the appearance of Kannan's paper in [15, 16], many authors created contractive conditions not requiring the continuity of the mappings and established fixed point results of such mappings. Now this line of research has a vast literature. Another reason for the importance of Kannan type mappings is that it characterizes completeness which the Banach contraction does not. It has been shown in [35, 36] that the necessary existence of fixed points for Kannan type mappings implies that the corresponding metric space is complete. The same is not true for Banach contractions. There is an example of an incomplete metric space where every contraction has a fixed point [7]. Kannan type mappings, its generalizations and extensions in various spaces have been considered in a large number of works, some of which may be noted in [4, 6, 14, 19, 20, 21, 32] and [35]. There are also similarities between Banach and Kannan type contractions. One is referred to [20] and [21] for similarity between contraction and Kannan type mappings.

In recent years cyclic contraction and cyclic contractive type mapping have appeared in several works.

Definition 1.2. *Let A and B be two non-empty sets. A cyclic mapping is a mapping $T : A\bigcup B \to A\bigcup B$ which satisfies:*

$$TA \subseteq B \text{ and } TB \subseteq A.$$

This line of research was initiated by Kirk, Srinivasan and Veeramani [22], where they, amongst other results, established the following generalization of the contraction mapping principle.

Theorem 1.1. [22] *Let A and B be two non-empty closed subsets of a complete metric space X and suppose $f : X \to X$ satisfies:*

(1) $fA \subseteq B$ and $fB \subseteq A$,

(2) $d(fx, fy) \leq kd(x, y)$ for all $x \in A$ and $y \in B$ where $k \in (0, 1)$.

Then f has a unique fixed point in $A\bigcap B$.

The problems of cyclic contractions have been strongly associated with proximity point problems. Some other results dealing with cyclic contractions and proximity point problems may be noted in [8, 11, 17, 37, 38] and [39].

A generalization of cyclic mapping is p-cyclic mapping. The definition is the following:

Definition 1.3. *Let $\{A_i\}_{i=1}^{p}$ be non-empty sets. A p-cyclic mapping is a mapping $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ which satisfies the following conditions :*

(*i*) $TA_i \subseteq A_{i+1}$ *for* $1 \le i < p$, $TA_p \subseteq A_1$.

In this case where $p = 2$, this reduces to cyclic mappings. Some fixed point results of p-cyclic maps have been obtained in [17, 18, 37]. In this paper we are interested in the fixed point properties of p-cyclic mappings in probabilistic metric spaces. In the following we describe the space briefly and to the extent of our requirement. Several aspects of this space has been described comprehensively by Schweizer and Sklar [33].

Definition 1.4. [12, 33] *A mapping $F : R \to R^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf_{t\in R} F(t) = 0$ and $\sup_{t\in R} F(t) = 1$, where R is the set of real numbers and R^+ denotes the set of non-negative real numbers.*

Definition 1.5. *t-norm* [12, 33]

A t-norm is a function $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ which satisfies the following conditions for all $a, b, c, d \in [0, 1]$

(i) $\Delta(1, a) = a$,

(ii) $\Delta(a, b) = \Delta(b, a)$,

(iii) $\Delta(c, d) \ge \Delta(a, b)$ *whenever* $c \ge a$ *and* $d \ge b$,

(iv) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

Definition 1.6. *Hadzic type t-norm* [12]

A t-norm Δ is said to be Hadzic type t-norm if the family $\{\Delta^p\}_{p\in N}$ of its iterates, defined for each $s \in (0, 1)$ as

$$\Delta^0(s) = 1,\ \Delta^{p+1}(s) = \Delta(\Delta^p(s), s) \text{ for all } p \ge 0,$$

is equi-continuous at $s = 1$, that is, given $\lambda > 0$ there exits $\eta(\lambda) \in (0, 1)$ such that

$$1 \ge s > \eta(\lambda) \Rightarrow \Delta^p(s) \ge 1 - \lambda \text{ for all } p \ge 0.$$

Definition 1.7. ***Menger space*** [12, 33]

A Menger space is a triplet (X, F, Δ), where X is a non empty set, F is a function defined on $X \times X$ to the set of distribution functions and Δ is a t-norm, such that the following are satisfied:

(i) $F_{x,y}(0) = 0$ *for all* $x, y \in X$,

(ii) $F_{x,y}(s) = 1$ *for all* $s > 0$ *if and only if* $x = y$,

(iii) $F_{x,y}(s) = F_{y,x}(s)$ *for all* $x, y \in X, s > 0$ *and*

(iv) $F_{x,y}(u + v) \geq \Delta(F_{x,z}(u), F_{z,y}(v))$ *for all* $u, v \geq 0$ *and* $x, y, z \in X$.

An interpretation of $F_{x,y}(t)$ is that it is the probability of the event that the distance between the points x and y is less than t. A metric space becomes a Menger space if we write $F_{x,y}(t) = H(t - d(x, y))$ where H is the Heaviside function given by

$$H(t) = 1, \text{ if } t > 0,$$
$$= 0, \text{ if } t \leq 0.$$

Definition 1.8. [12, 33] *A sequence* $\{x_n\} \subset X$ *is said to converge to some point* $x \in X$ *if given* $\epsilon > 0, \lambda > 0$ *we can find a positive integer* $N_{\epsilon,\lambda}$ *such that for all* $n > N_{\epsilon,\lambda}$

$$F_{x_n,x}(\epsilon) \geq 1 - \lambda. \tag{1.2}$$

Definition 1.9. [12, 33] *A sequence* $\{x_n\}$ *is said to be a Cauchy sequence in* X *if given* $\epsilon > 0, \lambda > 0$ *there exists a positive integer* $N_{\epsilon,\lambda}$ *such that*

$$F_{x_n,x_m}(\epsilon) \geq 1 - \lambda \quad \text{for all } m, n > N_{\epsilon,\lambda}. \tag{1.3}$$

Definition 1.8 and 1.9 can be equivalently written by replacing '$\geq$' with '$>$' in (1.2) and (1.3) respectively. More often than not, they are written in that way. We have given them in the present form for our convenience in the proofs of our theorems.

Definition 1.10. [12, 33] *A Menger space* (X, F, Δ) *is said to be complete if every Cauchy sequence is convergent in* X.

In [1] Choudhury and Das extended the concept of altering distance function in the context of Menger spaces and gave the following definition of ϕ-function.

Definition 1.11. ***Φ-function*** [1]

A function $\phi : R \to R^+$ is said to be a Φ-function if it satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$,

(ii) $\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$,

(iii) ϕ is left continuous in $(0, \infty)$,

(iv) ϕ is continuous at 0.

It has been shown in [1] that a ϕ-function can generate altering distance function. A cyclic contraction principle in Menger spaces have been proved with the help of ϕ function by the present authors in [5].

Theorem 1.2. [5] *Let (X, F, Δ) be a complete Menger space where Δ is the minimum t-norm and let there exist two closed subsets A and B of X where $A \bigcap B$ is nonempty such that the mapping $T : A \bigcup B \to A \bigcup B$ satisfies the following conditions :*

(i) *$TA \subseteq B$ and $TB \subseteq A$,*

(ii) $F_{Tx,Ty}(\phi(ct)) \geq F_{x,y}(\phi(t))$

for all $x \in A$ and $y \in B$ where $c \in (0, 1)$ and $t > 0$. Then T has a unique fixed point in $A \bigcap B$.

It has been shown that the Theorem 1.2 contains Theorem 1.1.

We will make use of the following function in our theorems.

Definition 1.12. ***Ψ-function*** [4]

A function $\psi : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a Ψ-function if

(i) ψ-is monotone increasing and continuous,

(ii) $\psi(x,x) \geq x$ *for all* $0 < x < 1$,

(iii) $\psi(1,1) = 1, \psi(0,0) = 0$.

2. Main Result

In this section we have two theorems and two examples.

Theorem 2.1. *Let* (X, F, Δ) *be a complete Menger space with a Hadzic type t- norm* Δ *such that whenever* $x_n \to x$ *and* $y_n \to y$, $F_{x_n,y_n}(t) \to F_{x,y}(t)$. *Let* $\{A_i\}_{i=1}^p$ *be non-empty closed subsets of* X *and the mapping* $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ *is a p-cyclic mapping, that is,*

$$TA_i \subseteq A_{i+1} \text{ for } 1 \leq i < p,\ TA_p \subseteq A_1 \tag{2.1}$$

and is such that

$$F_{Tx,Ty}(t) > \psi(F_{x,Tx}(\tfrac{t_1}{a}), F_{y,Ty}(\tfrac{t_2}{b})) \tag{2.2}$$

whenever $x \in A_i$ *and* $y \in A_{i+1}$ *for* $1 \leq i < p$

and

$$F_{Tz,Tw}(t) > \psi(F_{z,Tz}(\tfrac{t_1}{a}), F_{w,Tw}(\tfrac{t_2}{b})) \tag{2.3}$$

whenever $z \in A_p$ *and* $w \in A_1$

for $t_1, t_2, t > 0$ *with* $t = t_1 + t_2$, $a, b > 0$ *with* $0 < a + b < 1$ *and* ψ *is a* Ψ*-function. Then* $\bigcap_{i=1}^p A_i$ *is non-empty and* T *has a unique fixed point in* $\bigcap_{i=1}^p A_i$.

Proof. *Let* x_0 *be any arbitrary point in* A_1. *Now we define the sequence* $\{x_n\}_{n=0}^\infty$ *in* X *by* $x_n = Tx_{n-1}, n \in N$ *where* N *is the set of natural numbers.*

By (2.1), we have

$x_o \in A_1$, $x_1 \in A_2$, $x_2 \in A_3$,......,$x_{p-1} \in A_p$ *and in general* $x_{np} \in A_1$, $x_{np+1} \in A_2$,......., $x_{np+(p-1)} \in A_p$ *for all* $n \geq 0$.

Now, for $t, t_1, t_2 > 0$ *with* $t = t_1 + t_2$, *we have*

$$\begin{aligned} F_{x_{n+1},x_n}(t) &= F_{Tx_n,Tx_{n-1}}(t) \\ &= F_{Tx_{n-1},Tx_n}(t) \\ &> \psi(F_{x_{n-1},Tx_{n-1}}(\tfrac{t_1}{a}), F_{x_n,Tx_n}(\tfrac{t_2}{b}))\ (\text{ since } x_{n-1} \in A_n,\ x_n \in A_{n+1}\ ) \\ &= \psi(F_{x_{n-1},x_n}(\tfrac{t_1}{a}), F_{x_n,x_{n+1}}(\tfrac{t_2}{b})). \end{aligned} \tag{2.4}$$

Let $t_1 = \frac{at}{a+b}$, $t_2 = \frac{bt}{a+b}$ *and* $c = a + b$, *then obviously we have* $0 < c < 1$. *(2.5)*

Then, we have from (2.4),

$$F_{x_{n+1},x_n}(t) > \psi(F_{x_{n-1},x_n}(\tfrac{t}{c}), F_{x_n,x_{n+1}}(\tfrac{t}{c})). \tag{2.6}$$

We now claim that for all $t > 0$, $n \geq 0$,

$$F_{x_n,x_{n+1}}(\tfrac{t}{c}) \geq F_{x_{n-1},x_n}(\tfrac{t}{c}). \tag{2.7}$$

If possible, let for some $s > 0$, *and some* $n \geq 0$,

$F_{x_n,x_{n+1}}(\frac{s}{c}) < F_{x_{n-1},x_n}(\frac{s}{c})$, *then we have from (2.6),*

$$\begin{aligned} F_{x_{n+1},x_n}(s) &> \psi(F_{x_n,x_{n+1}}(\tfrac{s}{c}), F_{x_n,x_{n+1}}(\tfrac{s}{c})) \\ &= \psi(F_{x_{n+1},x_n}(\tfrac{s}{c}), F_{x_{n+1},x_n}(\tfrac{s}{c})) \\ &\geq F_{x_{n+1},x_n}(\tfrac{s}{c}) \\ &\geq F_{x_{n+1},x_n}(s), \end{aligned}$$

which is a contradiction, since $0 < c < 1$ *and* F *is non-decreasing.*

Therefore, for all $t > 0$, $n \geq 0$, *(2.7) holds.*

Now using (2.7), we have from (2.6), for all $t > 0$,

$$\begin{aligned} F_{x_{n+1},x_n}(t) &> \psi(F_{x_{n-1},x_n}(\tfrac{t}{c}), F_{x_n,x_{n+1}}(\tfrac{t}{c})) \\ &\geq \psi(F_{x_{n-1},x_n}(\tfrac{t}{c}), F_{x_{n-1},x_n}(\tfrac{t}{c})) \\ &= \psi(F_{x_n,x_{n-1}}(\tfrac{t}{c}), F_{x_n,x_{n-1}}(\tfrac{t}{c})) \\ &\geq F_{x_n,x_{n-1}}(\tfrac{t}{c}). \end{aligned} \tag{2.8}$$

By repeated applications of this inequality, for all $t > 0$, $n \geq 0$, *we obtain*

$$F_{x_n,x_{n+1}}(t) > F_{x_0,x_1}(\tfrac{t}{c^n}). \tag{2.9}$$

Taking limit as $n \to \infty$ *on both sides, for all* $t > 0$, *we have*

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(t) = 1. \tag{2.10}$$

Again, by repeated applications of (2.8), it follows that for all $t > 0$, $n \geq 0$ *and each* $i \geq 1$,

$$F_{x_{n+i},x_{n+i+1}}(t) > F_{x_n,x_{n+1}}(\tfrac{t}{c^i}). \tag{2.11}$$

We next prove that $\{x_n\}$ *is a Cauchy sequence (Definition 1.9), that is, we prove that for arbitrary* $\epsilon > 0$ *and* $0 < \lambda < 1$, *there exists* $N(\epsilon, \lambda)$ *such that*

$F_{x_n,x_m}(\epsilon) \geq 1 - \lambda$ *for all* $n, m \geq N(\epsilon, \lambda)$.

Without loss of generality we can assume that $m > n$.

Now,

$$\epsilon = \epsilon\tfrac{1-c}{1-c} > \epsilon(1-c)(1+c+c^2+.........+c^{m-n-1}).$$

Then, by the monotone increasing property of F, we have

$$F_{x_n,x_m}(\epsilon) \geq F_{x_n,x_m}(\epsilon(1-c)(1+c+c^2+.........+c^{m-n-1})),$$

that is,

$$F_{x_n,x_m}(\epsilon) \geq \Delta(F_{x_n,x_{n+1}}(\epsilon(1-c)), \Delta(F_{x_{n+1},x_{n+2}}(\epsilon c(1-c)), \Delta(.........,$$
$$\Delta(F_{x_{m-2},x_{m-1}}(\epsilon c^{m-n-2}(1-c)), F_{x_{m-1},x_m}(\epsilon c^{m-n-1}(1-c)))......))). \tag{2.12}$$

Putting $t = (1-c)\epsilon c^i$ in (2.11), we get

$$F_{x_{n+i},x_{n+i+1}}((1-c)\epsilon c^i) > F_{x_n,x_{n+1}}((1-c)\epsilon).$$

Then, by (2.12), we have

$$F_{x_n,x_m}(\epsilon) \geq \Delta(F_{x_n,x_{n+1}}(\epsilon(1-c)), \Delta(F_{x_n,x_{n+1}}(\epsilon(1-c)), \Delta(.........,$$
$$\Delta(F_{x_n,x_{n+1}}(\epsilon(1-c)), F_{x_n,x_{n+1}}(\epsilon(1-c)))......))),$$

that is,

$$F_{x_n,x_m}(\epsilon) \geq \Delta^{(m-n)} F_{x_n,x_{n+1}}(\epsilon(1-c)). \tag{2.13}$$

Since the t-norm Δ is a Hadzic type t-norm, the family $\{\Delta^p\}$ of its iterates is equicontinuous at the point $s = 1$, that is, there exists $\eta(\lambda) \in (0,1)$ such that for all $m > n$,

$$\Delta^{(m-n)}(s) \geq 1-\lambda \text{ whenever } \eta(\lambda) < s \leq 1. \tag{2.14}$$

Since, $F_{x_0,x_1}(t) \to 1$ as $t \to \infty$ and $0 < c < 1$, there exists an positive integer $N(\epsilon,\lambda)$ such that

$$F_{x_0,x_1}(\tfrac{(1-c)\epsilon}{c^n}) > \eta(\lambda) \text{ for all } n \geq N(\epsilon,\lambda). \tag{2.15}$$

From (2.15) and (2.11), with $n = 0$, $i = n$ and $t = (1-c)\epsilon$, we get

$$F_{x_n,x_{n+1}}(\epsilon(1-c)) > F_{x_0,x_1}(\tfrac{(1-c)\epsilon}{c^n}) > \eta(\lambda) \text{ for all } n \geq N(\epsilon,\lambda).$$

Then, from (2.14) with $s = F_{x_n,x_{n+1}}(\epsilon(1-c))$, we have

$$\Delta^{(m-n)}(F_{x_n,x_{n+1}}(\epsilon(1-c))) \geq 1-\lambda.$$

It then follows from (2.13) that

$$F_{x_n,x_m}(\epsilon) \geq 1-\lambda \text{ for all } m,n \geq N(\epsilon,\lambda).$$

Thus $\{x_n\}$ is a Cauchy sequence.

Since X is complete, we have

$$\lim_{n\to\infty} x_n = z. \tag{2.16}$$

By the construction of the sequence $\{x_n\}$, *we have* $x_p \in A_1$, $x_{2p} \in A_1$, ...$x_{np} \in A_1$. *Therefore the subsequence* $\{x_{np}\}$ *of* $\{x_n\}$ *which belongs to* A_1 *also converges to* z *in* A_1, *since* A_1 *is closed. Similarly subsequence* $\{x_{np+1}\}$ *belongs to* A_2 *also converges to* z *in* A_2. *Since* A_3, A_4, ...,A_p *are closed sets, similarly we get* $z \in A_3$, A_4,........,A_p. *Therefore* $z \in A_1 \bigcap A_2 \bigcap A_3 ...... \bigcap A_p$.

Now, we prove that $Tz = z$.

Putting $x = x_n$, $y = z$ *in (2.2), for all* $t > 0$, *we have*

$$F_{Tx_n,Tz}(t) > \psi(F_{x_n,Tx_n}(\tfrac{t_1}{a}), F_{z,Tz}(\tfrac{t_2}{b}))\ (x_n \in A_{n+1},\ z \in A_{n+2}).$$

Now, using (2.5), for all $t > 0$, *we get*

$$F_{Tx_n,Tz}(t) > \psi(F_{x_n,Tx_n}(\tfrac{t}{c}),\ F_{z,Tz}(\tfrac{t}{c})). \tag{2.17}$$

Taking limit as $n \to \infty$ *in (2.17), for all* $t > 0$, *we have*

$$\begin{aligned} F_{z,Tz}(t) &\geq \psi(F_{z,z}(\tfrac{t}{c}),\ F_{z,Tz}(\tfrac{t}{c})) \\ &= \psi(1, F_{z,Tz}(\tfrac{t}{c})) \\ &\geq \psi(F_{z,Tz}(\tfrac{t}{c}), F_{z,Tz}(\tfrac{t}{c})) \\ &\geq F_{z,Tz}(\tfrac{t}{c}). \end{aligned} \tag{2.18}$$

(since by our assumption $x_n \to x, y_n \to y$ *implies* $F_{x_n,y_n} \to F_{x,y}$*)*

By repeated applications of (2.18) n *times, for all* $t > 0$, *we obtain*

$$F_{z,Tz}(t) \geq F_{z,Tz}(\tfrac{t}{c^n}).$$

Taking limit as $n \to \infty$ *on both sides, for all* $t > 0$,

$$F_{z,Tz}(t) \geq \lim_{n\to\infty} F_{z,Tz}(\frac{t}{c^n}) = 1,$$

which implies

$$F_{z,Tz}(t) = 1.$$

Thus $z = Tz$.

To prove the uniqueness of the fixed point, let us take $v \in A_1 \bigcap A_2 \bigcap A_3 ...... \bigcap A_p$ *be another fixed point of* T, *that is,* $Tv = v$.

Now,

$$\begin{aligned} F_{z,v}(t) &= F_{Tz,Tv}(t) \\ &> \psi(F_{z,Tz}(\tfrac{t_1}{a}), F_{v,Tv}(\tfrac{t_2}{b})) \\ &= \psi(F_{z,z}(\tfrac{t_1}{a}), F_{v,v}(\tfrac{t_2}{b})) \\ &= \psi(1,1) = 1, \end{aligned}$$

which implies that $z = v$, that is, the fixed point is unique.

We prove our next theorem with the help of the control function ϕ and using the minimum t-norm instead of Hadzic type t-norm. It is important to note that to prove our result we can not follow the same argument as in Theorem 2.1.

Theorem 2.2. *Let (X, F, Δ) be a complete Menger space where Δ is the minimum t-norm. Let $\{A_i\}_{i=1}^p$ be non-empty closed subsets of X and the mapping $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is a p-cyclic mapping, that is,*

$$TA_i \subseteq A_{i+1} \text{ for } 1 \le i < p,\ TA_p \subseteq A_1 \tag{2.19}$$

and is such that

$$F_{Tx,Ty}(\phi(t)) > \psi(F_{x,Tx}(\phi(\tfrac{t_1}{a})), F_{y,Ty}(\phi(\tfrac{t_2}{b}))) \tag{2.20}$$

whenever $x \in A_i$, $y \in A_j$ and $1 \le i, j \le p$, $i \ne j$, $t_1, t_2, t > 0$ with $t = t_1 + t_2$, $a, b > 0$ with $0 < a + b < 1$, ψ is a Ψ-function and ϕ is a Φ-function. Then $\bigcap_{i=1}^p A_i$ is non-empty and T has a unique fixed point in $\bigcap_{i=1}^p A_i$.

Proof. Let x_0 be any arbitrary point in A_1. Now we define the sequence $\{x_n\}_{n=0}^\infty$ in X by $x_n = Tx_{n-1}, n \in N$ where N is the set of natural numbers.

By (2.19), we have
$x_o \in A_1$, $x_1 \in A_2$, $x_2 \in A_3$,......,$x_{p-1} \in A_p$ and in general $x_{np} \in A_1$, $x_{np+1} \in A_2$,......., $x_{np+(p-1)} \in A_p$ for all $n \ge 0$.
Now, for $t, t_1, t_2 > 0$ with $t = t_1 + t_2$, we have

$$\begin{aligned} F_{x_{n+1},x_n}(\phi(t)) &= F_{Tx_n,Tx_{n-1}}(\phi(t)) \\ &= F_{Tx_{n-1},Tx_n}(\phi(t)) \\ &> \psi(F_{x_{n-1},Tx_{n-1}}(\phi(\tfrac{t_1}{a})), F_{x_n,Tx_n}(\phi(\tfrac{t_2}{b}))) \ (\ x_{n-1} \in A_n,\ x_n \in A_{n+1}\ ) \end{aligned}$$

$$= \psi(F_{x_{n-1},x_n}(\phi(\tfrac{t_1}{a})), F_{x_n,x_{n+1}}(\phi(\tfrac{t_2}{b}))). \qquad (2.21)$$

Let $t_1 = \frac{at}{a+b}$, $t_2 = \frac{bt}{a+b}$ and $c = a + b$, then obviously we have $0 < c < 1$. (2.22)

Then, we have from (2.21),

$$F_{x_{n+1},x_n}(\phi(t)) > \psi(F_{x_{n-1},x_n}(\phi(\tfrac{t}{c})), F_{x_n,x_{n+1}}(\phi(\tfrac{t}{c}))). \qquad (2.23)$$

We now claim that for all $t > 0$, $n \geq 0$

$$F_{x_n,x_{n+1}}(\phi(\tfrac{t}{c})) \geq F_{x_{n-1},x_n}(\phi(\tfrac{t}{c})). \qquad (2.24)$$

If possible, let for some $s > 0$, and some $n \geq 0$,

$F_{x_n,x_{n+1}}(\phi(\frac{s}{c})) < F_{x_{n-1},x_n}(\phi(\frac{s}{c}))$, then we have from (2.23),

$$\begin{aligned} F_{x_{n+1},x_n}(\phi(s)) &> \psi(F_{x_n,x_{n+1}}(\phi(\tfrac{s}{c})), F_{x_n,x_{n+1}}(\phi(\tfrac{s}{c}))) \\ &= \psi(F_{x_{n+1},x_n}(\phi(\tfrac{s}{c})), F_{x_{n+1},x_n}(\phi(\tfrac{s}{c}))) \\ &\geq F_{x_{n+1},x_n}(\phi(\tfrac{s}{c})) \\ &\geq F_{x_{n+1},x_n}(\phi(s)), \end{aligned}$$

which is a contradiction, since $0 < c < 1$, ϕ is strictly increasing and F is non-decreasing. Therefore, for all $t > 0$ and $n \geq 0$, (2.24) holds.

Now, using (2.24) in (2.23) and by the properties of ψ, for all $t > 0$, $n \geq 0$, we have

$$\begin{aligned} F_{x_{n+1},x_n}(\phi(t)) &> \psi(F_{x_{n-1},x_n}(\phi(\tfrac{t}{c})), F_{x_n,x_{n+1}}(\phi(\tfrac{t}{c}))) \\ &\geq \psi(F_{x_{n-1},x_n}(\phi(\tfrac{t}{c})), F_{x_{n-1},x_n}(\phi(\tfrac{t}{c}))) \\ &= \psi(F_{x_n,x_{n-1}}(\phi(\tfrac{t}{c})), F_{x_n,x_{n-1}}(\phi(\tfrac{t}{c}))) \\ &\geq F_{x_n,x_{n-1}}(\phi(\tfrac{t}{c})). \end{aligned} \qquad (2.25)$$

By repeated applications of (2.25), for all $t > 0$, $n \geq 0$, we have

$$F_{x_{n+1},x_n}(\phi(t)) > F_{x_1,x_0}(\phi(\tfrac{t}{c^n})). \qquad (2.26)$$

Taking limit as $n \to \infty$ on both sides of (2.26), for all $t > 0$, we obtain

$$\lim_{n\to\infty} F_{x_{n+1},x_n}(\phi(t)) = 1. \qquad (2.27)$$

Again, by virtue of a property of ϕ, given $s > 0$ we can find $t > 0$ such that $s > \phi(t)$. Thus the above limit implies that for all $s > 0$,

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(s) = 1. \qquad (2.28)$$

We next prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ be not a Cauchy sequence. Then there exist $\epsilon > 0$ and $0 < \lambda < 1$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)}}(\epsilon) < 1 - \lambda. \tag{2.29}$$

We take $n(k)$ corresponding to $m(k)$ to be the smallest integer satisfying (2.29) so that

$$F_{x_{m(k)},x_{n(k)-1}}(\epsilon) \geq 1 - \lambda. \tag{2.30}$$

If $\epsilon_1 < \epsilon$, then we have

$$F_{x_{m(k)},x_{n(k)}}(\epsilon_1) \leq F_{x_{m(k)},x_{n(k)}}(\epsilon).$$

We conclude that it is possible to construct $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ with $n(k) > m(k) > k$ and satisfying (2.29) and (2.30) whenever ϵ is replaced by a smaller positive value. As ϕ is continuous at 0 and strictly monotone increasing with $\phi(0) = 0$, it is possible to obtain $\epsilon_2 > 0$ such that $\phi(\epsilon_2) < \epsilon$.

Then, by the above argument, it is possible to obtain an increasing sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$F_{x_{m(k)},x_{n(k)}}(\phi(\epsilon_2)) < 1 - \lambda \tag{2.31}$$

and

$$F_{x_{m(k)},x_{n(k)-1}}(\phi(\epsilon_2)) \geq 1 - \lambda. \tag{2.32}$$

By (2.31), we get

$$\begin{aligned} 1 - \lambda &> F_{x_{m(k)},x_{n(k)}}(\phi(\epsilon_2)) \\ &= F_{Tx_{m(k)-1},Tx_{n(k)-1}}(\phi(\epsilon_2)) \\ &> \psi(F_{x_{m(k)-1},Tx_{m(k)-1}}(\phi(\tfrac{t_1}{a})), F_{x_{n(k)-1},Tx_{n(k)-1}}(\phi(\tfrac{t_2}{b}))) \\ &= \psi(F_{x_{m(k)-1},x_{m(k)}}(\phi(\tfrac{t_1}{a})), F_{x_{n(k)-1},x_{n(k)}}(\phi(\tfrac{t_2}{b}))). \end{aligned} \tag{2.33}$$

(since $x_{m(k)-1} \in A_{m(k)}$ and $x_{n(k)-1} \in A_{n(k)}$, $m(k) \neq n(k)$)

By (2.28), for sufficiently large k and by virtue of property of ϕ we have

$$F_{x_{m(k)-1},x_{m(k)}}(\phi(\tfrac{t_1}{a})) \geq 1 - \lambda \tag{2.34}$$

and

$$F_{x_{n(k)-1},x_{n(k)}}(\phi(\tfrac{t_2}{b})) \geq 1 - \lambda. \tag{2.35}$$

Using (2.34) and (2.35) in (2.33), we have

$$1-\lambda > \psi(1-\lambda, 1-\lambda) \geq 1-\lambda,$$

which is a contradiction.

Thus $\{x_n\}$ is a Cauchy sequence.

Since X is complete, we have

$$\lim_{n\to\infty} x_n = z. \tag{2.36}$$

By the construction of the sequence $\{x_n\}$, we have $x_p \in A_1$, $x_{2p} \in A_1$, ...$x_{np} \in A_1$. Therefore the subsequence $\{x_{np}\}$ of $\{x_n\}$ which belongs to A_1 also converges to z in A_1, since A_1 is closed. Similarly subsequence $\{x_{np+1}\}$ belongs to A_2 also converges to z in A_2. Since A_3, A_4, ...,A_p are closed sets, similarly we get $z \in A_3$, A_4,........,A_p. Therefore $z \in A_1 \bigcap A_2 \bigcap A_3 ...... \bigcap A_p$.

Now, we prove that $Tz = z$.

Putting $x = x_n$, $y = z$ in the inequality (2.20), for all $t > 0$, we have

$$F_{Tx_n,Tz}(\phi(t)) > \psi(F_{x_n,Tx_n}(\phi(\tfrac{t_1}{a})), F_{z,Tz}(\phi(\tfrac{t_2}{b}))).$$

($x_n \in A_{n+1}$, $z \in A_i$, where $n+1 \neq i$)

Now, using (2.22), for all $t > 0$, we get

$$F_{Tx_n,Tz}(\phi(t)) > \psi(F_{x_n,Tx_n}(\phi(\tfrac{t}{c})),\ F_{z,Tz}(\phi(\tfrac{t}{c}))).$$

Taking liminf as $n \to \infty$ on both sides of the above inequality, for all $t > 0$, and by the properties of ϕ, we have

$$\liminf_{n\to\infty} F_{Tx_n,Tz}(\phi(t)) \geq \psi(\liminf_{n\to\infty} F_{x_n,Tx_n}(\phi(\frac{t}{c})),\ F_{z,Tz}(\phi(\tfrac{t}{c}))),$$

that is,

$$\begin{aligned} F_{z,Tz}(\phi(t)) &\geq \psi(F_{z,z}(\phi(\tfrac{t}{c})),\ F_{z,Tz}(\phi(\tfrac{t}{c}))) \text{ (by (2.36))} \\ &= \psi(1, F_{z,Tz}(\phi(\tfrac{t}{c}))) \\ &\geq \psi(F_{z,Tz}(\phi(\tfrac{t}{c})), F_{z,Tz}(\phi(\tfrac{t}{c}))) \end{aligned}$$

$\geq F_{z,Tz}(\phi(\frac{t}{c}))$.

By repeated applications of the above inequality for all $t > 0$ we have

$$F_{z,Tz}(\phi(t)) \geq F_{z,Tz}(\phi(\tfrac{t}{c^n})).$$

Now, taking limit $n \to \infty$ on both sides for all $t > 0$, we have

$$F_{z,Tz}(\phi(t)) \geq \lim_{n\to\infty} F_{z,Tz}(\phi(\frac{t}{c^n})) = 1.$$

Then, by the property of ϕ, we get $z = Tz$.

To prove the uniqueness of the fixed point, let us take $v \in A_1 \bigcap A_2 \bigcap A_3 ...... \bigcap A_p$ be another fixed point of T, that is, $Tv = v$.

Now,

$$\begin{aligned} F_{z,v}(\phi(t)) &= F_{Tz,Tv}(\phi(t)) \\ &> \psi(F_{z,Tz}(\phi(\tfrac{t_1}{a})), F_{v,Tv}(\phi(\tfrac{t_2}{b}))) \\ &= \psi(F_{z,z}(\phi(\tfrac{t_1}{a})), F_{v,v}(\phi(\tfrac{t_2}{b}))) \\ &= \psi(1,1) = 1. \end{aligned}$$

Again, by the properties of ϕ we can conclude that $z = v$.

Therefore $z \in A_1 \bigcap A_2 \bigcap A_3 ...... \bigcap A_p$ is a unique fixed point of T.

Taking two non-empty sets A and B of X we get the following corollary.

Corollary 2.1. *Let (X, F, Δ) be a complete Menger space where Δ is the minimum t- norm and let there exist two closed subsets A and B of X such that the mapping $T : A \bigcup B \to A \bigcup B$ satisfies the following conditions :*

$$\text{(i)} \quad TA \subseteq B \text{ and } TB \subseteq A, \tag{2.37}$$

$$\text{(ii)} \ F_{Tx,Ty}(\phi(t)) > \psi(F_{x,Tx}(\phi(\tfrac{t_1}{a})), F_{y,Ty}(\phi(\tfrac{t_2}{b}))) \tag{2.38}$$

for all $x \in A$ and $y \in B$, $t_1, t_2, t > 0$ with $t = t_1 + t_2$, $a, b > 0$ with $0 < a + b < 1$, ψ is a Ψ-function and ϕ is a Φ-function. Then $A \bigcap B$ is nonempty and T has a unique fixed point in $A \bigcap B$.

Now, we give the following example to illustrate Theorem 2.1 for $p = 2$.

Example 2.1. *Let* $X = \{x_1, x_2, x_3\}$, $A_1 = \{x_1, x_2\}$, $A_2 = \{x_1, x_3\}$, $\Delta(a, b) = min(a, b)$ *and* $F_{x,y}(t)$ *be defined as:*

$$F_{x_1,x_2}(t) = \begin{cases} 0, & if \quad t \leq 0, \\ 0.75, & if \quad 0 < t \leq 2, \\ 1, & if \quad t > 2. \end{cases}$$

$$F_{x_1,x_3}(t) = \begin{cases} 0, & if \quad t \leq 0, \\ 0.90, & if \quad 0 < t \leq 1, \\ 1, & if \quad t > 1. \end{cases}$$

$$F_{x_2,x_3}(t) = \begin{cases} 0, & if \quad t \leq 0, \\ 0.75, & if \quad 0 < t \leq 2, \\ 1, & if \quad t > 2. \end{cases}$$

Then (X, F, Δ) *be a complete Menger space. If we define a mapping* $T : A_1 \bigcup A_2 \to A_1 \bigcup A_2$ *satisfies all the conditions of Theorem 2.1 by taking* $Tx_1 = x_1, Tx_2 = x_3, Tx_3 = x_1$ *with* $\psi(x, y) = \min\{x, y\}$, $a = 0.85$, $b = 0.10$. *Here* x_1 *is the unique fixed point of* T *in* $A_1 \bigcap A_2$.

Now we give the following example to illustrate our Theorem 2.2 for $p = 3$.

Example 2.2. *Let* $X = \{x_1, x_2, x_3, x_4\}$, $p = 3$ *and* $A_1 = \{x_2, x_3\}$, $A_2 = \{x_2, x_1\}$, $A_3 = \{x_2, x_4\}$. *Also we take the t-norm* $\Delta(a, b) = min(a, b)$ *and* $F_{x,y}(t)$ *be defined as:*

$$F_{x_1,x_2}(t) = F_{x_1,x_3}(t) = F_{x_1,x_4}(t) = F_{x_2,x_4}(t) = F_{x_3,x_4}(t) = \begin{cases} 0, & if \quad t \leq 0, \\ 0.60, & if \quad 0 < t < 8, \\ 1, & if \quad t \geq 8. \end{cases}$$

$$F_{x_2,x_3}(t) = \begin{cases} 0, & if \quad t \leq 0, \\ 1, & if \quad t > 0. \end{cases}$$

Then (X, F, Δ) *be a complete Menger space. If we define* $T : \bigcup_{i=1}^{3} A_i \rightarrow \bigcup_{i=1}^{3} A_i$ *satisfies all the conditions of Theorem 2.2 by taking* $Tx_1 = x_2, Tx_2 = x_2, Tx_3 = x_2, Tx_4 = x_3$ *with* $\phi(t) = t$, $\psi(x, y) = \min\{x, y\}$, $a = 0.10$, $b = 0.85$. *Here* x_2 *is the unique fixed point of* T *in* $\bigcap_{i=1}^{3} A_i$.

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