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ON $*$-n-PARANORMAL OPERATORS ON BANACH SPACES

MUNEO CHŌ [1*], KÔTARÔ TANAHASHI[2]

[1]Department of Mathematics, Kanagawa University, Hiratsuka City 259-1293, Japan

[2]Department of Mathematics, Tohoku Pharmaceutical University, Sendai City 981-8558, Japan

Abstract. Let T be a $*$-n-paranormal operator on a complex Banach space $\mathscr{X}$. In this paper we show that T is isoloid and if α, β are distinct eigen-values of T, then $\ker(T-\alpha I) \perp \ker(T-\beta I)$. Also we show that if the dual space $\mathscr{X}^*$ is uniformly convex and $(T-\alpha I)x_k \longrightarrow 0$ for $(x_k, f_k) \in \Pi(\mathscr{X})$, then $(T-\alpha I)^* f_k \longrightarrow 0$.

Keywords: Banach space, $*$-n-paranormal operator, spectrum.

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1. Introduction

F. F. Bonsall and J. Duncan published books [2] and [3] concerning Banach spaces and spectral properties of Banach space operators. K. Mattila in [6] studied spectral properties of Banach space operators. In [5] she studied proper boundary points of the spectrum. In [4] M. Chō and S. Ôta introduced n-paranormal and $*$-n-paranormal operators on Banach spaces and studied properties of eigen-value of the spectrum. In [7] A. Uchiyama and K. Tanahashi studied spectral properties of $*$-paranormal Hilbert space operators. P. Aiena in [1] studied spectral properties of polynomially paranormal Banach space operators. In [9] L. Zhang, A. Uchiyama

*Corresponding author

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and K. Tanahashi and M. Chō showed that a polynomially *-paranormal operator T on a complex Hilbert space is isoloid (That is, an isolated point of the spectrum is an eigen-value.) and the spectral mapping theorem holds for the essential approximate point spectrum of T. In this paper we study spectral properties of $*$-n-paranormal Banach space operators.

2. Preliminaries

Let $\mathscr{X}$ be a complex Banach space and $T \in B(\mathscr{X})$. Let $\Pi(\mathscr{X})$ be

$$\Pi(\mathscr{X}) = \{(x,f) \in \mathscr{X} \times \mathscr{X}^* : \|f\| = f(x) = \|x\| = 1\},$$

where $\mathscr{X}^*$ is the dual space of $\mathscr{X}$. We define the numerical range $V(T)$ of T by

$$V(T) = \{f(Tx) \ : \ (x,f) \in \Pi(\mathscr{X})\}.$$

It is well known that following inclusion relations hold

$$\mathrm{co}\,\sigma(T) \subset \overline{V(T)} \subset \{z \in \mathbb{C} : |z| \leq \|T\|\},$$

where $\mathrm{co}\,\sigma(T), \overline{V(T)}$ and $\|T\|$ are the convex hull of the spectrum $\sigma(T)$, the closure of $V(T)$ and the norm of T, respectively. See Theorem 19.4 of [3].

Definition 1.

(1) T is said to be *n-paranormal* if $\|Tx\|^n \leq \|T^n x\| \cdot \|x\|^{n-1}$ for all $x \in \mathscr{X}$.

(2) T is said to be $*$-*n-paranormal* if $\|T^* f\|^n \leq \|T^n x\|$ for all $(x,f) \in \Pi(\mathscr{X})$, where T^* is the dual operator of T.

2-Paranormal operators are simply called paranormal. We denote the sets of all n-paranormal operators and $*$-n-paranormal operators by $\mathfrak{P}(n)$ and $\mathfrak{S}(n)$, respectively. In [4] M. Chō and S. Ôta proved that $\mathfrak{S}(n) \subset \mathfrak{P}(n+1)$ for every $n \in \mathbb{N}$ and $\mathfrak{P}(2) \subset \bigcap_{n=3}^{\infty} \mathfrak{P}(n) = \mathfrak{P}(3) \bigcap \mathfrak{P}(4)$.

Definition 2. Let A and B be subspaces of $\mathscr{X}$. A is *orthogonal* to B (denoted $A \perp B$) if

$$\|a\| \ \leq \ \|a+b\| \ \ (a \in A, b \in B).$$

3. Main results

Let $\ker(T)$ and $R(T)$ be the kernel and the range of T, respectively. Then it is well known that $\ker(T)\perp R(T)$ if and only if there exists $(x,f)\in\Pi(\mathscr{X})$ such that $x\in\ker(T)$ and $f\in\ker(T^*)$. See Lemma 20.3 of [3].

By the definition of (2) it is clear that if T is $*$-n-paranormal and $Tx=0$ ($\|x\|=1$), then, for any $f\in\mathscr{X}^*$ such that $(x,f)\in\Pi(\mathscr{X})$, $T^*f=0$.

Hence we have the following result.

Theorem 1. *Let T,S be $*$-n-paranormal operators. Then $\big(\ker(T)\bigcap\ker(S)\big)\perp\mathscr{M}$, where $\mathscr{M}$ is the smallest subspace containing $R(T)$ and $R(S)$.*

See page 14 of [6].

Next proposition is important in this paper. The paper [7] is for Hilbert space operators. But following results hold for Banach space operators.

Proposition 1 (Proposition 1 and Theorem 1, [7]**).** *Let T be an n-paranormal operator on $\mathscr{X}$.*
(1) *T is normaloid, that is, $\|T\|=r(T)$ (the spectral radius of T).*
(2) *If T is invertible, then $\|T^{-1}\|\le r(T^{-1})^{\frac{n(n-1)}{2}}\cdot r(T)^{\frac{(n+1)(n-2)}{2}}$.*

In [1] P. Aiena showed following result for paranormal operators.

Theorem 2. *Let T be an n-paranormal operator on $\mathscr{X}$. If $\sigma(T)=\{\alpha\}$, then $T=\alpha I$.*

Proof. If $\alpha=0$, then $\|T\|=r(T)=0$. Hence $T=0$. We assume $\alpha\neq 0$. Let $S=\dfrac{1}{\alpha}T$. Then S is invertible n-paranormal, $\|S\|=1$ and $\sigma(S)=\{1\}$. By Proposition 1, we have $\|S^{-1}\|=1$. Hence $\|S^n\|\le\|S\|^n\le 1$ and $\|S^{-n}\|\le\|S^{-1}\|^n\le 1$ for $n\in\mathbb{N}$. Hence $S=I$ and $T=\alpha I$ by Theorem 1 of [8]. This completes the proof.

Definition 3. T is said to be *polynomially n-paranormal* (*$*$-n-paranormal*) if there exists a non-constant polynomial $p(x)$ such that $p(T)$ is n-paranormal ($*$-n-paranormal).

Theorem 3. *Let T be a polynomially n-paranormal operator on $\mathscr{X}$. If $\sigma(T)=\{\alpha\}$, then $T-\alpha I$ is nilpotent.*

Proof. Let $p(x)$ be a polynomial such that $p(T)$ is n-paranormal. Let

$$p(x) - p(\alpha) = a(x-\alpha)^k \cdot (x-\alpha_1)\cdots(x-\alpha_m),$$

where $\alpha_j \neq \alpha$ for all $j = 1,2,...,m$ and $a \neq 0$. Since $p(T)$ is n-paranormal and $\sigma(p(T)) = \{p(\alpha)\}$, we have $p(T) = p(\alpha)I$ by Theorem 2. Hence

$$p(T) - p(\alpha)I = a(T-\alpha I)^k \cdot (T-\alpha_1 I)\cdots(T-\alpha_m I) = 0.$$

Since all $T-\alpha_j I \ \ (j = 1,2,...,m)$ are invertible, we have $(T-\alpha I)^k = 0$. This completes the proof.

If T is $*$-n-paranormal, then T is $(n+1)$-paranormal by Theorem 6 of [4]. Hence we have following corollary.

Corollary 1. *For $T \in B(\mathscr{X})$, let $\sigma(T) = \{\alpha\}$.*

(1) *If T be a $*$-n-paranormal operator, then $T = \alpha I$.*

(2) *If T be a polynomially $*$-n-paranormal operator, then $T-\alpha I$ is nilpotent.*

Theorem 4. *Let T be a $*$-n-paranormal (n-paranormal) operator on $\mathscr{X}$. If $\mathscr{M}$ is a closed invariant subspace for T, then $T\big|_{\mathscr{M}}$ is $*$-n-paranormal (n-paranormal).*

Proof. Let T be $*$-n-paranormal on $\mathscr{X}$ and $(x,f) \in \Pi(\mathscr{M})$. By the Hahn-Banach theorem, there exists $g \in \mathscr{X}^*$ such that $g\big|_{\mathscr{M}} = f$ and $\|g\| = \|f\|$. Hence $(x,g) \in \Pi(\mathscr{X})$ and

$$\left(\left(T\big|_{\mathscr{M}}\right)^* f\right)(x) = f\left(T\big|_{\mathscr{M}} x\right) = f(Tx) = g(Tx) = (T^* g)(x),$$

and so

$$\|\left(T\big|_{\mathscr{M}}\right)^* f\|^n \leq \|T^* g\|^n \leq \|T^n x\| = \|\left(T\big|_{\mathscr{M}}\right)^n x\|.$$

It is easy to see that if T is n-paranormal, then $T\big|_{\mathscr{M}}$ is n-paranormal. This completes the proof.

Theorem 5. *Let T be an n-paranormal operator on $\mathscr{X}$. Then T is isoloid.*

Proof. Let α be an isolated point of $\sigma(T)$ and P be the spectral projection associated with α. Since then $T\big|_{P(\mathscr{M})}$ is n-paranormal by Theorem 4 and $\sigma(T\big|_{P(\mathscr{M})}) = \{\alpha\}$, $T\big|_{P(\mathscr{M})} = \alpha I$. Hence α is an eigen-value of T. This completes the proof.

If T is $*$-n-paranormal, then T is $(n+1)$-paranormal. Hence the following corollary is direct from Theorem 5 and Corollary 1.

Corollary 2. *If T is $*$-n-paranormal, then T is isoloid.*

Theorem 6. *Let T be an operator on $\mathscr{X}$ and satisfy one of the following statements. If α is an isolated point of $\sigma(T)$, then α is a pole of the resolvent, that is, T is polaroid.*

(1) *T is n-paranormal.*

(2) *T is $*$-n-paranormal.*

(3) *T is polynomially n-paranormal.*

(4) *T is polynomially $*$-n-paranormal.*

Proof. Proof is same with Theorem 1.3 of [1].

An operator T is called *hereditarily polaroid* if any restriction to an invariant closed subspace is polaroid. Hence, the following result is clear.

Theorem 7. *Polynomially n-paranormal operators on $\mathscr{X}$ are hereditarily polaroid.*

Definition 4. $\alpha \in \sigma(T)$ is said to be *proper boundary point* of $\sigma(T)$ if there exists a bounded sequence $\{\alpha_n\} \subset \rho(T)$ (the resolvent set of T) such that $\|(\alpha-\alpha_n)(T-\alpha_n I)^{-1}\| \longrightarrow 1$.

Proposition 2 (Lemma 1, [5]**).** *If $\alpha \in \partial V(T) \bigcap \sigma(T)$, then α is a proper boundary point of $\sigma(T)$, where $\partial V(T)$ is the boundary of $V(T)$.*

Proposition 3 (Proposition 3.7, [6]**).** *If 0 is a proper boundary point of $\sigma(T)$ and $Tx = 0$ with $\|x\| = 1$, then $1 \leq \|x+Ty\|$ for every $y \in \mathscr{X}$. That is,* $\ker(T) \perp R(T)$.

Theorem 8. *Let T be n-paranormal operators on $\mathscr{X}$. If α, β are distinct eigenvalues of T, then* $\ker(T-\alpha I) \perp \ker(T-\beta I)$.

For the proof of Theorem 8 we prepare lemmas. For the completeness, we give proofs.

For an eigen-value α of T, let $K(\alpha) = \{x \in \mathscr{X} : Tx = \alpha x\}$.

Lemma 1. *Let $T \in B(\mathscr{X})$. Let α, β be distinct eigen-values of T. Then $K(\alpha)+K(\beta) = \{x+y : x \in K(\alpha), y \in K(\beta)\}$ is a closed subspace.*

Proof. Let $\mathscr{M} = K(\alpha) + K(\beta)$. Then it is easy $\mathscr{M}$ is a subspace. We show $\mathscr{M}$ is closed. Let $x_n + y_n \to z$, where $x_n \in K(\alpha), y_n \in K(\beta)$. Then

$$(T - \alpha I)(x_n + y_n) = (\beta - \alpha)y_n \to (T - \alpha I)z.$$

Since $K(\beta)$ is closed and $(\beta - \alpha)y_n \in K(\beta)$, this implies $(T - \alpha I)z \in K(\beta)$. Similarly $(T - \beta I)z \in K(\alpha)$. Thus

$$z = \frac{(T - \beta I)z}{\alpha - \beta} - \frac{(T - \alpha I)z}{\alpha - \beta} \in K(\alpha) + K(\beta).$$

Hence $\mathscr{M}$ is closed. This completes the proof.

Lemma 2. *Let $T \in B(\mathscr{X})$ and α, β be distinct eigen-values of T. If $\mathscr{M} = K(\alpha) + K(\beta)$, then*

$$\sigma(T_{|\mathscr{M}}) = \{\alpha, \beta\}.$$

Proof. By Lemma 1, $\mathscr{M}$ is a closed invariant subspace for T and it is obvious that

$$\alpha, \beta \in \sigma_p(T_{|\mathscr{M}}) \subset \sigma(T_{|\mathscr{M}}).$$

We show $T_{|\mathscr{M}} - \lambda I$ is bijective if $\lambda \neq \alpha, \beta$. Let $(T_{|\mathscr{M}} - \lambda I)(x + y) = 0$ where $x \in K(\alpha)$ and $y \in K(\beta)$. Then $(\alpha - \lambda)x + (\beta - \lambda)y = 0$.

Since $K(\alpha)$ and $K(\beta)$ are linear independent, $x = 0$ and $y = 0$. Hence $T_{|\mathscr{M}} - \lambda$ is injective. Let $x \in K(\alpha)$ and $y \in K(\beta)$. Then $\frac{x}{\alpha - \lambda} \in K(\alpha)$ and $\frac{y}{\beta - \lambda} \in K(\beta)$. Since

$$(T_{|\mathscr{M}} - \lambda I)\left(\frac{x}{\alpha - \lambda} + \frac{y}{\beta - \lambda}\right) = x + y,$$

$T_{|\mathscr{M}} - \lambda I$ is surjective. Hence $\sigma(T_{|\mathscr{M}}) = \{\alpha, \beta\}$. This completes the proof.

Proof of Theorem 8. We may assume that $|\alpha| \geq |\beta|$. Let $\mathscr{M} = K(\alpha) + K(\beta)$. Then $\mathscr{M}$ is a closed subspace and invariant for T by Lemma 1. Hence it holds $\sigma(T_{|\mathscr{M}}) = \{\alpha, \beta\}$ by Lemma 2. Since $T_{|\mathscr{M}}$ is $*$-n-paranormal by Theorem 4, $T_{|\mathscr{M}}$ is normaloid by Proposition 1. Hence $\|T_{|\mathscr{M}}\| = |\alpha|$ and

$$\alpha \in \sigma(T_{|\mathscr{M}}) \subset \overline{V(T_{|\mathscr{M}})} \subset \{z \in \mathbb{C} : |z| \leq |\alpha|\}.$$

Therefore, $\alpha \in \partial V(T_{|\mathscr{M}}) \bigcap \sigma(T_{|\mathscr{M}})$. So we have $\ker(T-\alpha I) \perp R(T-\alpha I)$ by Proposition 3. Let $x \in \ker(T-\alpha I)$ and $y \in \ker(T-\beta I)$ such that $\|x\|=1$. Then

$$1 \leq \|x+(\beta-\alpha)^{-1}(T-\alpha I)y\| = \|x+y+(\beta-\alpha)^{-1}(T-\beta I)y\| = \|x+y\|.$$

Therefore, $\ker(T-\alpha I) \perp \ker(T-\beta I)$. This completes the proof.

Since a $*$-n-paranormal operator T is $(n+1)$-paranormal, we have following corollary.

Corollary 3. *Let T be $*$-n-paranormal operators on $\mathscr{X}$. If α, β are distinct eigen-values of T, then* $\ker(T-\alpha I) \perp \ker(T-\beta I)$.

In [4] M. Chō and Ôta proved that if $\mathscr{X}^*$ is strictly convex and $Tx = \alpha x$ for some $(x,f) \in \Pi(\mathscr{X})$, then $T^* f = \alpha f$ (Theorem 15, [4]). Finally we extend this result for an approximate point spectrum of T on a uniformly convex space.

Definition 5. A Banach space $\mathscr{X}$ is said to be *uniformly convex* if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x\| = \|y\| = 1$ and $\|x-y\| \geq \varepsilon$, then

$$\|x+y\| \ \leq \ 2(1-\delta).$$

By the definition of uniformly convexity, it holds that if $\lim\|x_k\| = \lim\|y_k\| = 1$ and $\lim(\|x_k\|+\|y_k\|) = 2$, then $\lim(\|x_k - y_k\|) = 0$,

Theorem 9. *Let the dual space $\mathscr{X}^*$ be uniformly convex and T be $*$-n-paranormal on $\mathscr{X}$. If $(T-\alpha I)x_k \ \longrightarrow \ 0$ for $(x_k, f_k) \in \Pi(\mathscr{X})$, then $(T-\alpha I)^* f_k \ \longrightarrow \ 0$.*

Proof. If $\alpha = 0$, $\|T^* f_k\|^n \leq \|T^n x_k\|$ and hence $\lim_{k\to\infty} \|T^* f_k\| = 0$. So we may show the theorem for $\alpha \neq 0$. Since $\frac{1}{\alpha}T$ is $*$-n-paranormal, we may assume $\alpha = 1$. Hence we show that if $(T-I)x_k \longrightarrow 0$, then $(T-I)^* f_k \longrightarrow 0$. Since T is $*$-n-paranormal, it holds

$$\|T^* f_k\|^n \leq \|T^n x_k\| \ \longrightarrow \ 1.$$

Hence $\limsup\|T^* f_k\| \leq 1$. Since $f_k(Tx_k) \longrightarrow 1$, it holds

$$2 \geq \lim\sup(\|T^* f_k\| + \|f_k\|) \geq \liminf(\|T^* f_k\| + \|f_k\|)$$

$$\geq \liminf(\|T^* f_k + f_k\|) \geq \liminf|(T^* f_k + f_k)(x_k)| \ \longrightarrow \ 2.$$

Hence $\lim(\|T^* f_k\| + \|f_k\|) = 2$ and $\lim \|T^* f_k\| = 1$.

Since

$$2 \geq \limsup(\|T^* f_k\| + \|f_k\|) \geq \limsup(\|T^* f_k + f_k\|)$$

$$\geq \liminf(\|T^* f_k + f_k\|) \geq \liminf|(T^* f_k + f_k)(x_k)| \longrightarrow 2,$$

we have $\lim \|T^* f_k + f_k\| = 2$.

Since it is clear that $\lim\|T^* f_k\| = \lim\|f_k\| = 1$, by uniformly convexity it holds

$$\lim \|T^* f_k - f_k\| = 0, \text{ i.e., } (T-I)^* f_k \longrightarrow 0.$$

This completes the proof.

Since uniformly convex space is reflexive, following corollary is clear.

Corollary 4. *Let $\mathscr{X}$ be uniformly convex and $T \in B(\mathscr{X})$. If T^* is $*$-n-paranormal on $\mathscr{X}^*$ and $(T-\alpha I)^* f_k \longrightarrow 0$ for $(x_k, f_k) \in \Pi(\mathscr{X})$, then $(T-\alpha I)x_k \longrightarrow 0$.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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