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GENERALIZED HYERS-ULAM STABILITY OF A QUADRATIC FUNCTIONAL EQUATION IN (β, p)-BANACH SPACES

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Abstract. The main goal of this paper is the investigation of the general solution and the generalized Hyers-Ulam stability of the following quadratic functional equation

$$f(ax+y)+\frac{a}{2}[f(x-y)+f(y-x)]=a(a+1)f(x)+(a+1)f(y)$$

in (β, p)-Banach spaces, where a is a nonzero fixed integer such that $a \neq 0,-1,-2$.

Keywords: Hyers-Ulam stability; quadratic function; quadratic functional equation; (β, p)-Banach space.

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1. Introduction and preliminaries

In 1940, Ulam [30] gave a talk before the Mathematics Club of the University of Wisconsin in which proposed the following stability problem, well-known as Ulam stability problem.

Let G_1 be a group and let G_2 be a metric group with the metric $d(.,.)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

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In 1941, Hyers [10] gave a first affirmative answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Th.M. Rassias [23] for linear mappings by considering an unbounded Cauchy difference. Găvruta [9] provided a further generalization of the Rassias' theorem by using a general control function.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called the quadratic functional equation. Quadratic functional equation were used to characterize inner product spaces [1, 2, 11]. In particular every solution of the quadratic functional equation is said to be quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x)=B(x,x)$ for all x (see [1, 11]). The bi-additive mapping is given by

$$B(x,y)=\frac{1}{4}\left[f(x+y)-f(x-y)\right].$$

The generalized Hyers-Ulam stability problem for the above quadratic functional equation was proved by Skof [29] for mapping $f:X\to Y$, where X is a normed space and Y is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if relevant domain X is replaced by an abelian group. In [6], Cherwik proved the generalized Hyers-Ulam of the quadratic functional equation as above. Grabiec [8] has generalized these results mentioned above. Several functional equations have been investigated in [4, 16-19, 24-28].

The notion of quasi-β-normed space was introduced by Rassias and Kim in [22]. This notion is a generalization of that of quasi-normed space. We consider some basic concepts concerning quasi-β-normed space. We fix a real number β with $0<\beta\leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1. Let X be a linear space over $\mathbb{K}$. A quasi-β-norm $\|.\|$ is a real-valued function on X satisfying the following:

(1) $\|x\|\geq 0$ for all $x\in X$ and $\|x\|=0$ if and only if $x=0$,

(2) $\|\lambda x\|=|\lambda|^{\beta}\,\|x\|$ for all $\lambda\in\mathbb{K}$ and all $x\in X$,

(3) there is a constant $K\geq 1$ such that $\|x+y\|\leq K\left(\|x\|+\|y\|\right)$ for all $x,y\in X$.

The pair $(X, \|.\|)$ is called a quasi-β-normed space if is a quasi-β-norm on X. The smallest possible K is called the modulus of concavity of $\|.\|$. A quasi-β-Banach space is a complete quasi-β-normed space. A quasi-β-norm $\|.\|$ is called a (β, p)-norm $0 < p \leq 1$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, the quasi-β-Banach space is called a (β, p)-Banach space. We observe that if $x_1, x_2, ..., x_n$ are nonnegative real numbers, then

$$\left(\sum_{i=0}^{n} x_i\right)^p \leq \sum_{i=0}^{n} x_i^p,$$

where $0 < p \leq 1$ [15].

Rassias [20] investigated the stability of Ulam for the Euler-Lagrange quadratic functional equation

$$f(ax+by) + f(bx-ay) = (a^2+b^2)\,[f(x)+f(y)]$$

Gordji and Khodaei investigated the generalized Hyers-Ulam stability of other Euler-Lagrange quadratic Functional equation [7]. Jun et. al. [14] introduced and proved the general solution and the generalized Hyers-Ulam stability of the Euler-Lagrange quadratic functional equation

$$f(ax+y) + af(x-y) = (a+1)f(y) + a(a+1)f(x), \tag{1.2}$$

for any fixed integer a with $a \neq 0, -1$.

H.-M. Kim and M.-Y. Kim introduced and proved the general solution and the generalized Hyers-Ulam stability of the quadratic functional equation

$$f(ax+by) + af(x-by) = (a+1)b^2 f(y) + a(a+1)f(x)$$

in (β, p)-Banach spaces [13].

In 2010, Rassias [21] introduced the Euler-Lagrange type quadratic functional equation

$$f(x+y) + \frac{1}{2}[f(x-y) + f(y-x)] = 2f(x) + 2f(y), \tag{1.3}$$

and investigated the Rassias ”product-sum” stability of this equation.

Throughout this paper, assume that a is a fixed nonzero integr with $a \neq 0, -1, -2$, we introduce the following functional equation:

$$f(ax+y)+\frac{a}{2}[f(x-y)+f(y-x)] = a(a+1)f(x)+(a+1)f(y). \tag{1.4}$$

In this paper, we establish the general solution and the generalized Hyers-Ulam stability of (1.4) in (β, p)-Banach spaces.

2. General solution of (1.4)

First, we present the general solution of (1.4) in the class of all functions between vector spaces.

Theorem 2.1. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies (1.1) if and only if it satisfies (1.2).*

Proof. See the same proof in [14].

Theorem 2.2. *Let X and Y be vector spaces. A mapping $f : X \to Y$ satisfies (1.1) if and only if it satisfies (1.4).*

Proof. Assume that f satisfies (1.1), then we have

$$f(-x) = f(x) \tag{2.1}$$

for all $x \in X$, it follows from (2.1) and (1.2) that

$$\begin{aligned} f(ax+y)+\frac{a}{2}[f(x-y)+f(y-x)] &= f(ax+y)+\frac{a}{2}[f(x-y)+f(x-y)] \\ &= f(ax+y)+af(x-y) \\ &= a(a+1)f(x)+(a+1)f(y) \end{aligned} \tag{2.2}$$

for all $x, y \in X$. Then (1.1) satisfies (1.4). Conversely, assume that f satisfies (1.4). Letting $x = y = 0$ in (1.4), we get $f(0) = 0$. Letting $x = 0$ in (1.4), we have

$$f(y)+\frac{a}{2}[f(-y)+f(y)] = (a+1)f(y),$$

and so we conclude that f is even. So we have

$$f(ax+y)+af(x-y)=f(ax+y)+\frac{a}{2}\left[f(x-y)+f(x-y)\right], \tag{2.3}$$

for all $x,y\in X$, it follows from (1.4) that

$$f(ax+y)+af(x-y)=a(a+1)f(x)+(a+1)f(y)$$

for all $x,y\in X$. Then from Theorem 1.1, the functional equation (1.4) satisfies (1.1). This completes the proof.

3. Hyers-Ulam stability of (1.4)

For convenience, we define the difference operator $D_f : X\times X\to Y$ by

$$D_f(x,y):=f(ax+y)+\frac{a}{2}\left[f(x-y)+f(y-x)\right]-a(a+1)f(x)-(a+1)f(y), \tag{3.1}$$

for all $x,y\in X$ and a fixed integer such that $a\neq 0,-1,-2$, where $f:X\to Y$ is a given function.

From now on, let X be a vector space, and Y be a (β,p)-Banach space with (β,p)-norm $\|.\|_Y$. Let K be the modulus of concavity of $\|.\|_Y$. We will investigate the generalized Hyers-Ulam stability problem for the functional equation (1.4).

Theorem 3.1. *Let $\varphi : X^2\to[0,\infty)$ and $\Psi_1 : X^2\to[0,\infty)$ be two functions such that*

$$\begin{cases}\Psi_1(x,x):=\sum_{i=0}^{\infty}\frac{1}{|a+1|^{2\beta pi}}\varphi((a+1)^ix,(a+1)^ix)^p<\infty,\\ \lim_{n\to\infty}\frac{1}{|a+1|^{2\beta n}}\varphi\left((a+1)^nx,(a+1)^ny\right)=0,\end{cases} \tag{3.2}$$

for all $x,y\in X$. Let $f:X\to Y$ be a function such that

$$\left\|D_f(x,y)\right\|_Y\leq\varphi(x,y) \tag{3.3}$$

for all $x,y\in X$. Then there exists a unique quadratic function $F:X\to Y$ such that

$$\left\|F(x)+\frac{f(0)}{a+2}-f(x)\right\|_Y\leq\frac{1}{|a+1|^{2\beta}}\Psi_1(x,x)^{\frac{1}{p}} \tag{3.4}$$

for all $x\in X$. The function $F:X\to Y$ is given by

$$F(x)=\lim_{n\to\infty}\frac{f((a+1)^nx)}{(a+1)^{2n}} \tag{3.5}$$

for all $x, y \in X$.

Proof. Letting $y = x$ in (3.3), we get

$$\left\|(a+1)^2 f(x) + af(0) - f((a+1)x)\right\|_Y \leq \varphi(x,x) \tag{3.6}$$

for all $x \in X$. Multiplying both sides by $\frac{1}{|a+1|^{2\beta}}$, we have

$$\left\|\frac{\tilde{f}(x) - \tilde{f}((a+1)x)}{(a+1)^2}\right\|_Y \leq \frac{1}{|a+1|^{2\beta}}\varphi(x,x) \tag{3.7}$$

for all $x \in X$, where

$$\tilde{f}(x) = f(x) - \frac{f(0)}{a+2} \tag{3.8}$$

for all $x \in X$. It follows from (3.7) with $(a+1)^n x$ in place of x and multiplying both sides by $\frac{1}{|a+1|^{2\beta n}}$ that

$$\left\|\frac{\tilde{f}((a+1)^n x)}{(a+1)^{2n}} - \frac{\tilde{f}((a+1)^{n+1}x)}{(a+1)^{2(n+1)}}\right\|_Y \leq \frac{1}{|a+1|^{2\beta(n+1)}}\varphi((a+1)^n x, (a+1)^n x) \tag{3.9}$$

for all $x \in X$. Next we show that the sequence $\left\{\frac{\tilde{f}((a+1)^n x)}{(a+1)^{2n}}\right\}_{n\geq 0}$ is a Cauchy sequence. For any $m, n \in \mathbb{N}, m > n \geq 0$ and $x \in X$, it follows from (3.9) that

$$\begin{aligned}
&\left\|\frac{\tilde{f}((a+1)^n x)}{(a+1)^{2n}} - \frac{\tilde{f}((a+1)^{m+1}x)}{(a+1)^{2(m+1)}}\right\|_Y^p \\
&= \left\|\sum_{i=n}^{m}\left(\frac{\tilde{f}((a+1)^i x)}{(a+1)^{2i}} - \frac{\tilde{f}((a+1)^{i+1}x)}{(a+1)^{2(i+1)}}\right)\right\|_Y^p \\
&\leq \sum_{i=n}^{m}\left\|\frac{\tilde{f}((a+1)^i x)}{(a+1)^{2i}} - \frac{\tilde{f}((a+1)^{i+1}x)}{(a+1)^{2(i+1)}}\right\|_Y^p \\
&\leq \sum_{i=n}^{m}\frac{1}{|a+1|^{2\beta p(i+1)}}\left(\varphi((a+1)^i x, (a+1)^i x)\right)^p \\
&= \frac{1}{|a+1|^{2\beta p}}\sum_{i=n}^{m}\frac{1}{|a+1|^{2\beta pi}}\varphi((a+1)^i x, (a+1)^i x)^p
\end{aligned} \tag{3.10}$$

for all $x \in X$. It follows from (3.2) and (3.10) that the sequence $\left\{\frac{\tilde{f}((a+1)^n x)}{(a+1)^{2n}}\right\}_{n\geq 0}$ is a Cauchy sequence in Y, for all $x \in X$. Since Y is a (β, p)-Banach space, the sequence $\left\{\frac{\tilde{f}((a+1)^n x)}{(a+1)^{2n}}\right\}_{n\geq 0}$

converges for all $x \in X$. Therefore, we can define a function $F : X \to Y$ by

$$F(x) = \lim_{n\to\infty} \frac{\tilde{f}((a+1)^n x)}{(a+1)^{2n}} = \lim_{n\to\infty} \frac{1}{(a+1)^{2n}} \left(f((a+1)^n x) - \frac{f(0)}{a+2} \right) = \lim_{n\to\infty} \frac{f((a+1)^n x)}{(a+1)^{2n}}$$

for all $x \in X$. Taking $m \to \infty$ and $n = 0$ in (3.10), we have

$$\begin{aligned}\left\|F(x) - \tilde{f}(x)\right\|_Y^p &\leq \frac{1}{|a+1|^{2\beta p}} \sum_{i=0}^{\infty} \frac{1}{|a+1|^{2\beta p i}} \varphi((a+1)^i x, (a+1)^i x)^p \\ &= \frac{1}{|a+1|^{2\beta p}} \Psi_1(x,x),\end{aligned}$$

for all $x \in X$. Therefore,

$$\left\|F(x) + \frac{f(0)}{a+2} - f(x)\right\|_Y \leq \frac{1}{|a+1|^{2\beta}} \Psi_1(x,x)^{\frac{1}{p}},$$

for all $x \in X$, that is the function F satisfies (3.4). It follows from (3.5), (3.3) and (3.2) that

$$\begin{aligned}\|D_F(x,y)\|_Y &= \lim_{n\to\infty} \frac{1}{(a+1)^{2n}} \left\|D_f((a+1)^n x, (a+1)^n y)\right\|_Y \\ &\leq \lim_{n\to\infty} \frac{1}{|a+1|^{2\beta n}} \varphi((a+1)^n x, (a+1)^n y) \\ &= 0,\end{aligned}$$

for all $x, y \in X$. Therefore, F satisfies (1.4), and so the function F is quadratic.

To prove the uniqueness of the quadratic function F, let us assume that there exists a quadratic function $G : X \to Y$ which satisfies (1.4) and the inequality (3.4). Then it follows easly that by setting $x = y$ in (1.4), we have $G((a+1)x) = (a+1)^2 G(x)$ and $G((a+1)^n x) = (a+1)^{2n} G(x)$, for all $x \in X$ and all $n \in \mathbb{N}$. Thus one proves by the last equality and (3.4) that

$$\begin{aligned}\left\|\frac{\tilde{f}((a+1)^n x)}{(a+1)^{2n}} - G(x)\right\|_Y^p &= \frac{1}{|a+1|^{2\beta np}} \left\|\tilde{f}((a+1)^n x) - G((a+1)^n x)\right\|_Y^p \\ &\leq \frac{1}{|a+1|^{2\beta np}} . \frac{1}{|a+1|^{2\beta p}} \Psi_1((a+1)^n x, (a+1)^n x),\end{aligned}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Therefore, letting $n \to \infty$, one has $F(x) - G(x) = 0$ for all $x \in X$, completing the proof of uniqueness.

Theorem 3.2. *Let $\varphi : X^2 \to [0,\infty)$ and $\Psi_2 : X^2 \to [0,\infty)$ be two functions such that*

$$\begin{cases} \Psi_2(x,x) := \sum_{i=0}^{\infty} |a+1|^{2\beta pi}\, \varphi(\frac{x}{(a+1)^{i+1}}, \frac{x}{(a+1)^{i+1}})^p < \infty, \\ \lim_{n\to\infty} |a+1|^{2\beta n}\, \varphi(\frac{x}{(a+1)^{n+1}}, \frac{y}{(a+1)^{n+1}}) = 0, \end{cases} \tag{3.11}$$

for all $x,y \in X$. Let $f : X \to Y$ be a function such that

$$\left\|D_f(x,y)\right\|_Y \le \varphi(x,y), \tag{3.12}$$

for all $x,y \in X$. Then there exists a unique quadratic function $F : X \to Y$ such that

$$\|F(x) - f(x)\|_Y \le \Psi_2(x,x)^{\frac{1}{p}}, \tag{3.13}$$

for all $x \in X$. The function $F : X \to Y$ is given by

$$F(x) = \lim_{n\to\infty} (a+1)^{2n} f\left(\frac{x}{(a+1)^n}\right), \tag{3.14}$$

for all $x \in X$.

Proof. In this case, $f(0) = 0$ since $\sum_{i=0}^{\infty} |a+1|^{2\beta pi}\, \varphi(0,0)^p < \infty$, and so $\varphi(0,0) = 0$ by assumption. It follows from (3.6) with $\frac{x}{(a+1)^{n+1}}$ in place of x and multiplying both sides by $|a+1|^{2\beta n}$ that

$$\begin{aligned} &\left\|(a+1)^{2n} f\left(\frac{x}{(a+1)^n}\right) - (a+1)^{2(n+1)} f\left(\frac{x}{(a+1)^{n+1}}\right)\right\|_Y \\ &\le |a+1|^{2\beta n}\, \varphi(\frac{x}{(a+1)^{n+1}}, \frac{x}{(a+1)^{n+1}}) \end{aligned}$$

for all $x \in X$.

The rest of proof is similar to proof of theorem 3.1. This completes the proof.

Theorem 3.3. *Let $\varphi : X^2 \to [0,\infty)$ and $\Psi_3 : X^2 \to [0,\infty)$ be two functions such that*

$$\begin{cases} \Psi_3(x,x) := \sum_{i=0}^{\infty} \frac{K^i \varphi((a+1)^i x, (a+1)^i x)}{|a+1|^{2\beta i}} < \infty, \\ \lim_{n\to\infty} \frac{K^n \varphi((a+1)^n x, (a+1)^n y)}{|a+1|^{2\beta n}} = 0, \end{cases} \tag{3.15}$$

for all $x,y \in X$. Let $f : X \to Y$ be a function such that

$$\left\|D_f(x,y)\right\|_Y \le \varphi(x,y) \tag{3.16}$$

for all $x,y \in X$. Then there exists a unique quadratic function $F : X \to Y$ such that

$$\left\| F(x) + \frac{f(0)}{a+2} - f(x) \right\|_Y \leq \frac{K}{|a+1|^{2\beta}} \Psi_3(x,x) \tag{3.17}$$

for all $x \in X$. The function $F : X \to Y$ is given by

$$F(x) = \lim_{n\to\infty} \frac{f((a+1)^n x)}{(a+1)^{2n}}. \tag{3.18}$$

Proof. Letting $y = x$ in (3.16), we get

$$\left\| (a+1)^2 f(x) + af(0) - f((a+1)x) \right\|_Y \leq \varphi(x,x) \tag{3.19}$$

for all $x \in X$. Multiplying both sides by $\dfrac{1}{|a+1|^{2\beta}}$, we have

$$\left\| \frac{\tilde{f}(x) - \tilde{f}((a+1)x)}{(a+1)^2} \right\|_Y \leq \frac{1}{|a+1|^{2\beta}} \varphi(x,x) \tag{3.20}$$

for all $x \in X$, where

$$\tilde{f}(x) = f(x) - \frac{f(0)}{a+2} \tag{3.21}$$

for all $x \in X$. It follows from (3.20) with $(a+1)^n x$ in place of x and multiplying both sides by $\dfrac{1}{|a+1|^{2\beta n}}$ that

$$\left\| \frac{\tilde{f}((a+1)^n x)}{(a+1)^{2n}} - \frac{\tilde{f}((a+1)^{n+1} x)}{(a+1)^{2(n+1)}} \right\|_Y \leq \frac{1}{|a+1|^{2\beta(n+1)}} \varphi((a+1)^n x, (a+1)^n x) \tag{3.22}$$

for all $x \in X$. By itertive method, we get

$$\begin{aligned} \left\| \frac{\tilde{f}((a+1)^n x)}{(a+1)^{2n}} - \frac{\tilde{f}((a+1)^{m+1} x)}{(a+1)^{2(m+1)}} \right\|_Y &\leq \frac{1}{K^{n-1} |a+1|^{2\beta}} \sum_{i=n}^{m-1} \frac{K^i \varphi((a+1)^i x, (a+1)^i x)}{|a+1|^{2\beta i}} \\ &\quad + \frac{1}{K^n |a+1|^{2\beta}} \cdot \frac{K^m \varphi((a+1)^m x, (a+1)^m x)}{|a+1|^{2\beta m}} \end{aligned} \tag{3.23}$$

for all $x \in X$ and for any $m > n \geq 0$. Thus it follows that the sequence $\left\{ \frac{\tilde{f}((a+1)^n x)}{(a+1)^{2n}} \right\}_{n\geq 0}$ is a Cauchy sequence in Y, for all $x \in X$. Since Y is a (β, p)-Banach space, the sequence $\left\{ \frac{\tilde{f}((a+1)^n x)}{(a+1)^{2n}} \right\}_{n\geq 0}$ converges for all $x \in X$. Therefore, we can define a mapping $F : X \to Y$ by

$$F(x) = \lim_{n\to\infty} \frac{\tilde{f}((a+1)^n x)}{(a+1)^{2n}} = \lim_{n\to\infty} \frac{1}{(a+1)^{2n}} \left(f((a+1)^n x) - \frac{f(0)}{a+2} \right) = \lim_{n\to\infty} \frac{f((a+1)^n x)}{(a+1)^{2n}}$$

for all $x \in X$. Taking $m \to \infty$ and $n = 0$ in (3.12), we have

$$\begin{aligned} \left\| F(x) - \tilde{f}(x) \right\|_Y &\leq \frac{K}{|a+1|^{2\beta}} \sum_{i=0}^{\infty} \frac{K^i \varphi((a+1)^i x, (a+1)^i x)}{|a+1|^{2\beta i}} \\ &= \frac{K}{|a+1|^{2\beta}} \Psi_3(x,x), \end{aligned} \tag{3.24}$$

for all $x \in X$. Therefore,

$$\left\| F(x) + \frac{f(0)}{a+2} - f(x) \right\|_Y \leq \frac{K}{|a+1|^{2\beta}} \Psi_3(x,x),$$

for all $x \in X$, that is the mapping F satisfies (3.17). It follows from (3.18), (3.16) and (3.15) that

$$\begin{aligned} \|D_F(x,y)\|_Y &= \lim_{n\to\infty} \left\| \frac{1}{(a+1)^{2n}} D_f((a+1)^n x, (a+1)^n y) \right\|_Y \\ &= \lim_{n\to\infty} \frac{1}{|a+1|^{2\beta n}} \left\| D_f((a+1)^n x, (a+1)^n y) \right\|_Y \\ &\leq \lim_{n\to\infty} \frac{1}{|a+1|^{2\beta n}} \varphi((a+1)^n x, (a+1)^n y) \\ &\leq \lim_{n\to\infty} \frac{K^n}{|a+1|^{2\beta n}} \varphi((a+1)^n x, (a+1)^n y) \\ &= 0, \end{aligned}$$

for all $x, y \in X$. Therefore, F satisfies (1.4), and so the mapping F is quadratic.

To prove the uniqueness of the quadratic function F, let us assume that there exists a quadratic function $G : X \to Y$ which satisfies (1.4) and the inequality (3.16). Then it follows easly that by setting $x = y$ in (1.4), we have $G((a+1)x) = (a+1)^2 G(x)$ and $G((a+1)^n x) = (a+1)^{2n} G(x)$, for all $x \in X$ and all $n \in \mathbb{N}$. Thus one proves by the last equality and (3.17) that

$$\begin{aligned} \left\| \frac{\tilde{f}((a+1)^n x)}{(a+1)^{2n}} - G(x) \right\|_Y &= \frac{1}{|a+1|^{2\beta n}} \left\| \tilde{f}((a+1)^n x) - G((a+1)^n x) \right\|_Y \\ &\leq \frac{1}{|a+1|^{2\beta n}} \cdot \frac{K}{|a+1|^{2\beta}} \Psi_3((a+1)^n x, (a+1)^n x), \end{aligned}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Therefore, letting $n \to \infty$, one has $F(x) - G(x) = 0$ for all $x \in X$, completing the proof of uniqueness.

Theorem 3.4. *Let $\varphi : X^2 \to [0,\infty)$ and $\Psi_4 : X^2 \to [0,\infty)$ be two functions such that*

$$\begin{cases} \Psi_4(x,x) := \sum_{i=0}^{\infty} \left(K\,|a+1|^{2\beta}\right)^i \varphi\left(\frac{x}{(a+1)^{i+1}}, \frac{x}{(a+1)^{i+1}}\right) < \infty, \\ \lim_{n\to\infty} \left(K\,|a+1|^{2\beta}\right)^n \varphi\left(\frac{x}{(a+1)^{n+1}}, \frac{y}{(a+1)^{n+1}}\right) = 0, \end{cases} \tag{3.25}$$

for all $x,y \in X$. Let $f : X \to Y$ be a function such that

$$\left\|D_f(x,y)\right\|_Y \leq \varphi(x,y) \tag{3.26}$$

for all $x,y \in X$. Then there exists a unique quadratic function $F : X \to Y$ such that

$$\|F(x) - f(x)\|_Y \leq K\Psi_4(x,x) \tag{3.27}$$

for all $x \in X$. The function $F : X \to Y$ is given by

$$F(x) = \lim_{n\to\infty} (a+1)^{2n} f\left(\frac{x}{(a+1)^n}\right) \tag{3.28}$$

for all $x \in X$.

Proof. In this case, $f(0) = 0$ since $\sum_{i=0}^{\infty} \left(K\,|a+1|^{2\beta}\right)^i \varphi(0,0) < \infty$, and so $\varphi(0,0) = 0$ by assumption. It follows from (3.19) and the similar method to (3.23) that

$$\begin{aligned} &\left\|(a+1)^{2n} f\left(\frac{x}{(a+1)^n}\right) - (a+1)^{2(n+1)} f\left(\frac{x}{(a+1)^{n+1}}\right)\right\|_Y \\ &\leq |a+1|^{2\beta n} \varphi\left(\frac{x}{(a+1)^{n+1}}, \frac{x}{(a+1)^{n+1}}\right) \end{aligned}$$

for all $x \in X$. By itertive method, we get

$$\begin{aligned} \left\|\frac{\tilde{f}((a+1)^n x)}{(a+1)^{2n}} - \frac{\tilde{f}((a+1)^{m+1} x)}{(a+1)^{2(m+1)}}\right\|_Y &\leq \frac{1}{K^{n-1}} \sum_{i=n}^{m-1} \left(K\,|a+1|^{2\beta}\right)^i \varphi\left(\frac{x}{(a+1)^{i+1}}, \frac{x}{(a+1)^{i+1}}\right) \\ &\quad + \frac{\left(K\,|a+1|^{2\beta}\right)^m}{K^n} \varphi\left(\frac{x}{(a+1)^{m+1}}, \frac{x}{(a+1)^{m+1}}\right) \end{aligned}$$

for all $x \in X$ and for any $m > n \geq 0$. Therefore we see that a function $F : X \to Y$ defined by

$$F(x) = \lim_{n\to\infty} |a+1|^{2n} f\left(\frac{x}{(a+1)^n}\right)$$

is well defined for all $x \in X$. The rest assertion does through by the similar way to corresponding part of theorem 3.3. This completes the proof.

In the following corollary, we get the stability of (1.4) in the sens of J.M. Rassias.

Corollary 3.5. *Let X be a quasi-α-normed space with (α,p)-norm $\|.\|$ for fixed real number α with $0<\alpha\le 1$. Let δ,γ_1,γ_2 be real numbers such that $\gamma_1\gamma_2\ge 0$, $\delta\ge 0$, $K|a+1|^{2\beta}\neq |a+1|^{\alpha(\gamma_1+\gamma_2)}$ if $\gamma_1,\gamma_2>0$ and $K|a+1|^{\alpha(\gamma_1+\gamma_2)}\neq |a+1|^{2\beta}$ if $\gamma_1,\gamma_2<0$. Assume that a function $f:X\to Y$ satisfies the inequality*

$$\left\|D_f(x,y)\right\|_Y\le \delta\|x\|^{\gamma_1}\|y\|^{\gamma_2} \tag{3.29}$$

for all $x,y\in X$ and $X\backslash\{0\}$ if $\gamma_1,\gamma_2<0$. Then there exists a unique quadratic function $F:X\to Y$ wich satisfies (1.4) and the inequality

$$\left\|F(x)+\frac{f(0)}{a+2}-f(x)\right\|_Y\le\begin{cases}\dfrac{K\delta\|x\|^{(\gamma_1+\gamma_2)}}{|a+1|^{\alpha(\gamma_1+\gamma_2)}-K|a+1|^{2\beta}}, & \text{if } K|a+1|^{2\beta}<|a+1|^{\alpha(\gamma_1+\gamma_1)}\\ \text{and }\ \gamma_1,\gamma_2>0 & \\ \dfrac{K\delta\|x\|^{(\gamma_1+\gamma_2)}}{|a+1|^{2\beta}-K|a+1|^{\alpha(\gamma_1+\gamma_2)}}, & \text{if } K|a+1|^{\alpha(\gamma_1+\gamma_1)}<|a+1|^{2\beta}\\ \text{and }\ \gamma_1,\gamma_2<0 & \end{cases}$$

for all $x\in X$ and $X\backslash\{0\}$ if $\gamma_1,\gamma_2<0$, where $f(0)=0$ if $\gamma_1,\gamma_2>0$. The function F is given by

$$F(x)=\begin{cases}\lim\limits_{n\to\infty}(a+1)^{2n}f\left(\dfrac{x}{(a+1)^n}\right), & \text{if } K|a+1|^{2\beta}<|a+1|^{\alpha(\gamma_1+\gamma_1)}\text{and }\ \gamma_1,\gamma_2>0\\ \lim\limits_{n\to\infty}\dfrac{f((a+1)^nx)}{(a+1)^{2n}}, & \text{if } K|a+1|^{\alpha(\gamma_1+\gamma_1)}<|a+1|^{2\beta}\text{and }\ \gamma_1,\gamma_2<0\end{cases}$$

for all $x\in X$ and $X\backslash\{0\}$ if $\gamma_1,\gamma_2<0$.

Proof.

(1) If $\gamma_1,\gamma_1>0$, and $K|a+1|^{2\beta}<|a+1|^{\alpha(\gamma_1+\gamma_1)}$, we put $x=y=0$ in (3.29) and get $f(0)=0$. Let

$$\varphi(x,y)=\delta\|x\|^{\gamma_1}\|y\|^{\gamma_2}$$

for all $x,y\in X$. Then

$$\begin{aligned}\Psi_4(x,x):=&\sum_{i=0}^{\infty}\left(K|a+1|^{2\beta}\right)^i\delta\left\|\frac{x}{(a+1)^{i+1}}\right\|^{\gamma_1}\left\|\frac{x}{(a+1)^{i+1}}\right\|^{\gamma_2}\\ &=\frac{\delta\|x\|^{\gamma_1+\gamma_2}}{|a+1|^{\alpha(\gamma_1+\gamma_1)}}\sum_{i=0}^{\infty}\frac{\left(K|a+1|^{2\beta}\right)^i}{|a+1|^{\alpha(\gamma_1+\gamma_1)i}}\\ &<\infty,\end{aligned}$$

for all $x \in X$, and

$$\begin{aligned}
&\lim_{n\to\infty} \left(K\left|a+1\right|^{2\beta}\right)^n \varphi\left(\frac{x}{(a+1)^{n+1}}, \frac{y}{(a+1)^{n+1}}\right) \\
&= \lim_{n\to\infty} \left(K\left|a+1\right|^{2\beta}\right)^n \delta \left\|\frac{x}{(a+1)^{n+1}}\right\|^{\gamma_1} \left\|\frac{y}{(a+1)^{n+1}}\right\|^{\gamma_2} \\
&= \lim_{n\to\infty} \frac{\delta \|x\|^{\gamma_1} \|y\|^{\gamma_2}}{|a+1|^{\alpha(\gamma_1+\gamma_1)}} \left(\frac{K\left|a+1\right|^{2\beta}}{|a+1|^{\alpha(\gamma_1+\gamma_1)}}\right)^n \\
&= 0
\end{aligned}$$

for all $x, y \in X$. By Theorem 3.4, there exists a unique quadratic mapping $F : X \to Y$ such that

$$\begin{aligned}
\|F(x) - f(x)\|_Y &\le K\Psi_4(x,x) \\
&= \frac{\delta K \|x\|^{(\gamma_1+\gamma_2)}}{|a+1|^{\alpha(\gamma_1+\gamma_2)} - K\left|a+1\right|^{2\beta}}
\end{aligned}$$

for all $x \in X$.

(2) If $\gamma_1, \gamma_1 < 0$, and $K\left|a+1\right|^{\alpha(\gamma_1+\gamma_1)} < |a+1|^{2\beta}$. Let

$$\varphi(x,y) = \delta \|x\|^{\gamma_1} \|y\|^{\gamma_2}$$

for all $x, y \in X\backslash\{0\}$. Then

$$\begin{aligned}
\Psi_3(x,x) &:= \sum_{i=0}^{\infty} \left(\frac{K}{|a+1|^{2\beta}}\right)^i \delta \left\|(a+1)^i x\right\|^{\gamma_1} \left\|(a+1)^i x\right\|^{\gamma_2} \\
&= \delta \|x\|^{\gamma_1+\gamma_2} \sum_{i=0}^{\infty} \left(K\left|a+1\right|^{\alpha(\gamma_1+\gamma_2)-2\beta}\right)^i \\
&= \frac{\delta \|x\|^{\gamma_1+\gamma_2}}{1 - |a+1|^{\alpha(\gamma_1+\gamma_1)-2\beta}} \\
&< \infty,
\end{aligned}$$

for all $x \in X\backslash\{0\}$, and

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{K}{|a+1|^{2\beta}}\right)^n \delta \|(a+1)^n x\|^{\gamma_1} \|(a+1)^n y\|^{\gamma_2} &= \lim_{n\to\infty} \delta \|x\|^{\gamma_1} \|y\|^{\gamma_2} \left(K\left|a+1\right|^{\alpha(\gamma_1+\gamma_2)-2\beta}\right)^n \\
&= 0
\end{aligned}$$

for all $x, y \in X \backslash \{0\}$. By Theorem 3.3, there exists a unique quadratic mapping $F : X \to Y$ such that

$$\begin{aligned}\|F(x) - f(x)\|_Y &\leq \frac{K}{|a+1|^{2\beta}} \Psi_3(x,x) \\ &= \frac{\delta K \|x\|^{(\gamma_1+\gamma_2)}}{|a+1|^{2\beta} - K|a+1|^{\alpha(\gamma_1+\gamma_2)}}\end{aligned}$$

for all $x \in X \backslash \{0\}$.

This completes the proof.

The following corollary, we obtain a stability result of (1.4) in the sens of Rassias.

Corollary 3.6. *Let X be a quasi-α-normed space with (α, p)-norm $\|.\|$ for fixed real number α with $0 < \alpha \leq 1$. Let δ, γ be real numbers such that $\delta \geq 0$, $K|a+1|^{2\beta} \neq |a+1|^{\gamma\alpha}$ if $\gamma > 0$ and $K|a+1|^{\gamma\alpha} \neq |a+1|^{2\beta}$ if $\gamma < 0$. Assume that a function $f : X \to Y$ satisfies the inequality*

$$\left\|D_f(x,y)\right\|_Y \leq \delta \left(\|x\|^\gamma + \|y\|^\gamma\right) \tag{3.30}$$

for all $x \in X$ and $X \backslash \{0\}$ if $\gamma < 0$. Then there exists a unique quadratic function $F : X \to Y$ wich satisfies (1.4) and the inequality

$$\left\|F(x) - \frac{f(0)}{a+2} - f(x)\right\|_Y \leq \begin{cases} \dfrac{2K\delta\|x\|^\gamma}{|a+1|^{\gamma\alpha} - K|a+1|^{2\beta}}, & \text{if } \ K|a+1|^{2\beta} < |a+1|^{\gamma\alpha} \text{ and } \ \gamma > 0 \\ \dfrac{2K\delta\|x\|^\gamma}{|a+1|^{2\beta} - K|a+1|^{\gamma\alpha}}, & \text{if } \ K|a+1|^{\gamma\alpha} < |a+1|^{2\beta} \text{ and } \ \gamma < 0 \end{cases}$$

for all $x \in X$ and $X \backslash \{0\}$ if $\gamma < 0$, where $f(0) = 0$ if $\gamma > 0$. The function F is given by

$$F(x) = \begin{cases} \lim\limits_{n\to\infty} (a+1)^{2n} f\left(\dfrac{x}{(a+1)^n}\right), & \text{if } \ K|a+1|^{2\beta} < |a+1|^{\gamma\alpha} \text{ and } \ \gamma > 0 \\ \lim\limits_{n\to\infty} \dfrac{f((a+1)^n x)}{(a+1)^{2n}}, & \text{if } \ K|a+1|^{\gamma\alpha} < |a+1|^{2\beta} \text{ and } \ \gamma < 0 \end{cases}$$

for all $x \in X$ and $X \backslash \{0\}$ if $\gamma < 0$.

Proof. If $\gamma > 0$, we put $x = y = 0$ in (3.30) and get $f(0) = 0$. Let

$$\varphi(x,y) := \delta \left(\|x\|^\gamma + \|y\|^\gamma\right)$$

for all $x \in X$. Then applying Theorem 3.4 and Theorem 3.3, we obtain the desired results.

In Corollary 3.7, we obtain the stability of (1.4) in the sens of Hyers-Ulam.

Corollary 3.7. *Assume that for some $\delta \geq 0$ a function $f : X \to Y$ satisfies the inequality*

$$\left\|D_f(x,y)\right\|_Y \leq \delta \tag{3.31}$$

for all $x,y \in X$. Then there exists a unique quadratic function $F : X \to Y$ wich satisfies (1.4) and the inequality

$$\left\|F(x) - \frac{f(0)}{a+2} - f(x)\right\|_Y \leq \frac{K\delta}{|a+1|^{2\beta} - K}, \quad if\ K < |a+1|^{2\beta}$$

for all $x \in X$. The function F is given by

$$F(x) = \lim_{n\to\infty} \frac{f((a+1)^n x)}{(a+1)^{2n}}$$

for all $x \in X$.

Proof. Let $\varphi(x,y) := \delta$. Then φ satisfies the condition (3.15), and so we get the desired result.

Conflict of Interests

The authors declare that there is no conflict of interests.

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